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GENERAL METHOD OF LYAPUNOV FUNCTIONALS CONSTRUCTION IN STABILITY INVESTIGATIONS OF NONLINEAR STOCHASTIC DIFFERENCE EQUATIONS WITH CONTINUOUS TIME

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The general method of Lyapunov functionals construction has been developed during the last decade for stability investigations of stochastic differential equations with aftereffect and stochastic difference equations. After some modification of the basic Lyapunov type theorem this method was successfully used also for difference Volterra equations with continuous time. The latter often appear as useful mathematical models. Here this method is used for a stability investigation of some nonlinear stochastic difference equation with continuous time.

Keywords: Nonlinear stochastic difference equations; continuous time; stability investigations; method of Lyapunov functionals construction.

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1. Introduction. Auxiliary Results

Many processes in automatic regulation, physics, mechanics, biology, economics, etc. can be modelled by functional differential or difference equations (also called hereditary equations or equations with memory or equations with delay or, in particular, Volterra equations). Consequently, a considerable amount of results concerning their theoretical analysis and applications has been developed [1, 5, 6, 8–12, 14, 15, 17, 18, 30, 33, 35, 36, 44]. One of the main problems in the theory of equations with memory and their applications is connected with stability. Different methods for performing stability investigations were obtained, in particular, by modifications of the direct Lyapunov method [31]. Using this method many stability results were obtained by constructing appropriate Lyapunov functionals. However, it is a very difficult problem to find suitable Lyapunov functionals for a given hereditary system. In many cases each such functional appeared to be a guess (or an art) of its author. To solve this problem the general method of Lyapunov functionals construction for stability investigations of hereditary systems was proposed and successfully applied during the last decade (see [7, 13, 16, 19–28, 34, 39]) both for stochastic differential equations with aftereffect and stochastic difference equations. Applications of this method to some biological and mechanical problems are given in [2, 4, 40]. It was also shown [41–43] that after some modification of the basic Lyapunov type stability theorem this method can be successfully used also for difference Volterra equations with continuous time, which are a quite popular topic of research [3, 29, 32, 37, 38]. In the current article the general method of Lyapunov functionals construction developed in [42] is used for the investigation of mean-square stability of some nonlinear stochastic difference equation with continuous time.

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a probability space, $\{\mathfrak{F}_t, t \geq t_0\}$ be a nondecreasing family of sub-σ-algebras of $\mathfrak{F}$, i.e. $\mathfrak{F}_{t_1} \subset \mathfrak{F}_{t_2}$ for $t_1 < t_2$, $\mathbf{E}$ be the expectation with respect to the measure $\mathbf{P}$ and $\mathbf{E}_t = \mathbf{E}(./\mathfrak{F}_t)$ be the conditional expectation with respect to the σ-algebra $\mathfrak{F}_t$.

Consider the stochastic difference equation

$$\begin{aligned} x(t+h_0) &= a_1(t, x(t), x(t-h_1), x(t-h_2), \dots) \\ &\quad + a_2(t, x(t), x(t-h_1), x(t-h_2), \dots)\xi(t+h_0), \quad t > t_0 - h_0, \end{aligned} \tag{1.1}$$

with the initial condition

$$x(\theta) = \phi(\theta), \quad \theta \in \Theta = \Big[t_0 - h_0 - \max_{j\geq 1} h_j, t_0\Big]. \tag{1.2}$$

Here $x \in \mathbb{R}^n, h_0, h_1, \dots$ are positive constants and the functionals $a_1 \in \mathbb{R}^n$ and $a_2 \in \mathbb{R}^{n\times m}$ satisfy the condition

$$|a_l(t, x_0, x_1, x_2, \dots)|^2 \leq \sum_{j=0}^{\infty} a_{lj}|x_j|^2, \quad A = \sum_{l=1}^{2}\sum_{j=0}^{\infty} a_{lj} < \infty\,.$$

Further, $\phi(\theta)$, $\theta \in \Theta$, is an $\mathfrak{F}_{t_0}$-measurable function and the perturbation $\xi(t) \in \mathbb{R}^m$ is a $\mathfrak{F}_t$-measurable stationary stochastic process such that $\mathbf{E}_t\xi(t+h_0) = 0$, $\mathbf{E}_t\xi(t+h_0)\xi'(t+h_0) = I$, $t > t_0 - h_0$.

A solution of problem (1.1), (1.2) is an $\mathfrak{F}_t$-measurable process $x(t) = x(t; t_0, \phi)$, which is equal to the initial function $\phi(t)$ from (1.2) for $t \leq t_0$ and with probability 1 is defined by Eq. (1.1) for $t > t_0$.

Definition 1.1. The trivial solution of Eqs. (1.1), (1.2) is called mean square stable if for any $\varepsilon > 0$ and t_0 there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\mathbf{E}|x(t; t_0, \phi)|^2 < \varepsilon$ for all $t \geq t_0$ if $\|\phi\|^2 = \sup_{\theta\in\Theta} \mathbf{E}|\phi(\theta)|^2 < \delta$.

Definition 1.2. The trivial solution of Eqs. (1.1), (1.2) is called asymptotically mean-square quasistable if it is mean-square stable and for each $t \in [t_0, t_0 + h_0)$ and each initial function ϕ one has $\lim_{j\to\infty} \mathbf{E}|x(t + jh_0; t_0, \phi)|^2 = 0$.

Definition 1.3. The solution of Eq. (1.1) with initial condition (1.2) is called mean-square integrable if for each initial function ϕ one has $\int_{t_0}^{\infty} \mathbf{E}|x(t; t_0, \phi)|^2\, dt < \infty$.

Theorem 1.1. ([42]) *Let there exist a non-negative functional* $V(t) = V(t, x(t), x(t-h_1), x(t-h_2), \ldots)$ *and positive numbers* c_1, c_2, *such that*

$$\mathbf{E}V(t) \leq c_1 \sup_{s \leq t} \mathbf{E}|x(s)|^2, \quad t \in [t_0, t_0 + h_0), \tag{1.3}$$

$$\mathbf{E}\Delta V(t) \leq -c_2 \mathbf{E}|x(t)|^2, \qquad t \geq t_0, \tag{1.4}$$

where $\Delta V(t) = V(t + h_0) - V(t)$. *Then the trivial solution of Eqs.* (1.1), (1.2) *is asymptotically mean square quasistable.*

Remark 1.1. ([42]) If the conditions of Theorem 1.1 hold then the solution of Eq. (1.1) for each initial function (1.2) is mean-square integrable.

From Theorem 1.1 and Remark 1.1 it follows that an investigation of the behavior of the solution of Eq. (1.1) can be reduced to the construction of appropriate Lyapunov functionals. In the remainder of this article we will apply the method of Lyapunov functionals construction for the stability investigation of some classes of nonlinear stochastic difference equations.

Remark 1.2. Suppose that in Eq. (1.1) one sets $h_0 = h > 0, h_j = jh, j = 1, 2, \ldots$. Putting $t = t_0 + sh, y(s) = x(t_0 + sh), \eta(s) = \xi(t_0 + sh)$, one can write Eq. (1.1) in the form

$$\begin{aligned} y(s+1) &= b_1(s, y(s), y(s-1), y(s-2), \ldots) \\ &\quad + b_2(s, y(s), y(s-1), y(s-2), \ldots)\eta(s+1), \quad s > -1, \\ y(\theta) &= \phi(\theta), \quad \theta \leq 0. \end{aligned} \tag{1.5}$$

Subsequently equations of the type (1.5) will be considered.

2. Nonlinear Stochastic Difference Equations

Consider the scalar difference equation

$$\begin{aligned} x(t+1) = -\sum_{j=0}^{[t]+r} a(t,j) g(x(t-j)) + \sum_{j=0}^{[t]+r} \sigma(t,j) f(x(t-j)) \xi(t+1), \quad t > -1, \\ x(s) = \phi(s), \quad s \in [-(r+1), 0], \end{aligned} \tag{2.1}$$

where the functions $g(x)$, $f(x)$ and the stochastic process $\xi(t)$ satisfy the conditions

$$\begin{aligned} &0 < c_1 \leq \frac{g(x)}{x} \leq c_2, \quad x \neq 0, \qquad |f(x)| \leq c_3 |x|, \\ &\mathbf{E}_t \xi(t+1) = 0, \quad \mathbf{E}_t \xi^2(t+1) = 1, \qquad t > -1. \end{aligned} \tag{2.2}$$

Put

$$F(t, j, a, \sigma) = |a(t,j)| \sum_{k=0}^{[t]+r} |\sigma(t,k)|, \tag{2.3}$$

$$G(a, \sigma) = \frac{1}{2} \sup_{t \geq 0} \sum_{j=0}^{\infty} \left(F(t+j, j, a, \sigma) + F(t+j, j, \sigma, a) \right). \tag{2.4}$$

Theorem 2.1. *Let the coefficients* $a(t,j), \sigma(t,j), t > -1, j = 0, 1, \ldots, [t]+r$, *satisfy the conditions*

$$a(t,j) \geq 0, \quad \alpha(t,j) = a(t,j) - a(t,j+1) \geq 0, \tag{2.5}$$

$$\begin{aligned} \alpha(t,j-1) - \alpha(t+1,j) = a(t+1,j+1) - a(t+1,j) \\ - a(t,j) + a(t,j-1) \geq 0, \end{aligned} \tag{2.6}$$

$$\begin{aligned} a &= \sup_{t>-1} (a(t+1,0) + a(t,0) - a(t+1,1)) \\ &< 2\left[\frac{1}{c_2} - \frac{c_3}{c_1}\left(c_3 G(\sigma,\sigma) + \frac{c_2 - c_1}{2} G(a,\sigma)\right)\right] \end{aligned} \tag{2.7}$$

(*here and everywhere below it is assumed that* $a(t,-1) = a$ *and* $a(t,j) = 0$ *for* $j > [t]+r$). *Then the trivial solution of Eq.* (2.1) *is asymptotically mean square quasistable.*

Proof. It is enough to construct for Eq. (2.1) a Lyapunov functional $V(t)$ satisfying the conditions of Theorem 1.1. Following the general method of Lyapunov functionals construction (GMLFC) let us first consider a simple auxiliary difference equation

$$x(t+1) = -a(t,0)g(x(t)). \tag{2.8}$$

This equation is an equation without delay. Obtaining a Lyapunov function for the auxiliary equation (2.8) via GMLFC one can extend it and construct a Lyapunov functional for the original equation (2.1). Consider, for instance, the function $v(t) = x(t)g(x(t)) + p\alpha(t,0)g^2(x(t))$, where p is a positive number that will be defined below. From conditions (2.2), (2.5), (2.7) it follows that the function $v(t)$ is non-negative and satisfies condition (1.3). Calculating and estimating $\Delta v(t)$ one can show that $v(t)$ also satisfies condition (1.4). In fact,

$$\begin{aligned} \Delta v(t) = v(t+1) - v(t) = x(t+1)g(x(t+1)) - x(t)g(x(t)) \\ + p[\alpha(t+1,0)g^2(x(t+1)) - \alpha(t,0)g^2(x(t))]. \end{aligned}$$

To estimate $\Delta v(t)$ note that via (2.2), (2.6) we have $x(t)g(x(t)) \geq c_1 x^2(t)$ and

$$\begin{aligned} \alpha(t+1,0)g^2(x(t+1)) - \alpha(t,0)g^2(x(t)) &= (\alpha(t+1,0) - \alpha(t,-1))g^2(x(t+1)) \\ &\quad + \alpha(t,-1)g^2(x(t+1)) - \alpha(t,0)g^2(x(t)) \\ &\leq q(t), \end{aligned}$$

where

$$\begin{aligned} q(t) &= \alpha(t,-1)g^2(x(t+1)) + \alpha(t,0)(g(x(t+1)) + g(x(t)))^2 - \alpha(t,0)g^2(x(t)) \\ &= (\alpha(t,-1) + \alpha(t,0))g^2(x(t+1)) + 2\alpha(t,0)g(x(t))g(x(t+1)). \end{aligned}$$

Note that for Eq. (2.8) one has that $a(t,j) = 0$ for $j > 0$. Therefore, from (2.5) it follows that $\alpha(t,0) = a(t,0)$ and $\alpha(t,-1) + \alpha(t,0) = a$. Then using (2.2) for

$x \neq 0$ we have $g^2(x) = \frac{g(x)}{x} x g(x) \leq c_2 x g(x)$. So, via (2.8) one obtains $q(t) \leq -(2 - ac_2)x(t+1)g(x(t+1))$. From (2.7) it follows that $ac_2 < 2$. As a result, by putting $p = (2 - ac_2)^{-1}$ we obtain $\Delta v(t) \leq -c_1 x^2(t) + [1 - p(2 - ac_2)]x(t+1)g(x(t+1)) = -c_1 x^2(t)$.

So, the function $v(t)$ is a Lyapunov function for the auxiliary equation (2.8). Following the GMLFC we will construct a Lyapunov functional for the original equation (2.1) in the form of a sum of the functional $V_1(t) = v(t)$ and some additional functionals that will be constructed below. Similar to the estimation of $\Delta v(t)$ for $\Delta V_1(t)$ we have

$$\begin{aligned}\Delta V_1(t) \leq &-c_1 x^2(t) + 2p\alpha(t,0)g(x(t+1))g(x(t)) \\ &+ [1 + p(\alpha(t,-1) + \alpha(t,0))c_2]x(t+1)g(x(t+1)).\end{aligned}$$

Note that using (2.5) for Eq. (2.1) we obtain $\alpha(t,-1) + \alpha(t,0) = a - a(t,1), \alpha(t,0) = a(t,0) - a(t,1)$. So,

$$\begin{aligned}\Delta V_1(t) &\leq -c_1 x^2(t) + 2p(a(t,0) - a(t,1))g(x(t+1))g(x(t)) \\ &\quad + [1 + p(a - a(t,1))c_2]x(t+1)g(x(t+1)) \\ &= -c_1 x^2(t) + (1 + pac_2)x(t+1)g(x(t+1)) + 2pg(x(t+1))a(t,0)g(x(t)) \\ &\quad - pa(t,1)g(x(t+1))(2g(x(t)) + c_2 x(t+1)).\end{aligned}$$

Substituting $a(t,0)g(x(t))$ from (2.1) and using $p = (2 - ac_2)^{-1}$ we have

$$\begin{aligned}\Delta V_1(t) \leq &-c_1 x^2(t) + (1 + pac_2)x(t+1)g(x(t+1)) + 2pg(x(t+1)) \\ &\times \left[-x(t+1) - \sum_{j=1}^{[t]+r} a(t,j)g(x(t-j)) + \sum_{j=0}^{[t]+r} \sigma(t,j)f(x(t-j))\xi(t+1)\right] \\ &- pa(t,1)g(x(t+1))(2g(x(t)) + c_2 x(t+1)) \\ = &- c_1 x^2(t) - pa(t,1)g(x(t+1))(2g(x(t)) + c_2 x(t+1)) \\ &+ 2pg(x(t+1))\left[-\sum_{j=1}^{[t]+r} a(t,j)g(x(t-j)) + \sum_{j=0}^{[t]+r} \sigma(t,j)f(x(t-j))\xi(t+1)\right].\end{aligned}$$

Thus, the achieved estimate for $\Delta V_1(t)$ contains only one "good" term, i.e. $-c_1 x^2(t)$. All the other terms need to be neutralized using some additional functionals. Usually this can be done with additional functionals of some standard type.

Let us show that the first sum on the right of the estimate for $\Delta V_1(t)$ can be neutralized via the additional functional $V_2(t)$, which is of the form

$$V_2(t) = p \sum_{j=1}^{[t]+r} \alpha(t,j) \left(\sum_{k=0}^{j} g(x(t-k)) \right)^2.$$

In fact, for the functional

$$V_0(t) = V_1(t) + V_2(t) = x(t)g(x(t)) + p \sum_{j=0}^{[t]+r} \alpha(t,j) \left(\sum_{k=0}^{j} g(x(t-k)) \right)^2$$

using (2.6) we have

$$\begin{aligned}\Delta V_0(t) &= -x(t)g(x(t)) + x(t+1)g(x(t+1)) \\ &\quad + p\sum_{j=0}^{[t]+1+r}\alpha(t+1,j)\left(\sum_{k=0}^{j} g(x(t+1-k))\right)^2 \\ &\quad - p\sum_{j=0}^{[t]+r}\alpha(t,j)\left(\sum_{k=0}^{j} g(x(t-k))\right)^2 \\ &\le -c_1x^2(t) + x(t+1)g(x(t+1)) \\ &\quad + p\sum_{j=0}^{[t]+1+r}(\alpha(t+1,j)-\alpha(t,j-1))\left(\sum_{k=0}^{j} g(x(t+1-k))\right)^2 + pQ(t) \\ &\le -c_1x^2(t) + x(t+1)g(x(t+1)) + pQ(t),\end{aligned}$$

where

$$Q(t) = \sum_{j=0}^{[t]+1+r}\alpha(t,j-1)\left(\sum_{k=0}^{j} g(x(t+1-k))\right)^2 - \sum_{j=0}^{[t]+r}\alpha(t,j)\left(\sum_{k=0}^{j} g(x(t-k))\right)^2.$$

Now, we transform $Q(t)$ in the following way

$$\begin{aligned}Q(t) &= \alpha(t,-1)g^2(x(t+1)) - \sum_{j=0}^{[t]+r}\alpha(t,j)\left(\sum_{k=0}^{j} g(x(t-k))\right)^2 \\ &\quad + \sum_{j=1}^{[t]+1+r}\alpha(t,j-1)\left(\sum_{k=0}^{j} g(x(t+1-k))\right)^2 \\ &= \alpha(t,-1)g^2(x(t+1)) - \sum_{j=0}^{[t]+r}\alpha(t,j)\left(\sum_{k=0}^{j} g(x(t-k))\right)^2 \\ &\quad + \sum_{j=0}^{[t]+r}\alpha(t,j)\left(g(x(t+1) + \sum_{k=1}^{j+1} g(x(t+1-k))\right)^2 = \alpha(t,-1)g^2(x(t+1)) \\ &\quad + \sum_{j=0}^{[t]+r}\alpha(t,j)\left[\left(g(x(t+1)) + \sum_{k=0}^{j} g(x(t-k))\right)^2 - \left(\sum_{k=0}^{j} g(x(t-k))\right)^2\right] \\ &= \alpha(t,-1)g^2(x(t+1)) + \sum_{j=0}^{[t]+r}\alpha(t,j)\left[g^2(x(t+1)) + 2g(x(t+1))\sum_{k=0}^{j} g(x(t-k))\right] \\ &= g^2(x(t+1))\sum_{j=-1}^{[t]+r}\alpha(t,j) + 2g(x(t+1))\sum_{k=0}^{[t]+r} g(x(t-k))\sum_{j=k}^{[t]+r}\alpha(t,j).\end{aligned}$$

From (2.5) and $a(t,[t]+r+1) = 0$ it follows that $\sum_{j=k}^{[t]+r} \alpha(t,j) = a(t,k)$, $k = -1, 0, \dots, [t]+r$. Since $a(t,-1) = a$ we obtain

$$\Delta V_0(t) \le -c_1 x^2(t) + x(t+1)g(x(t+1)) + pag^2(x(t+1)) + 2pg(x(t+1)) \sum_{k=0}^{[t]+r} a(t,k)g(x(t-k)).$$

Therefore using (2.2), (2.1) and $p = (2-ac_2)^{-1}$, for $x(t+1) \ne 0$ yields

$$\begin{aligned}\Delta V_0(t) &\le -c_1 x^2(t) + (1+pac_2)x(t+1)g(x(t+1)) \\ &\quad + 2pg(x(t+1)) \left(-x(t+1) + \sum_{j=0}^{[t]+r} \sigma(t,j) f(x(t-j))\xi(t+1) \right) \\ &= -c_1 x^2(t) + 2pg(x(t+1)) \sum_{j=0}^{[t]+r} \sigma(t,j) f(x(t-j))\xi(t+1).\end{aligned}$$

As a result for the functional V_0 we have

$$\mathbf{E}\Delta V_0(t) \le -c_1 \mathbf{E} x^2(t) + 2pZ(t), \tag{2.9}$$

where

$$Z(t) = \left| \sum_{j=0}^{[t]+r} \sigma(t,j) \mathbf{E} f(x(t-j))\xi(t+1)g(x(t+1)) \right|.$$

The estimate for $\mathbf{E}\Delta V_0(t)$ contains only one "bad" summand: $2pZ(t)$. To neutralize it let us estimate $Z(t)$. Put $\eta(t) = \frac{g(x(t))}{x(t)}$, $t > 0$. Using (2.1), we obtain

$$\begin{aligned}&\mathbf{E} f(x(t-j))\xi(t+1)g(x(t+1)) \\ &\quad = \mathbf{E} f(x(t-j)) \mathbf{E}_t[x(t+1)\eta(t+1)\xi(t+1)] \\ &\quad = \mathbf{E} f(x(t-j)) \left[-\sum_{k=0}^{[t]+r} a(t,k) g(x(t-k)) \mathbf{E}_t[\eta(t+1)\xi(t+1)] \right. \\ &\qquad \left. + \sum_{k=0}^{[t]+r} \sigma(t,k) f(x(t-k)) \mathbf{E}_t[\eta(t+1)\xi^2(t+1)] \right].\end{aligned}$$

From (2.2) it follows that $\mathbf{E}_t[\eta(t+1)\xi^2(t+1)] \le c_2$ and therefore

$$\begin{aligned}Z(t) &\le \mathbf{E} \left| \sum_{j=0}^{[t]+r} \sigma(t,j) f(x(t-j)) \right| \left| \sum_{k=0}^{[t]+r} a(t,k) g(x(t-k) \right| |\mathbf{E}_t[\eta(t+1)\xi(t+1)]| \\ &\quad + c_2 \mathbf{E} \left(\sum_{j=0}^{[t]+r} \sigma(t,j) f(x(t-j)) \right)^2.\end{aligned}$$

To estimate the expression $|\mathbf{E}_t[\eta(t+1)\xi(t+1)]|$ consider the measure $\mathbf{P}_t$ corresponding to the conditional expectation $\mathbf{E}_t$ and put $\Omega_t^+ = \{\omega\colon \xi(t+1,\omega) \geq 0\}$, $\Omega_t^- = \{\omega\colon \xi(t+1,\omega) < 0\}$. From (2.2) it follows that $0 < c_1 \leq \eta(t) \leq c_2$. Therefore

$$\begin{aligned}\mathbf{E}_t[\eta(t+1)\xi(t+1)] &= \int_\Omega \eta(t+1,\omega)\xi(t+1,\omega)\mathbf{P}_t(d\omega)\\ &= \int_{\Omega_t^+} \eta(t+1,\omega)\xi(t+1,\omega)\mathbf{P}_t(d\omega)\\ &\quad + \int_{\Omega_t^-} \eta(t+1,\omega)\xi(t+1,\omega)\mathbf{P}_t(d\omega)\\ &\leq c_2 \int_{\Omega_t^+} \xi(t+1,\omega)\mathbf{P}_t(d\omega) + c_1 \int_{\Omega_t^-} \xi(t+1,\omega)\mathbf{P}_t(d\omega).\end{aligned}$$

Using (2.2), we have

$$\mathbf{E}_t\xi(t+1) = \int_{\Omega_t^+} \xi(t+1,\omega)\mathbf{P}_t(d\omega) + \int_{\Omega_t^-} \xi(t+1,\omega)\mathbf{P}_t(d\omega) = 0.$$

So,

$$\int_{\Omega_t^+} \xi(t+1,\omega)\mathbf{P}_t(d\omega) = -\int_{\Omega_t^-} \xi(t+1,\omega)\mathbf{P}_t(d\omega) = \int_{\Omega_t^-} |\xi(t+1,\omega)|\mathbf{P}_t(d\omega)$$

and

$$\begin{aligned}\mathbf{E}_t|\xi(t+1)| &= \int_{\Omega_t^+} \xi(t+1,\omega)\mathbf{P}_t(d\omega) + \int_{\Omega_t^-} |\xi(t+1,\omega)|\mathbf{P}_t(d\omega)\\ &= 2\int_{\Omega_t^+} \xi(t+1,\omega)\mathbf{P}_t(d\omega).\end{aligned}$$

Therefore, via (2.2)

$$\begin{aligned}\mathbf{E}_t[\eta(t+1)\xi(t+1)] &\leq (c_2 - c_1)\int_{\Omega_t^+} \xi(t+1,\omega)\mathbf{P}_t(d\omega) = \frac{c_2 - c_1}{2}\mathbf{E}_t|\xi(t+1)|\\ &\leq \frac{c_2 - c_1}{2}\sqrt{\mathbf{E}_t\xi^2(t+1)} = \frac{c_2 - c_1}{2}.\end{aligned}$$

Similarly

$$\begin{aligned}&\mathbf{E}_t[\eta(t+1)\xi(t+1)]\\ &\quad= \int_{\Omega_t^+} \eta(t+1,\omega)\xi(t+1,\omega)\mathbf{P}_t(d\omega) + \int_{\Omega_t^-} \eta(t+1,\omega)\xi(t+1,\omega)\mathbf{P}_t(d\omega)\\ &\quad\geq c_1 \int_{\Omega_t^+} \xi(t+1,\omega)\mathbf{P}_t(d\omega) + c_2 \int_{\Omega_t^-} \xi(t+1,\omega)\mathbf{P}_t(d\omega)\\ &\quad= (c_1 - c_2)\int_{\Omega_t^+} \xi(t+1,\omega)\mathbf{P}_t(d\omega) = \frac{c_1 - c_2}{2}\mathbf{E}_t|\xi(t+1)|\\ &\quad\geq \frac{c_1 - c_2}{2}\sqrt{\mathbf{E}_t\xi^2(t+1)} = \frac{c_1 - c_2}{2}.\end{aligned}$$

Thus, $|\mathbf{E}_t[\eta(t+1)\xi(t+1)]| \le \frac{1}{2}(c_2 - c_1)$. As a result, we obtain

$$Z(t) \le c_2 \mathbf{E}\left(\sum_{j=0}^{[t]+r} \sigma(t,j) f(x(t-j))\right)^2 + \frac{c_2 - c_1}{2}\mathbf{E}\left|\sum_{j=0}^{[t]+r} \sigma(t,j) f(x(t-j))\right| \left|\sum_{k=0}^{[t]+r} a(t,k) g(x(t-k))\right|. \quad (2.10)$$

Using (2.2), (2.3), we have

$$\mathbf{E}\left(\sum_{j=0}^{[t]+r} \sigma(t,j) f(x(t-j))\right)^2 \le c_3^2 \mathbf{E}\left(\sum_{j=0}^{[t]+r} |\sigma(t,j)||x(t-j)|\right)^2 \le c_3^2 \sum_{j=0}^{[t]+r} F(t,j,\sigma,\sigma)\mathbf{E}x^2(t-j) \quad (2.11)$$

and

$$\begin{aligned} &\mathbf{E}\left|\sum_{j=0}^{[t]+r} \sigma(t,j) f(x(t-j))\right| \left|\sum_{k=0}^{[t]+r} a(t,k) g(x(t-k))\right| \\ &\le c_2 c_3 \sum_{j=0}^{[t]+r}\sum_{k=0}^{[t]+r} |\sigma(t,j)||a(t,k)|\mathbf{E}|x(t-j)||x(t-k)| \\ &\le \frac{1}{2}c_2 c_3 \left[\sum_{k=0}^{[t]+r} F(t,k,a,\sigma)\mathbf{E}x^2(t-k) + \sum_{j=0}^{[t]+r} F(t,j,\sigma,a)\mathbf{E}x^2(t-j)\right] \\ &= \frac{1}{2}c_2 c_3 \sum_{j=0}^{[t]+r} [F(t,j,a,\sigma) + F(t,j,\sigma,a)]\mathbf{E}x^2(t-j). \end{aligned} \quad (2.12)$$

Using (2.9)–(2.12), finally we obtain

$$\mathbf{E}\Delta V_0(t) \le -c_1 \mathbf{E}x^2(t) + \sum_{j=0}^{[t]+r} Q(t,j)\mathbf{E}x^2(t-j),$$

where

$$Q(t,j) = 2pc_2c_3\left[c_3 F(t,j,\sigma,\sigma) + \frac{1}{4}(c_2 - c_1)(F(t,j,a,\sigma) + F(t,j,\sigma,a))\right].$$

The square form on the right-hand side of this inequality can be neutralized via the standard additional functional

$$V_3(t) = \sum_{k=1}^{[t]+r} x^2(t-k) \sum_{j=k}^{\infty} Q(t-k+j,j).$$

In fact,

$$\begin{aligned}\Delta V_3(t) &= \sum_{k=1}^{[t]+1+r} x^2(t+1-k)\sum_{j=k}^{\infty} Q(t+1-k+j,j) - V_3(t)\\&= x^2(t)\sum_{j=1}^{\infty} Q(t+j,j) + \sum_{k=2}^{[t]+1+r} x^2(t+1-k)\sum_{j=k}^{\infty} Q(t+1-k+j,j) - V_3(t)\\&= x^2(t)\sum_{j=1}^{\infty} Q(t+j,j)\\&\quad + \sum_{l=1}^{[t]+r} x^2(t-l)\left[\sum_{j=l+1}^{\infty} Q(t-l+j,j) - \sum_{j=l}^{\infty} Q(t-l+j,j)\right]\\&= x^2(t)\sum_{j=1}^{\infty} Q(t+j,j) - \sum_{l=1}^{[t]+r} Q(t,l)x^2(t-l).\end{aligned}$$

As a result, we obtain for the functional $V(t) = V_1(t) + V_2(t) + V_3(t)$ the estimate

$$\begin{aligned}\mathbf{E}\Delta V(t) &\le -c_1\mathbf{E}x^2(t) + \mathbf{E}x^2(t)\sum_{j=1}^{\infty} Q(t+j,j) + \mathbf{E}x^2(t)Q(t,0)\\&= \left[-c_1 + \sum_{j=0}^{\infty} Q(t+j,j)\right]\mathbf{E}x^2(t)\\&\le \left[-c_1 + 2pc_2c_3\left(c_3G(\sigma,\sigma) + \frac{c_2-c_1}{2}G(a,\sigma)\right)\right]\mathbf{E}x^2(t),\end{aligned}$$

where $G(a,\sigma)$ is defined by (2.4), (2.3). From condition (2.7) it follows that the expression in square brackets is negative. So, the functional $V(t)$ constructed above satisfies the conditions of Theorem 1.1. Therefore, the trivial solution of Eq. (2.1) is asymptotically mean-square quasistable. The theorem is proved.

Remark 2.1. Due to Remark 1.1 the conditions of Theorem 2.1 imply that the solution of Eq. (2.1) is mean square integrable.

Remark 2.2. Suppose that the parameters of Eq. (2.1) do not depend on t, i.e. $a(t,j) = a(j)$, $\sigma(t,j) = \sigma(j)$. It is easy to see that in this case $G(a,\sigma) = \hat{a}\hat{\sigma}$, where $\hat{a} = \sum_{j=0}^{\infty} a(j)$, $\hat{\sigma} = \sum_{j=0}^{\infty} |\sigma(j)|$. So, the stability conditions (2.5)–(2.7) have the form

$$a(j) \ge 0, \quad a(j+1) - a(j) \le 0, \quad a(j+2) - 2a(j+1) + a(j) \ge 0, \quad j = 0,1,\ldots,$$
$$a(0) - \frac{a(1)}{2} < \frac{1}{c_2} - \frac{c_3}{c_1}\hat{\sigma}\left(c_3\hat{\sigma} + \frac{c_2-c_1}{2}\hat{a}\right).$$

This means, in particular, that the sequence $a(j)$ has non-negative members with nonpositive first differences and non-negative second differences.

Remark 2.3. Without loss of generality in condition (2.2) we can put $c_3 = 1$ and $c_1 \le c_2 = 1$ or $c_2 \ge c_1 = 1$. In fact, if this is not the case we can put for instance $a(t,j)g(x) = \tilde{a}(t,j)\tilde{g}(x)$, where $\tilde{a}(t,j) = c_2 a(t,j)$, $\tilde{g}(x) = c_2^{-1} g(x)$. In this case the function $\tilde{g}(x)$ satisfies condition (2.2) with $c_2 = 1$.

Example 2.1. Consider the equation

$$x(t+1) = \sum_{j=0}^{[t]+1} a(t,j)[-\lambda_1 g(x(t-j)) + \lambda_2 f(x(t-j))\xi(t+1)], \tag{2.13}$$

where $a(t,j) = \frac{t+1-j}{(t+2)^2}$, $\lambda_l > 0$, $l = 1,2$, the functions $f(x)$, $g(x)$ and the stochastic process $\xi(t)$ satisfy conditions (2.2).

Let us construct a stability condition for this equation using Theorem 1.1. It is easy to check that conditions (2.5), (2.6) hold. Calculating a, we have

$$\begin{aligned} a &= \lambda_1 \sup_{t>-1}\left[\frac{t+2}{(t+3)^2} + \frac{t+1}{(t+2)^2} - \frac{t+1}{(t+3)^2}\right] \\ &= \lambda_1 \sup_{t>-1}\left[\frac{1}{(t+3)^2} + \frac{t+1}{(t+2)^2}\right] = A\lambda_1. \end{aligned}$$

Here $A = 0.383$ and A takes this value at the point $t = -0.45$. Calculating $G(a,\sigma)$, we have

$$\begin{aligned} G(a,\sigma) &= \lambda_1\lambda_2 \sup_{t\ge 0}\left[\sum_{j=0}^{\infty} a(t+j,j) \sum_{k=0}^{[t]+j+1} a(t+j,k)\right] \\ &= \lambda_1\lambda_2 \sup_{t\ge 0}\left[\sum_{j=0}^{\infty} \frac{t+1}{(t+j+2)^2} \sum_{k=0}^{[t]+j+1} \frac{t+j+1-k}{(t+j+2)^2}\right] \\ &= \frac{\lambda_1\lambda_2}{2} \sup_{t\ge 0}\left[(t+1)\sum_{j=0}^{\infty} \frac{(2t-[t]+j+1)([t]+j+2)}{(t+j+2)^4}\right]. \end{aligned}$$

Note that $(2t-[t]+j+1)([t]+j+2) \le (t+j+2)^2$. So,

$$\begin{aligned} G(a,\sigma) &\le \frac{\lambda_1\lambda_2}{2} \sup_{t\ge 0}\left[(t+1)\sum_{j=0}^{\infty} \frac{1}{(t+j+2)^2}\right] \\ &\le \frac{\lambda_1\lambda_2}{2} \sup_{t\ge 0}\left[(t+1)\left(\frac{1}{(t+2)^2} + \int_1^{\infty} \frac{ds}{(t+1+s)^2}\right)\right] \\ &= \frac{\lambda_1\lambda_2}{2} \sup_{t\ge 0}\left[(t+1)\left(\frac{1}{(t+2)^2} + \frac{1}{t+2}\right)\right] \\ &= \frac{\lambda_1\lambda_2}{2} \sup_{t\ge 0}\left[\frac{(t+1)(t+3)}{(t+2)^2}\right] = \frac{\lambda_1\lambda_2}{2}. \end{aligned}$$

Similarly, $G(\sigma,\sigma) \leq \frac{1}{2}\lambda_2^2$. So, if the condition

$$\left(Ac_1 + \frac{c_2 - c_1}{2}c_3\lambda_2\right)\frac{\lambda_1}{2} + \frac{c_3^2\lambda_2^2}{2} < \frac{c_1}{c_2}$$

holds then the trivial solution of Eq. (2.13) is asymptotically mean square quasistable.

3. Conclusion

Sufficient conditions for asymptotic mean square stability of the trivial solution of a class of nonlinear stochastic difference Volterra equations with continuous time are obtained via the general method of Lyapunov functionals construction.

References

1. R. Bellman and K. Cooke, *Differential-Difference Equations* (Academic Press, 1963).
2. E. Beretta, V. Kolmanovskii and L. Shaikhet, Stability of epidemic model with time delays influenced by stochastic perturbations, *Mathematics and Computers in Simulation, Special Issue "Delay Systems"* **45** (1998) 269–277.
3. M. G. Blizorukov, On the construction of solutions of linear difference systems with continuous time, *Differentsialniye Uravneniya* **32** (1996) 127–128; *Translation in Differential Eqs.* **32** (1996) 133–134.
4. P. Borne, V. Kolmanovskii and L. Shaikhet, Stabilization of inverted pendulum by control with delay, *Dynamic Systems and Applications* **9** (2000) 501–515.
5. T. A. Burton, *Volterra Integral and Differential Equations* (Academic Press, 1983).
6. J. M. Cushing, *Integrodifferential Equations and Delay Models in Population Dynamics*, Lectures Notes in Biomathematics, Vol. 20 (Springer-Verlag, 1977).
7. N. J. Ford, J. T. Edwards, J. A. Roberts and L. E. Shaikhet, Stability of a difference analogue for a nonlinear integro-differential equation of convolution type, *Stability and Control: Theory Appl.* **3** (2000) 24–37.
8. K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics* (Kluwer, 1992).
9. A. Halanay, *Differential Equations: Stability, Oscillations, Time Lags* (Academic Press, 1966).
10. J. K. Hale, Oscillations in neutral functional differential equations, in *Nonlinear Mechanics* (CIME, 1972).
11. J. K. Hale, *Theory of Functional Differential Equations* (Springer-Verlag, 1977).
12. J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations* (Springer-Verlag, 1993).
13. V. B. Kolmanovskii, The stability of certain discrete-time Volterra equations, *J. Appl. Math. Mech.* **63** (1999) 537–543.
14. V. B. Kolmanovskii, J. F. Lafay and J. P. Richard, *Riccati Equations in Stability Theory of Difference Equations with Memory*, ECC'99, European Control Conference, Karlstruhe, Germany, September, 1999.
15. V. B. Kolmanovskii and A. D. Myshkis, *Applied Theory of Functional Differential Equations* (Kluwer, 1992).
16. V. B. Kolmanovskii and A. D. Myshkis, Stability in the first approximation of some Volterra difference equations, *J. Difference Eqs. Appl.* **3** (1998) 563–569.

17. V. B. Kolmanovskii and V. R. Nosov, *Stability of Functional Differential Equations* (Academic Press, 1986).
18. V. B. Kolmanovskii and L. E. Shaikhet, *Control of Systems with Aftereffect*, Translations of Mathematical Monographs, Vol. 157 (American Mathematical Society, 1996).
19. V. B. Kolmanovskii and L. E. Shaikhet, A method for constructing Lyapunov functionals for stochastic systems with aftereffect, *Differentialniye Uravneniya* **29** (1993) 1909–1920; *Transl. Differential Eqs.* **29** (1993) 1657–1666.
20. V. B. Kolmanovskii and L. E. Shaikhet, Stability of stochastic systems with aftereffect, *Avt. Telemekh.* No. 7 (1993) 66–85; *Transl. Automat. Remote Control* **54** (1993) 1087–1107.
21. V. Kolmanovskii and L. Shaikhet, New results in stability theory for stochastic functional-differential equations (SFDEs) and their applications, *Dynamic Publishers Inc.*, in *Proc. of Dynamic Systems and Applications*, Atlanta **1** (1994) 167–171.
22. V. B. Kolmanovskii and L. E. Shaikhet, A method for constructing Lyapunov functionals for stochastic differential equations of neutral type, *Differentialniye Uravneniya* **31** (1995) 1851–1857; *Trans. Differential Eqs.* **31** (1995) 1819–1825.
23. V. Kolmanovskii and L. Shaikhet, General method of Lyapunov functionals construction for stability investigations of stochastic difference equations, in *Dynamical Systems and Applications*, World Scientific Series in Applicable Analysis, Vol. 4 (World Scientific, 1995), pp. 397–439.
24. V. B. Kolmanovskii and L. E. Shaikhet, Asymptotic behaviour of some discrete-time systems, *Avtomatika i Telemekhanika* N. 12 (1996) 58–66; *Transl. Automat. Remote Control* **57** (1996) 1735–1742.
25. V. Kolmanovskii and L. Shaikhet, Matrix Riccati equations and stability of stochastic linear systems with nonincreasing delays, *Funct. Differential Eqs.* **4** (1997) 279–293.
26. V. Kolmanovskii and L. Shaikhet, Some peculiarities of the general method of Lyapunov functionals construction, *Appl. Math. Lett.* **15** (2002) 355–360.
27. V. Kolmanovskii and L. Shaikhet, Construction of Lyapunov functionals for stochastic hereditary systems: A survey of some recent results, *Math. Computer Model.* **36** (2002) 691–716.
28. V. Kolmanovskii and L. Shaikhet, About one application of the general method of Lyapunov functionals construction, *Int. J. Robust and Nonlinear Control, Special Issue on Time Delay Systems* **13** (2003) 805–818.
29. D. G. Korenevskii, Stability criteria for solutions of systems of linear deterministic or stochastic delay difference equations with continuous time, *Mat. Zam.* **70** (2001) 213–229; *Transl. Math. Notes* **70** (2001) 192–205.
30. G. S. Ladde, V. Lakshmikantham and B. G. Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments* (Marcel Dekker, 1987).
31. J. LaSalle and S. Lefschetz, *Stability by Lyapunov's Direct Method with Applications* (Academic Press, 1961).
32. Yu. L. Maistrenko and A. N. Sharkovsky, Difference equations with continuous time as mathematical models of the structure emergences, Dynamical systems and environmental models, in *Eisenach*, *Mathem. Ecol.* (Akademie-Verlag, 1986), pp. 40–49.
33. M. Malek-Zavarei and M. Jamshidi, *Time Delay Systems*: *Analysis*, *Optimization and Applications*, Systems and Control Series, Vol. 9 (North-Holland, 1988).
34. X. Mao and L. Shaikhet, Delay-dependent stability criteria for stochastic differential delay equations with Markovian switching, *Stability Control*: *Theory Appl.* **3** (2000) 88–101.
35. S.-E. A. Mohammed, *Stochastic Functional Differential Equations* (Longman, 1986).
36. A. D. Myshkis, *General Theory of Differential Equations with Delay*, Translations of Mathematical Monographs, Vol. 55 (American Mathematical Society, 1951).

37. H. Peics, Representation of solutions of difference equations with continuous time, *Electronic J. Qualitative Theory of Differential Equations*, in *Proc. of the 6th Colloquium of Differential Equations*, Vol. 21 (2000) 1–8.
38. G. P. Pelyukh, A certain representation of solutions to finite difference equations with continuous argument, *Differentsialniye Uravneniya* **32** (1996) 256–264; *Transl. Differential Eqs.* **32** (1996) 260–268.
39. L. E. Shaikhet, Necessary and sufficient conditions of asymptotic mean square stability for stochastic linear difference equations, *Appl. Math. Lett.* **10** (1997) 111–115.
40. L. Shaikhet, Stability of predator-prey model with aftereffect by stochastic perturbations, *Stability and Control*: *Theory Appl.* **1** (1998) 3–13.
41. L. Shaikhet, About Lyapunov functionals construction for difference equations with continuous time, *Appl. Math. Lett.* **17** (2004) 985–991.
42. L. Shaikhet, Construction of Lyapunov functionals for stochastic difference equations with continuous time, *Math. Comput. Simul.* **66** (2004) 509–521.
43. L. Shaikhet, Lyapunov functionals construction for stochastic difference second kind Volterra equations with continuous time, *Adv. Difference Eqs.* (2004) 67.
44. V. Volterra, *Lesons sur la Theorie Mathematique de la Lutte Pour la vie* (Gauthier-Villars, 1931).