Coloring H-free hypergraphs

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Abstract

Fix $r \geq 2$ and a collection of r-uniform hypergraphs $\mathcal{H}$. What is the minimum number of edges in an $\mathcal{H}$-free r-uniform hypergraph with chromatic number greater than k? We investigate this question for various $\mathcal{H}$. Our results include the following:
• An (r, l)-system is an r-uniform hypergraph with every two edges sharing at most l vertices. For k sufficiently large, there is an (r, l)-system with chromatic number greater than k and number of edges at most $c(k^{r-1} \log k)^{l/(l-1)}$, where

$$c = 2\left(\frac{100(r)_l^2}{l!}\right)^{1/(l-1)}\left(\frac{10(r-1)}{l-1}\right)^{l/(l-1)}.$$

This improves on the previous best bounds of Kostochka-Mubayi-Rödl-Tetali [8]. The upper bound is sharp apart from the constant c as shown in [8].
• The minimum number of edges in an r-uniform hypergraph with independent neighborhoods and chromatic number greater than k is of order $\tilde{k}^{r+1/(r-1)}$ as $k \to \infty$. This generalizes (aside from logarithmic factors) a result of Gimbel and Thomassen [6] for triangle-free graphs.
• Let T be an r-uniform hypertree of t edges. Then every T-free r-uniform hypergraph has chromatic number at most $2(r-1)(t-1)+1$. This generalizes the well known fact that every T-free graph has chromatic number at most t.

Several open problems and conjectures are also posed.

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1 Introduction

An r-graph is a hypergraph whose edges all have size r. The chromatic number of an r-graph is the minimum number of colors required to partition its vertex set so that no edge is monochromatic. The starting point of our investigations is the following basic question:

What is the minimum number $m_k(r)$ of edges in an r-graph with chromatic number greater than k?

In general this problem is very difficult to solve exactly, and so we seek asymptotic results as one or both of k, r tend to infinity. It is easy to observe that $m_k(2) = \binom{k+1}{2}$, but already determining $m_k(3)$ is a challenging open question: $m_2(3) = 7$ achieved by the Fano plane, but $m_3(3)$ is unknown. For fixed r and large k, the best known bounds are due to Alon and are still far apart.

Theorem 1 (Alon [1]) *For every $k, r \geq 2$,*

$$(r-1)\left\lceil \frac{k}{r} \right\rceil \left(\frac{r-1}{r}k\right)^{r-1} < m_k(r) < \binom{kr+1}{r}\frac{5\log r}{r}.$$

Note that this implies that for fixed r and $k \to \infty$ we have $m_k(r) = \Theta(k^r)$. In the opposite direction, determining $m_2(r)$ is the well known problem concerning the minimum number of edges in r-graphs without Property B. The best upper bound follows from an old probabilistic construction of Erdős [4] while Radhakrishnan and Srinivasan [10] proved the lower bound below (for large r) which is the best to date.

$$0.7\sqrt{r/\log r} \times 2^r < m_2(r) < r^2 2^r.$$

Here we consider the same question, but we impose a natural restriction on the underlying r-graph.

Definition 2 *Fix $r \geq 2$ and a collection of r-graphs $\mathcal{H}$. Let $m_k(\mathcal{H})$ be the minimum number of edges in an $\mathcal{H}$-free r-graph with chromatic number greater than k.*

Note that $\mathcal{H}$-free in the definition above refers to not necessarily induced subhypergraphs. Also, if $\mathcal{H} = \{H\}$, we will abuse notation by writing $m_k(H)$.

Our goal is to determine $m_k(\mathcal{H})$ for various $\mathcal{H}$. Special cases of this parameter have already been studied, and lead to difficult problems. For example, Gimbel and Thomassen [6] proved that $m_2(K_3)$ has order of magnitude $(\log k)^2 k^3$ as $k \to \infty$. However, determining

$m_k(K_4)$ and $m_k(C_4)$ are open problems. In fact, these problems seem harder than determining the Ramsey numbers for the corresponding graphs, and the growth rates of these Ramsey numbers are well-studied and not known. Call a hypergraph *nontrivial* if it has at least two edges.

Definition 3 *Let H be a nontrivial r-graph. Then*

$$\rho(H) = \max_{H' \subset H} \frac{e' - 1}{v' - r},$$

where H' is nontrivial with v' vertices and e' edges. For a finite family $\mathcal{H}$ of nontrivial r-graphs, $\rho(\mathcal{H}) = \min_{H \in \mathcal{H}} \rho(H)$.

The parameter ρ appears to be the crucial hypergraph invariant for our problem. Our main result stated below provides a very general upper bound for $m_k(\mathcal{H})$. As we will show, in many cases this general upper bound seems to give the correct order of magnitude for fixed r as $k \to \infty$.

Theorem 4 *Let $\mathcal{H} = \{H_1, H_2, \ldots, H_\ell\}$ be a finite family of nontrivial r-graphs with $v_i = v(H_i)$ and $e_i = e(H_i)$. Let $\rho_i = \rho(H_i)$ and assume that $\rho_i \leq \rho_{i+1}$ for $1 \leq i \leq \ell$. Define s by $\rho = \rho_1 = \rho_2 = \cdots = \rho_s < \rho_{s+1}$ and assume that $\rho > 1/(r-1)$. For each i and each edge $e \in H_i$ let $\alpha_i(e)$ be the number of automorphisms of H_i that map e to itself. Let $\alpha_i = \min_e \alpha_i(e)$.*
Let c_1 be the solution to

$$\sum_{i=1}^{s} \frac{e_i x^{e_i - 1}}{\alpha_i} = \frac{1}{50r!}.$$

and let

$$c_2 = \max_{i=1}^{s} \left(\frac{\alpha_i}{5 e_i c_1^{e_i - 1} r!} \right)^{1/(r-1)} \geq 10^{1/(r-1)}.$$

Then for large k,

$$m_k(\mathcal{H}) < c_{\mathcal{H}} (k^{r-1} \log k)^{(r-1/\rho)/(r-1-1/\rho)}$$

where

$$c_{\mathcal{H}} = 2 \left(\frac{r!}{c_1} \right)^{1/(r-1-1/\rho)} c_2^{(r-1)(r-1/\rho)/(r-1-1/\rho))} \left(\frac{r-1}{r-1-1/\rho} \right)^{(r-1/\rho)/(r-1-1/\rho)}$$

(We can if we wish replace the 2 by a constant arbitrarily close to 1).

Note that the exponent of k in Theorem 4 is always greater than r.

1.1 (r, l)-systems

An (r, l)-system is an r-graph with every two edges sharing at most l vertices. Let $m_k(r, l)$ denote the minimum number of edges in an (r, l)-system with chromatic number greater than k. Erdős and Lovász [7] studied $m_k(r, 2)$, indeed the Local lemma was originally developed and used to give lower bounds for this parameter. Recently, Kostochka et.al. [8] proved that $m_k(r, l)$ has order of magnitude $(k^{r-1} \log k)^{l/(l-1)}$ as $k \to \infty$. They proved the upper bound $m_k(r, l) < b_{r,l}(k^{r-1} \log k)^{l/(l-1)}$ where

$$b_{r,l} = \frac{2(2r^{3l})^{l/(l-1)}}{(r)_l}.$$

Using Theorem 4 we can substantially improve this constant.

Theorem 5 *Fix $2 \le l < r$ and let k be sufficiently large. Then $m_k(r, l) < c_{r,l}(k^{r-1} \log k)^{l/(l-1)}$, where*

$$c_{r,l} = 2\left(\frac{100(r)_l^2}{l!}\right)^{1/(l-1)} \left(\frac{10(r-1)}{l-1}\right)^{l/(l-1)}.$$

Note that for large r, $b_{r,l}$ grows like r^{2l} whereas $c_{k,l}$ grows like r^3.

1.2 Independent neighborhoods

A triangle-free graph is one whose neighborhoods are all independent sets. Generalizing to r-graphs, one can study r-graphs with independent neighborhoods. If S is a set of vertices in an r-graph $G = (V, E)$ and $|S| = r - 1$, then its neighborhood $N_G(S) = \{v \in V - S : S \cup \{v\} \in E\}$. The degree $\deg_G(S) = |N_G(S)|$.

An r-graph has independent neighborhoods if it contains no copy of F_r, where F_r is the r-graph comprising $r+1$ edges $\{E_0, E_1, \dots, E_r\}$. Here, if $A = \cap_{i=1}^r E_i$ then (i) $|A| = r-1$ and (ii) $E_0 = \bigcup_{i=1}^r (E_i \setminus A)$. Thus $F_2 = K_3$. Gimbel and Thomassen [6] proved that the order of magnitude of $m_k(F_2)$ is $k^3(\log k)^2$. Although we are unable to determine the correct logarithmic factors, we generalize this result as follows; the upper bound follows directly from Theorem 4.

Theorem 6 *Fix $r \ge 3$ and let k be sufficiently large. The minimum number of edges in an r-graph with independent neighborhoods and chromatic number greater than k satisfies*

$$b_{\mathcal{I}} k^{r+1/(r-1)} < m_k(F_r) < c_{\mathcal{I}} k^{r+1/(r-1)} (\log k)^{1+r/(r-1)^2},$$

where

$$
\begin{aligned}
b_{\mathcal{I}} &= \frac{1}{40r^2 2^r} \\
c_{\mathcal{I}} &= \left(r! \left(\frac{50r!(r+1)}{(r-1)!^2} \right)^{1/r} \right)^{1/(r-3/2)} \left(\frac{10(r-1)}{r-3/2} \right)^{(r-1/2)/(r-3/2)}.
\end{aligned}
$$

Note that as $r \to \infty$, $c_{\mathcal{I}} = O(r)$.

It is hard to even make a conjecture about the correct growth rate of $m_k(F_r)$. Most likely neither the upper nor lower bounds give the correct order of magnitude. However, improving either bound seems difficult, since the corresponding improvement for the graph case involved deep results of Kim and Johansson on the independence number and chromatic number of triangle-free graphs. Currently, hypergraph versions of these two results do not exist.

1.3 Excluding a hypertree

A cycle of length $t \geq 3$ in an r-graph is a collection of t distinct vertices $X = \{x_1, \ldots, x_t\}$ and t distinct edges $E_1, \ldots, E_t$ such that (i) $E_i \cap X = \{x_i, x_{i+1}\}$ and (ii) $E_i \cap E_{i+1} = \{x_{i+1}\}$ (indices taken modulo t). A cycle of length two is a collection of two edges with at least two points in common. An r-forest is an r-graph with no cycles. It is easy to see that if H contains a cycle, then $\rho(H) > 1/(r-1)$ and Theorem 4 applies. On the other hand, if H is an r-forest, then it is easy to show that every H-free r-graph G has chromatic number at most c_H, so there can be no analogue of the upper bound in Theorem 4. It is an easy exercise to produce a proper coloring of G where the number of colors is exponential in the size of H. The next Theorem shows that we can reduce this bound substantially. An r-tree is a connected r-forest, where connected means that for every two vertices x, y, there is a sequence of edges $E_1, \ldots E_l$ such that $x \in E_1$, $y \in E_l$, and $E_i \cap E_{i+1} \neq \emptyset$ for all $i = 1, 2, \ldots, l-1$. The statement below applies to r-trees, but a similar statement can be proved for r-forests as well.

Theorem 7 *Let T be an r-tree with $(r-1)t+1$ vertices and suppose that G is an r-graph not containing T. Then the chromatic number of G is at most $2(r-1)(t-1)+1$.*

It is a well-known fact that if every T-free graph G has chromatic number at most t, where T has t edges. Indeed, this follows from the observation that every subgraph of G has a vertex of degree less than t. For $r \geq 3$ such a statement is false. For example, let T be the 3-tree comprising three edges, not all containing the same vertex. Let G be the 3-graph on n vertices, n large, all of whose edges contain a fixed vertex. Then clearly $T \not\subset G$ and G has minimum degree $n-2$, which can be arbitrarily large. This is the reason that Theorem 7 is not trivial. Nevertheless, the best lower bound on the chromatic number of a T-free r-graph that we have is t. It would be very interesting to narrow the gap for this problem, and we believe that Theorem 7 is far from the truth.

1.4 Graphs vs hypergraphs

Our final result shows the limitations of Theorem 4 in the case $r > 2$. Let K_t^r be the complete r-graph on t vertices. Then Theorem 4 implies that $m_k(K_t^r) < k^{r+\varepsilon}$ for some positive ε depending on r and t. However, for $r \geq 3$, this can be improved.

Theorem 8 *Fix $t > r \geq 3$. Then $m_k(K_t^r) = k^{r+o(1)}$, where $o(1) \to 0$ as $k \to \infty$. On the other hand, for each $s \geq 3$, there exists $\varepsilon = \varepsilon_s > 0$ such that $m_k(K_s) > k^{2+\varepsilon}$.*

Theorem 8 shows an interesting difference between graphs and hypergraphs. In fact, we conjecture that a similar result holds if we forbid much less than a clique. Call an r-graph simple if every two of its edges share at most one vertex; in the notation of Section 1.1, an r-graph is simple if and only if it is an $(r, 2)$-system. Simple hypergraphs are often studied due their similarity to graphs. We believe that there exist simple r-graphs H such that $m_k(H) = k^{r+o(1)}$. For $r \geq 4$ this follows from recent unpublished results of Rödl-Schacht and the third author, however, this remains open for $r = 3$.

Conjecture 9 *There exists a simple 3-graph H for which $m_k(H) = k^{3+o(1)}$.*

Let F be the Fano plane, which is the 3-graph with seven vertices and seven edges obtained from the points and lines of the projective geometry of dimension two over the finite field of order two. Perhaps one can even strengthen Conjecture 9 by proving that $m_k(F) = k^{3+o(1)}$?

In the next section we present the proof of Theorem 4. Sections 3, 4, 5 and 6 contain the proofs of theorems 5, 6, 7 and 8 respectively. The last section has several concluding remarks and open problems.

2 General upper bound: Proof of Theorem 4

In this section we prove Theorem 4. Our proof uses the method developed by Krivelevich [9] to obtain bounds for off diagonal Ramsey numbers. The main idea is to take a random hypergraph with appropriate edge probability and judiciously delete all copies of H from it. The additional requirement for us is to keep track of the total number of edges.

Proof of Theorem 4. Let

$$p = c_1 n^{-1/\rho}$$

and let G_p be the random r-graph on n vertices with edge probability p. Let $E_p = |E(G_p)|$. Then

$$E_p \leq \frac{2c_1 n^{r-1/\rho}}{r!} \ \textbf{whp}. \tag{1}$$

Here $\sim$ denotes $= (1 + o(1))$ as $n \to \infty$.

Next let

$$t = c_2 \left(\frac{r! \log n}{p}\right)^{1/(r-1)} = c_2 \left(\frac{r! \log n}{c_1} n^{1/\rho}\right)^{1/(r-1)}.$$

Now, using the Chernoff bounds to get the first inequality below, we have

$$\begin{aligned}
\mathsf{P}\left(\exists S : |S| = t \textit{ and } |E(S)| \leq E_0 = \binom{t}{r} p/2\right) &\leq \binom{n}{t} \exp\left\{-\frac{1}{8}\binom{t}{r} p\right\} \\
&\leq \left(\frac{ne}{t} \exp\left\{-\frac{t^{r-1}}{10r!} p\right\}\right)^t \\
&= \left(\frac{ne}{t} \exp\left\{-\frac{1}{10} c_2^{r-1} \log n\right\}\right)^t \\
&= o(1).
\end{aligned}$$

So, **whp**

$$\text{Every } t\text{-set contains at least } E_0 \text{ edges.} \tag{2}$$

Now, for $|S| = t$ let $Y_{S,i}$ be the number of edges in copies of H_i containing at least one edge of S. Let $Z_{S,i}$ be the number of edges in a maximal collection of pair-wise disjoint copies of H_i, each containing at least one edge of S.

Clearly,

$$Z_{S,i} \leq Y_{S,i}.$$

Let

$$\mu_i = \binom{t}{r}\binom{n}{v_i - r} e_i p^{e_i} \frac{r!(v_i - r)!}{\alpha_i}.$$

Thus

$$\mathsf{E}(Y_{S,i}) \leq \mu_i.$$

Explanation: We choose an edge e of H_i and r-subset R of S to fix an edge that will be e in a copy of H_i. Then we choose $v_i - r$ other vertices for the remainder of our copy. This accounts for $e_i\binom{t}{r}\binom{n}{v_i-r}$ choices. We then choose a copy of H_i in these v_i vertices for which R is an edge. The number of ways of doing this is $\frac{r!(v_i-r)!}{\alpha_i(e)} \leq \frac{r!(v_i-r)!}{\alpha_i}$. Finally, we multiply by p^{e_i}, the probability that the e_i edges chosen actually exist.

Now for $A > 0$,

$$\mathsf{P}(Z_{S,i} \geq A\mu_i) \leq \frac{\mathsf{E}(Z_{S,i})^{A\mu_i}}{(A\mu_i)!} \leq \frac{\mu_i^{A\mu_i}}{(A\mu_i)!} \leq \left(\frac{e}{A}\right)^{A\mu_i}.$$

(Here we are using an inequality of Erdős and Tetali [5], see for example Lemma 8.4.1 of [3]).

Suppose now that $\delta_i = \rho_{i+1} - \rho_i$ for $i = 1, 2, \ldots, \ell - 1$ and $0 = \delta_1 = \delta_2 = \cdots = \delta_s < \delta_{s+1}$. Now let

$$A_i = \begin{cases} 9 & i = 1 \\ 10 & 2 \leq i \leq s \\ \frac{\mu_1}{2^{i-s}} & i > s \end{cases}.$$

Then

$$\mathsf{P}(\exists S : Z_{S,i} \geq A_i\mu_i) \leq \binom{n}{t}\left(\frac{e}{A_i}\right)^{A_i\mu_i}. \tag{3}$$

Since n is sufficiently large,

$$\begin{aligned} \mu_i &= \frac{\theta_i e_i c_1^{e_i} t^r n^{v_i - r - e_i/\rho}}{\alpha_i} \qquad\qquad \frac{9}{10} \leq \theta_i \leq 1 \\ &= \frac{\theta_i e_i c_1^{e_i-1} c_2^{r-1} r! n^{v_i - r - (e_i-1)/\rho} \log n}{\alpha_i} t \\ &= \frac{\theta_i e_i c_1^{e_i-1} c_2^{r-1} r! n^{(v_i-r)(1-\rho_i/\rho)} \log n}{\alpha_i} t \end{aligned} \tag{4}$$

We see immediately that if $i \leq s$ then from the definitions of c_1, c_2,

$$\mathsf{P}(\exists S : Z_{S,i} \geq A_i\mu_i) \leq \left(\frac{ne}{t}\left(\frac{e}{9}\right)^{8e_i c_1^{e_i-1} c_2^{r-1} r! \alpha_i^{-1} \log n}\right)^t = o(1). \tag{5}$$

Now for $i > s$

$$\frac{A_i\mu_1}{\mu_i} \geq n^{(v_i-r)(1-\rho_i/\rho)-o(1)}.$$

So,

$$\mathsf{P}(\exists S : Z_{S,i} \geq A_i\mu_1) \leq \binom{n}{t} n^{-((v_i-r)(1-\rho_1/\rho)-o(1))t\log n} = o(1). \tag{6}$$

It follows from (5), (6) that **whp**

$$\sum_{i=1}^{\ell} Z_{S,i} \leq 10 \sum_{i=1}^{s} \mu_i, \qquad \forall |S| = t. \tag{7}$$

If we remove every edge from a maximal collection of edge disjoint copies of H_i, $i = 1, 2, \ldots, \ell$ then we destroy all copies of $H_i, i = 1, 2, \ldots, \ell$. Furthermore, no t-set will be independent if

$$E_0 > 10 \sum_{i=1}^{s} \mu_i.$$

This is equivalent to

$$10 \sum_{i=1}^{s} \binom{t}{r} \binom{n}{v_i - r} e_i p^{e_i} \frac{r!(v_i - r)!}{\alpha_i} < \frac{1}{2} \binom{t}{r} p$$

or

$$\sum_{i=1}^{s} \binom{n}{v_i - r} e_i p^{e_i - 1} \frac{r!(v_i - r)!}{\alpha_i} < \frac{1}{20}$$

and this is implied by

$$\sum_{i=1}^{s} \frac{e_i c_1^{e_i - 1}}{\alpha_i} < \frac{1}{20 r!}.$$

This follows from the definition of c_1 and so

$$after\ removal\ of\ edges,\ \alpha(G) \leq t. \tag{8}$$

Thus the chromatic number is at least

$$k = \frac{n}{t}.$$

We re-express things to eliminate n. As $k, n \to \infty$ we have

$$k \sim \frac{1}{c_2} \left(\frac{c_1}{r! \log n} \right)^{1/(r-1)} n^{1 - 1/(\rho(r-1))}.$$

$$k^{(r-1)/(r-1-1/\rho)} \sim \frac{1}{c_2^{(r-1)/(r-1-1/\rho)}} \left(\frac{c_1}{r! \log n} \right)^{1/(r-1-1/\rho)} n.$$

Now we see from this that

$$\frac{r-1}{r-1-1/\rho} \log k \sim \log n.$$

So,

$$n \sim k^{(r-1)/(r-1-1/\rho)} c_2^{(r-1)/(r-1-1/\rho)} \left(\frac{r!(r-1)\log k}{c_1(r-1-1/\rho)} \right)^{1/(r-1-1/\rho)}.$$

This gives

$$E_p \le 2 \left(\frac{r!}{c_1} \right)^{1/(r-1-1/\rho)} c_2^{(r-1)(r-1/\rho)/(r-1-1/\rho))} \left(\frac{r-1}{r-1-1/\rho} \right)^{(r-1/\rho)/(r-1-1/\rho)} \times k^{(r-1)(r-1/\rho)/(r-1-1/\rho)} (\log k)^{(r-1/\rho)/(r-1-1/\rho)}.$$

Note that $n \to \infty$ implies $k \to \infty$ and so this completes the proof of Theorem 4. □

3 (r, ℓ)-systems

In this section we give the short poof of Theorem 5. The only observation we need, which is very simple, is that an (r, ℓ)-system is one where a particular finite list hypergraphs is forbidden.

Proof of Theorem 5. We use Theorem 4. Let H_i, $i = 1, 2, \ldots, r - \ell$ be the hypergraph consisting of two edges intersecting in $\ell + i - 1$ vertices. Then an (r, ℓ)-system is one which is $\mathcal{H}$-free, where $\mathcal{H} = \{H_1, \ldots, H_{r-\ell}\}$. Using the notation of the previous section we have

$$\begin{aligned} \rho &= \frac{1}{r-\ell} \\ s &= 1 \\ \alpha_1 &= \ell!(r-\ell)!^2 \\ c_1 &= \frac{\ell!(r-\ell)!^2}{100r!} \\ c_2 &= 10^{1/(r-1)}. \end{aligned}$$

Plugging these values into the expression for $c_{\mathcal{H}}$ in Theorem 4 gives us our expression for $c_{r,l}$. This completes the proof of Theorem 5. □

4 Independent neighborhoods

In this section we prove Theorem 6. We need the following three Lemmas. The first was proved in [7] and follows immediately from the Local Lemma.

Lemma 10 ([7]) *Let $r \geq 2$ and let G be an r-graph with maximum degree at most $k^{r-1}/4r$. Then the chromatic number of G is at most k.*

The next Lemma has been proved by several researchers. In the form below it essentially appears in [8].

Lemma 11 *Let $0 < \alpha \leq 1/2$ and let G be a hypergraph on n vertices. Suppose that every subhypergraph P of G (including G itself) with m vertices has an independent set of size m^α. Then G has chromatic number at most $2n^{1-\alpha}$.*

Our final Lemma is fairly straightforward, and generalizes the easy argument that an n vertex triangle-free graph has an independent set of size at least $\sqrt{n}$ (actually, much more is guaranteed for graphs).

Lemma 12 *Let $r \geq 3$ and let G be an n-vertex r-graph with independent neighborhoods. Then G has an independent set of size at least $n^{1/r}$.*

Proof. Let Δ be the maximum size of a neighborhood of an $(r-1)$-set of vertices. Then

$$\sum_{v \in V(G)} d(v) = \sum_{|S|=r-1} d(S) \leq \binom{n}{r-1}\Delta < \frac{\Delta n^{r-1}}{2}.$$

Consequently, the average degree d of G satisfies $d \leq \Delta n^{r-2}/2$. Now by Turán's theorem, G has an independent set of size at least $(1-1/r)n/d^{1/(r-1)}$. Therefore, we have an independent set of size at least $\max\{\Delta, (1-1/r)n/d^{1/(r-1)}\} \geq n^{1/r}$. $\square$

Proof of Theorem 6. For the upper bound, we apply Theorem 4 with $\mathcal{H} = \{F_r\}$. In the notation of the proof of Theorem 4, we have

$$\begin{aligned} \rho &= 2 \\ s &= 1 \\ \alpha_1 &= (r-1)!^2 \\ c_1 &= \left(\frac{(r-1)!^2}{50r!(r+1)}\right)^{1/r} \\ c_2 &= 10^{1/(r-1)}. \end{aligned}$$

Plugging these values into the expression for $c_{\mathcal{H}}$ in Theorem 4 gives us our expression for $c_{\mathcal{I}}$. For the lower bound, suppose that G is an r-graph with independent neighborhoods and $|G| = bk^{r+1/(r-1)}$ where $b = 1/(40r^2 2^r)$. Let k be sufficiently large and even (a similar argument works for odd k) and let A be the set of vertices in G with degree less than $d = k^{r-1}/(2r2^r)$. By Lemma 10, we can color the induced subhypergraph $G[A]$ properly by $k/2$ colors. Let $G' \subset G$ be the r-graph induced by the uncolored vertices. Since every vertex of G' has degree (in G) at least d, the number of vertices n of G' satisfies $n \le rbk^{r+1/(r-1)}/d < k^{r/(r-1)}/20$. Applying Lemmas 12 and 11, we conclude that G' has a proper coloring where the number of colors is at most

$$2n^{1-1/r} < (2/5)k < k/2.$$

Putting these two colorings together yields a proper coloring of G with at most k colors. □

5 Excluding a hypertree

In this section we prove Theorem 7. Recall that an r-tree is a connected r-forest, where connected means that for every two vertices x, y, there is a sequence of edges $E_1, \dots, E_l$ such that $x \in E_1$, $y \in E_l$, and $E_i \cap E_{i+1} \neq \emptyset$ for all $i = 1, 2, \dots, l-1$. If T is an r-tree, then an edge $e \in T$ is a *leaf* if e intersects at most one other edge of T. If e is a leaf of T then the vertices in e of degree 1 in T are called *leaf vertices* of T. For example, if T is a nontrivial r-tree then each leaf contains $r-1$ leaf vertices.

Proof of Theorem 7. We begin by inductively defining a sequence of collections of r-trees. Set $\mathcal{F}_0 = \{T\}$. For $i = 1, \dots, t-1$ let $\mathcal{F}_i$ be the collection of r-trees given by deleting a leaf edge from some r-tree in $\mathcal{F}_{i-1}$. Note that each r-tree in $\mathcal{F}_i$ has $t-i$ edges and spans $(r-1)(t-i)+1$ vertices.

Let V be the vertex set of the graph G that does not contain a copy of T. We use the sets $\mathcal{F}_1, \dots, \mathcal{F}_{t-1}$ to define a partition of V. Set $G_1 = G$ and let A_1 be the set of vertices $v \in V$ with the property that there exists some r-tree $T' \in \mathcal{F}_1$ such that G contains a copy of T' with v as a leaf vertex. For each such vertex v let X_v be the set of vertices (other than v) spanned by **one** of these r-trees $T' \in \mathcal{F}_1$ that contain v as a leaf vertex. Note that $G_2 := G[V \setminus A_1]$ does not contain any copies of any r-tree in $\mathcal{F}_1$ (such a copy would include a leaf vertex and all such vertices were gathered into A_1).

Now, suppose disjoint sets $A_1, \dots, A_i \subseteq V$ have been defined with the following properties:

(i) If $v \in A_j$ then there is a copy of an r-tree $T' \in \mathcal{F}_j$ in G_j with v as a leaf vertex. The set of vertices (other than v) spanned by **one** such r-tree is X_v.

(ii) The graph

$$G_{i+1} = G\left[V \setminus \cup_{j=1}^{i} A_j\right]$$

does not contain any copy of an r-tree in $\mathcal{F}_i$.

Let A_{i+1} be the set of leaf vertices of copies of r-trees in $\mathcal{F}_{i+1}$ in G_{i+1}. For each $v \in A_{i+1}$ let X_v be the vertex set of **one** of the r-trees $T' \in \mathcal{F}_{i=1}$ that lies in G_{i+1} and contain v as a leaf vertex.

Now consider the graph H_G with vertex set V and edge set

$$\bigcup_{v \in V} \{\{u, v\} : u \in X_v\}.$$

In words we put an edge between each vertex v and every vertex in the set X_v. Note that every induced subgraph of H_G has average degree bounded above $2(r-1)(t-1)$ (as all edges in the subgraph induced by Y are 'generated' by one of the vertices in Y and each such vertex 'generates' at most $(t-1)(r-1)$ edges). It follows that G_H is $2(r-1)(t-1)$-degenerate and can be colored with $2(r-1)(t-1)+1$ colors. Let f be a proper coloring of H_G with $2(r-1)(t-1)+1$ colors.

We claim that f is also a proper coloring of G. Let e be an edge in G. Let $v \in e \cap A_i$ where i is the smallest index such that $e \cap A_i \neq \emptyset$. Consider the r-tree $T' \in \mathcal{F}_i$ that contains v as a leaf vertex and spans $\{v\} \cup X_v$. If $e \cap X_v = \emptyset$ then there is a copy of an r-tree in $\mathcal{F}_{i-1}$ in G_i, which contradicts the properties of the sets $A_1, \ldots, A_{t-1}$. Therefore e contains some vertex $u \in X_v$. As $\{u, v\} \in E(H_G)$, f assigns u and v different colors. $\square$

6 Cliques

In this section we prove Theorem 8. We must provide a construction that has fewer edges than the one in Theorem 4 when $r \geq 3$. It is motivated by similar constructions in Ramsey-Turán theory.

Construction. Fix $r \geq 3$. Let G be the r-graph with vertex set $V = [n]$ obtained by the following random process. For each $i \in [n]$, randomly partition $\{i+1, \ldots, n\}$ into $r-1$ sets $V_1^i, \ldots, V_{r-1}^i$, each of size $\lceil \frac{n-i}{r-1} \rceil$ or $\lfloor \frac{n-i}{r-1} \rfloor$. Now add all edges of the form $\{i, v_1, \ldots, v_{r-1}\}$, where $i < v_j$ and $v_j \in V_j^i$ for all j. □

Proof of Theorem 8. Let us first observe that G contains no copy of K_{r+1}^r. Indeed, if K is such a copy, let i denote its smallest vertex. Since there are r other vertices in K, by the pigeonhole principle, two of these, say w and y lie in V_j^i for some j. But this means that there is no edge of G containing all three of i, w, y, and in particular, at least one (in fact many) edge of K is missing in G. This contradiction implies that G contains no $(r+1)$-clique.

Next we count the edges in G from their leftmost endpoint.

$$|G| \leq \sum_{i=1}^{n-r+1} \left(\frac{n-i}{r-1}+1\right)^{r-1} < \frac{1}{(r-1)^{r-1}} \sum_{j=1}^{n} j^{r-1} < \frac{n^r}{(r-1)^{r-1}}.$$

Let us obtain an upper bound on the independence number of G. For any r-tuple $f = \{i_1, \ldots, i_r\}$ with $i_1 < \cdots < i_r$, let $\mathcal{E}_f$ be the event that $f \in G$. If $f = \{i_1, \ldots, i_r\}$ and $f' = \{i'_1, \ldots, i'_r\}$ with $i_1 < i_2 < \cdots < i_r$ and $i'_1 < i'_2 < \cdots i'_r$, then

$$\mathcal{E}_f \textit{ and } \mathcal{E}_{f'} \textit{ are independent if and only if } i_1 \neq i'_1. \tag{9}$$

Now pick a set $S = \{v_1, \ldots, v_s\} \subset V$ with $v_1 < v_2 < \cdots < v_s$. Let G_i be the set of edges in $G[S]$ whose smallest vertex is v_i. Then

$$\begin{aligned}
\mathsf{P}(S \text{ is independent}) &= \mathsf{P}(G_i = \emptyset \text{ for all } i = 1, \ldots, s) \\
&< \prod_{i=1}^{s/2} \mathsf{P}(G_i = \emptyset) \\
&= \prod_{i=1}^{s/2} \mathsf{P}(\exists j :\ V_j^i \cap \{v_{i+1}, \ldots, v_s\} = \emptyset) \\
&< \prod_{i=1}^{s/2} \frac{(r-1)\binom{(r-2)\lceil (n-v_i)/(r-1)\rceil}{s-i}}{\binom{n-v_i}{s-i}} \\
&< \prod_{i=1}^{s/2} r\left(\frac{r-2}{r-1}\right)^{s-i} \\
&< \left(re^{-s/(2r)}\right)^{s/2}
\end{aligned}$$

where the first inequality holds due to (9). Consequently, the expected number of independent sets of size s in G is at most

$$\binom{n}{s} \cdot \left(re^{-s/(2r)}\right)^{s/2} < \left(\frac{ner^{1/2}}{s} e^{-s/(4r)}\right)^{s} < 1$$

as long as $s > 4r\log n$. This shows that there exists such a G with chromatic number k at least $n/(4r\log n)$. Since the number of edges in G is at most $\frac{n^r}{(r-1)^{r-1}}$, this construction gives

$$m_k(K_{r+1}^r) < d_r(k\log k)^r$$

where $d_r \le 5^r r^r/(r-1)^{r-1}$.

Now we prove the statement about graphs. By standard results in Ramsey theory, every K_s-free graph on n vertices has an independent set of size at least n^{δ_s}, where $\delta_s > 0$. Now choose $\varepsilon = \varepsilon_s$ such that $0 < \varepsilon < 1/(1-\delta_s) - 1$. Suppose that G is a K_s-free graph with independent neighborhoods and $k^{2+\varepsilon}$ edges, where k is sufficiently large. Let A be the set of vertices in G with degree less than $k/2-1$. We can greedily color the induced subgraph $G[A]$ properly by $k/2$ colors. Let $G' \subset G$ be the subgraph induced by the uncolored vertices. Since every vertex of G' has degree (in G) at least $k/2$, the number of vertices n of G' satisfies $n \le 4k^{1+\varepsilon}$. By the choice of δ_s, every m-vertex subgraph of G' has an independent set of size at least m^{δ_s}. Hence by Lemma 11, we conclude that G' has a proper coloring where the number of colors is at most

$$2n^{1-\delta_s} < 2(4k^{1+\varepsilon})^{1-\delta_s} < k/2,$$

where the last inequality holds by the choice of ε and the fact that k is sufficiently large. Putting these two colorings together yields a proper coloring of G with at most k colors. $\Box$

7 Concluding remarks and open problems

In this section we repeat some of the open questions mentioned throughout the paper and state a couple of new ones as well.

• Attempts to improve the lower bound in Theorem 6 lead to the following question which is independently interesting. Suppose that G is an r-graph with independent neighborhoods and maximum degree Δ. What are the best upper bounds one can obtain on the chromatic number of G? The Local lemma gives $O(\Delta^{1/(r-1)})$, but the results for the graph case ($r=2$) suggest that one should be able to improve this to $O((\Delta/\log\Delta)^{1/(r-1)})$. The $r=2$ case, that triangle-free graphs with maximim degree Δ have chromatic number at most $O(\Delta/\log\Delta)$, is a deep result due to Johansson, but those ideas do not extend to $r>2$. When $r=3$ we pose the following weaker statement.

Problem. Let G be a 3-graph with independent neighborhoods and maximum degree Δ. Prove that the chromatic number of G is $o(\sqrt{\Delta})$.

A much stronger statement for graphs has been conjectured by Alon-Krivelevich-Sudakov [2].

As we mentioned earlier, we do not believe that the order of magnitude of the upper bound in Theorem 6 is correct either. Perhaps some generalization of Kim's construction for $R(3, t)$ would improve the log factors.

• Let T be an r-tree with t edges and G be an r-graph containing no copy of T. When $r = 2$, it is well-known that the chromatic number of G is at most t, and this is sharp. Theorem 7 gives an upper bound of about rt, but again the best lower bound we have is roughly t. It would be very interesting to narrow this gap, in particular to determine whether the coefficient of t depends on r in an essential way.

• Our final question is perhaps too ambitious given the current state of knowledge, and pertains to Theorem 8.

Problem. Characterize all 3-graphs H such that $m_k(H) = k^{3+o(1)}$.

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