Solving $ay'' + by' + cy = 0$ Without Characteristic Equations, Complex Numbers, or Hats

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Some calculus courses feature an introduction to ODEs for those students who do not need a full-fledged differential equations course. First order linear equations of the form

$$y'(t) + p(t)y(t) = q(t) \tag{1}$$

and second order linear equations with constant coefficients, of the form

$$ay''(t) + by'(t) + cy(t) = 0 \tag{2}$$

are typically covered. The standard approach is to apply a solution method for (2) which is arguably imported without context from the differential equations curriculum. This approach involves the *characteristic equation*

$$ar^2 + br + c = 0$$

associated with (2); it requires complex numbers, and it necessarily raises the problematic issue of whether there is a natural way to discover that $y(t) = te^{rt}$ is a solution to (2) when $b^2 - 4ac = 0$.

However, in the context of a calculus course, one might consider it preferable to bring to bear as much actual differentiation theory, and antidifferentiation theory, as possible upon the enterprise of solving differential equations. There is a way to solve (2) using only ideas and tools already familiar to the calculus student, save one piece of unfamiliar theory, namely, that the complete solution set to a linear ODE may be generated by linear combinations of (the right number of) particular solutions.

The pedagogical approach we advocate here is to solve (1) and then generalize the basic strategy to (2). The method of solving (2) is not new, and is far from unknown, but it is seldom used. It has been relegated to exercises, its pedagogical merits largely unrecognized. We reiterate that this approach is best suited to a curriculum in which no ODEs more general than (2) will be considered, and for which it is preferable to omit any theory which is otherwise only pertinent to differential equations courses.

We start by noting the similarity of the left hand side of (1) to the right hand side of

$$\frac{d}{dt}(fg) = fg' + f'g,$$

the Product Rule for differentiation. If y plays the role of g, then for

$$y' + py$$

to be the derivative of an actual product, we would need $f \equiv 1$, which of course will not work. But if we multiply (1) by an as-yet-unknown and never-zero function $w(t)$, we get

$$wy' + wpy = wq,$$

and then we need $w' = wp$ in order to transform (1) into the equivalent equation

$$\frac{d}{dt}(wy) = wq, \tag{3}$$

which we can then solve via antidifferentiation; in other words, (3) is a differential equation of the simplest type,

$$\frac{d}{dt}(\text{unknown function}) = \text{given function}.$$

And of course, the "multiplier" function w which serves as an ingredient to help us reduce (1) to (3) is

$$w(t) = \exp\left(\int p(t)dt\right).$$

Now let's consider equations of type (2), but let us choose the alternate standard form

$$y'' + ay' + by = 0. \tag{4}$$

Now multiply but an as-yet-unknown, never-zero function $w(t)$:

$$wy'' + awy' + bwy = 0. \tag{5}$$

We now have two derivatives involved. Piggybacking on our strategy for (1), we might try reducing (5) to the simple differential equation

$$\frac{d^2}{dt^2}(w(t)y(t)) = 0,$$

which we could then solve with two antidifferentiation steps. So let's see if that will work. The Product Rule for second derivatives is

$$(fg)'' = fg'' + 2f'g' + f''g,$$

so we want to fit the left hand side of (5) into this mould. This would mean that y should play the role of g, and then w should play the role of f, just as before. But now we need both of

- $w' = \frac{a}{2}w$
- $w'' = bw$

to hold for

$$fg'' + 2f'g' + f''g = wy'' + awy' + bwy$$

to be true. The first condition requires $w(t) = e^{at/2}$. But then we would have $w''(t) = \frac{a^2}{4}e^{at/2} = \frac{a^2}{4}w(t)$, and so the second condition only holds if $b = \frac{a^2}{4}$.

So it is not generally possible to express the left hand side of (5) as the second derivative of a product. However, we *can* rewrite (5) as

$$wy'' + awy' + \frac{a^2}{4}wy + \left(b - \frac{a^2}{4}\right)wy = 0. \tag{6}$$

Then let $w(t) = e^{at/2}$, and recognize the first three terms of (6),

$$wy'' + awy' + \frac{a^2}{4}wy,$$

as the second derivative of $w(t)y(t)$. Therefore (6) becomes

$$\frac{d^2}{dt}(w(t)y(t)) + \left(b - \frac{a^2}{4}\right)wy = 0.$$

Setting $u = wy$, we get

$$u'' + \left(b - \frac{a^2}{4}\right)u = 0.$$

One can point out to the students that the preceding steps are similar to the technique of *completing the square*, used to solve (quadratic) algebraic equations.

At this point, one can discuss how to solve simple second order equations of the form

$$u'' + cu = 0,$$

or the equivalent form

$$u'' = \beta u. \tag{7}$$

(The instructor may even wish to discuss this type of equation much earlier so it is already familiar to the students.) Regardless of the value of β, the calculus student can pretty well solve (7) "by inspection". We simply note that if $\beta = 0$, solutions are linear and can be obtained by two antidifferentiations; if $\beta > 0$, the exponential solutions

$$u_1(t) = \exp(\sqrt{\beta}t) \text{ and } u_2(t) = \exp(-\sqrt{\beta}t)$$

are easy to find; and if $\beta < 0$, we can easily find solutions

$$u_1(t) = \cos(\sqrt{-\beta}t) \text{ and } u_2(t) = \sin(\sqrt{-\beta}t).$$

So the strategy is to have students convert the problem of solving any given instance of (4) to that of solving an equation of the form (7). Note what happens if $\beta = 0$, which corresponds to $b - \frac{a^2}{4} = 0$. Then solutions to (6) are all of the form

$$u(t) = C_1 t + C_2,$$

and then, converting back to y, we get solutions to (4) all of the form

$$y(t) = C_1 t e^{-at/2} + C_2 e^{-at/2}.$$

And so we find the particular solution $y(t) = te^{-at/2}$ without either (i) doing anything special to look for it, e.g., reduction of order, or using the Wronskian, or (ii) simply producing it like a rabbit from a hat. All of the calculus texts in the references do (i) or (ii).

If $\beta \neq 0$, then solving (7) is not quite as simple as performing two antidifferentiations. But we *are* able to find two particular solutions, whether of exponential or trigonometric type. So, just as in the usual treatment of (2) in calculus courses, we do need one piece of differential equations theory at this point, namely the result that all solutions to a second order linear ODE may be generated by linear combinations of two particular solutions, neither a constant multiple of the other.

The only memorization students must do to apply this solution approach is to remember to multiply (4) through by $e^{-at/2}$. And complex numbers are not needed; in particular, we never need to explain to students why a function like

$$y(t) = e^{(\alpha + i\beta)t}$$

should make any sense to anyone. Ultimately we seek real-valued solutions, so not having to deal with complex-valued solutions, and complex linear combinations thereof, is a plus. Certainly the characteristic equation approach, generalizable as it is to higher order ODEs, together with the complex-valued solutions one must admit, is useful and powerful in a differential equations course. But the approach is cumbersome and unnatural if (4) is the terminus of one's ODE studies. Perhaps some calculus instructors will find the method outlined here to be an attractive alternative.

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