22/01/2014

Problem solving seminar

Tomasz Tkocz

Inequalities II

The worst case scenario

1. Prove the AM-GM inequality using the *worst case scenario* method (push cleverly the atoms).

2. Given $\alpha > 0$ find inf and sup of $\int_0^1 (f(x))^2 \mathrm{d}x$ subject to integrable nonincreasing functions $f\colon [0,1] \longrightarrow [0,\infty)$ with $\int_0^1 f(x)\mathrm{d}x = \alpha$.

3. Prove that for a nonincreasing sequence $(x_i)_{i=1}^n$ of positive numbers we have

$$\sum_{i=1}^{n-1} \frac{1}{\sqrt{i}} \sqrt{\sum_{j=i+1}^{n} x_j^2} < \frac{\pi}{2} \sum_{i=1}^{n} x_i.$$

4. Let $\phi\colon [0,\infty) \longrightarrow \mathbb{R}$ be a convex function and $\phi(0) = 0$, $\phi(x) \xrightarrow[x\to+\infty]{} +\infty$. For $t \geq 0$ define

$$T_1(t) = \int_t^\infty e^{-\phi(x)}\mathrm{d}x, \qquad T_2(t) = \int_t^\infty e^{-\alpha x}\mathrm{d}x,$$

where $\alpha > 0$ is chosen so that $T_1(0) = T_2(0)$. Prove that for all $t \geq 0$,

$$T_1(t) \leq T_2(t).$$

5. Given a nonincreasing differentiable function $f\colon (0,+\infty) \longrightarrow (0,+\infty)$ prove that

$$\int_0^\infty e^{-t-f(t)} \sqrt{1 + (f'(t))^2} dt \geq \sqrt{\alpha^2 + (\alpha - e^{-f(0)})^2},$$

where $\alpha = \int_0^\infty e^{-t-f(t)} dt$.