Problem solving seminar

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Inequalities I

Warm-up

1. Let $0 < a < b$. Prove that

$$\int_a^b (x^2+1)e^{-x^2} \geq e^{-a^2} - e^{-b^2}.$$

Averaging

2. Given 2014 points $P_1, \ldots, P_{2014}$ in the unit disk D on the plane, prove that there exists a point $P \in D$ such that

$$\sum_{i=1}^{2014} |PP_i| \geq 2014.$$

3. Let $n \geq 2$ and let $A = [a_{ij}]_{i,j=1}^n$ be a real matrix with $a_{ii} = 0$, $i = 1, \ldots, n$. Prove that there is a subset $I \subset \{1, \ldots, n\}$ such that

$$\sum_{i \in I, j \notin I} a_{ij} + \sum_{i \notin I, j \in I} a_{ij} \geq \frac{1}{2} \sum_{i \neq j} a_{ij}.$$

Integrals

4. Let $\{D_1, \ldots, D_n\}$ be a family of disks on the plane and let $a_{ij} = |D_i \cap D_j|$ be the surface area of the intersection $D_i \cap D_j$ for $i, j = 1, \ldots, n$. Prove that for every real numbers $x_1, \ldots, x_n$,

$$\sum_{i,j=1}^n a_{ij} x_i x_j \geq 0.$$

5 (†). Let a, b, c, x, y, z, q be positive numbers and $1 \leq x, y, z \leq 4$. Show that

$$\frac{x}{(2a)^q} + \frac{y}{(2b)^q} + \frac{z}{(2c)^q} \geq \frac{y+z-x}{(b+c)^q} + \frac{z+x-y}{(c+a)^q} + \frac{x+y-z}{(a+b)^q}.$$

Weights

6 (†). Prove that for positive numbers $a_1, a_2, \ldots$ such that $\sum_{i=1}^\infty a_i < \infty$ we have

$$\sum_{n=1}^\infty (a_1 \cdot \ldots \cdot a_n)^{1/n} < e \sum_{n=1}^\infty a_n \qquad \text{(Carleman's inequality)}.$$

Remark. † questions may be slightly harder.