ABOUT SOME FEATURES OF GENERAL METHOD OF LYAPUNOV FUNCTIONALS CONSTRUCTION

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Abstract. Many stability results in the theory of stochastic hereditary systems and their applications were obtained by construction of appropriate Lyapunov functionals. One general method of Lyapunov functionals construction was proposed and developed by the authors during last decade for stability investigation of deterministic and stochastic functional-differential and difference equations. In this paper a survey of some typical examples of this method application and at the same time some new features of this method for stochastic functional differential equations of neutral type are shown, which allow to use the method more effectively. The considered method is illustrated by a lot of figures of stability regions obtained by numerical calculations.

Keywords: Stochastic systems, delay, stability, Lyapunov functionals construction

1 Statement of the problem

A lot of processes in automatic regulation, physics, mechanics, biology, ecology, economy etc. can be modelled by functional differential equations [1-14]. Many stability results in the theory of functional differential equations and their applications were obtained using appropriate Lyapunov functionals. One general method of Lyapunov functionals construction both for deterministic and stochastic functional-differential and difference equations was proposed and developed by authors in [15-29]. Some applications of this method for stability investigation of the mathematical models of some real biological and mechanical systems are considered in [30-32]. Here a survey of some typical examples and some new features of this method for stochastic differential equations of neutral type are shown, which allow to use the method more effectively. A lot of figures with stability regions obtained by numerical calculations illustrate the considered method.

Consider the stochastic differential equation of neutral type

$$d(x(t) - G(t, x_t)) = a_1(t, x_t)dt + a_2(t, x_t)d\xi(t),$$
$$t \geq 0, \quad x(t) \in R^n, \quad x(s) = \varphi(s), \quad s \leq 0. \tag{1.1}$$

Here $x_t = x(t+s)$, $s \leq 0$, $\xi(t) \in R^m$ is a standard Wiener process, $G(t, 0) \equiv 0$, $a_i(t, 0) \equiv 0$, $i = 1, 2$. It is supposed that conditions for existence and unique of the solution of equation (1.1) hold, in particular,

$$|G(t, \varphi) - G(t, \psi)| \leq \int_0^\infty |\varphi(-s) - \psi(-s)|dK(s), \quad \int_0^\infty dK(s) < 1. \tag{1.2}$$

Definition 1.1. The trivial solution of equation (1.1) is called mean square stable if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $\mathbb{E}|x(t)|^2 < \epsilon$, $t \geq 0$, if $\|\varphi\|^2 = \sup_{s \leq 0} \mathbb{E}|\varphi(s)|^2 < \delta$. If, besides, $\lim_{t \to \infty} \mathbb{E}|x(t)|^2 = 0$ for every initial function $\varphi(s)$ then the trivial solution of equation (1.1) is called asymptotically mean square stable.

Theorem 1.1 [8]. Let condition (1.2) holds and there exists the functional

$$V(t, \varphi) = W(t, \varphi) + |\varphi(0) - G(t, \varphi)|^2,$$

such that

$$0 \leq \mathbb{E}W(t, x_t) \leq c_1 \|x_t\|^2,$$

$$\mathbb{E}LV(t, x_t) \leq -c_2 \mathbb{E}|x(t)|^2,$$

where $c_i > 0$, $i = 1, 2$, L is the generator of equation (1.1). Then the trivial solution of equation (1.1) is asymptotically mean square stable.

2 Procedure of Lyapunov functionals construction

This procedure consists of four steps.

Step 1. Transform equation (1.1) to the form

$$dz(t, x_t) = (b_1(t, x(t)) + c_1(t, x_t))dt + (b_2(t, x(t)) + c_2(t, x_t))d\xi(t), \tag{2.1}$$

where $z(t, x_t)$ is some functional on x_t, $z(t, 0) = 0$, functionals b_i, $i = 1, 2$, depend on t and $x(t)$ only and do not depend on the previous values $x(t+s)$, $s < 0$, of the solution, $b_i(t, 0) = 0$.

Step 2. Assume that the trivial solution of the auxiliary equation without memory

$$dy(t) = b_1(t, y(t))dt + b_2(t, y(t))d\xi(t). \tag{2.2}$$

is asymptotically mean square stable and therefore there exists a Lyapunov function $v(t, y)$, for which the condition $L_0 v(t, y) \leq -|y|^2$ holds. Here L_0 is the generator of equation (2.2).

Step 3. A Lyapunov functional $V(t, x_t)$ is constructed in the form $V = V_1 + V_2$, where $V_1(t, x_t) = v(t, z(t, x_t))$. Here the argument y of the function $v(t, y)$ is replaced on the functional $z(t, x_t)$ from the left-hand part of equation (2.1).

Step 4. Usually, the functional V_1 almost satisfies the conditions of Theorem 1.1. In order to satisfy these conditions completely it is necessary to calculate LV_1 and estimate it. Then the additional component V_2 can be easily chosen in a standard way.

Note that representation (2.1) is not unique. This fact allows us, using different representations (2.1) or different ways of LV_1 estimation, to construct different Lyapunov functionals and, as a result, to get different sufficient conditions of asymptotic mean square stability.

3   Application of Lyapunov functionals construction procedure

3.1   Linear differential equation of neutral type

Using the proposed procedure let us construct conditions of asymptotic mean square stability for the equation of neutral type

$$\dot{x}(t) + ax(t) + bx(t-h) + c\dot{x}(t-h) + \sigma x(t-\tau)\dot{\xi}(t) = 0, \quad |c| < 1. \tag{3.1}$$

3.1.1   *The first way of Lyapunov functional construction.*

Following Step 1 of the procedure, rewrite equation (3.1) in the form

$$\dot{z}(x_t) = -ax(t) - bx(t-h) - \sigma x(t-\tau)\dot{\xi}(t),$$
$$z(x_t) = x(t) + cx(t-h).$$

Suppose that $a > 0$. Then the function $v = y^2$ is a Lyapunov function for the auxiliary equation $\dot{y}(t) = -ay(t)$, since $\dot{v} = -2ay^2$. Thus, the trivial solution of the auxiliary equation is asymptotically stable. Put $V_1 = z^2(x_t)$. Then

$$LV_1 = 2z(x_t)(-ax(t) - bx(t-h)) + \sigma^2 x^2(t-\tau) =$$
$$= -2ax^2(t) - 2bcx^2(t-h) - 2(ac+b)x(t)x(t-h) + \sigma^2 x^2(t-\tau) \leq$$
$$\leq (-2a + |ac+b|)x^2(t) + \rho x^2(t-h) + \sigma^2 x^2(t-\tau),$$

where $\rho = |ac+b| - 2bc$ if $|ac+b| > 2bc$ and $\rho = 0$ if $|ac+b| \leq 2bc$.

Let

$$V_2 = \rho \int_{t-h}^{t} x^2(s)ds + \sigma^2 \int_{t-\tau}^{t} x^2(s)ds,$$

Then for the functional $V = V_1 + V_2$ we obtain

$$LV \leq (-2a + |ac+b| + \rho + \sigma^2)x^2(t).$$

So, if $|ac+b| + \rho + \sigma^2 < 2a$ then the trivial solution of equation (3.1) is asymptotically mean square stable. Using two representations for ρ, we obtain two stability conditions:

$$2bc \geq |ac+b|, \qquad \sigma^2 + |ac+b| < 2a \tag{3.2}$$

and

$$2bc < |ac+b|, \qquad p + |ac+b| - bc < a, \qquad p = \frac{\sigma^2}{2}. \tag{3.3}$$

From (3.2) and $a > 0$ we have $bc = |bc|$ and $|ac+b| = a|c| + |b|$. So, inequalities (3.2) take the form $2|bc| \geq a|c| + |b|$ and $\sigma^2 + a|c| + |b| < 2a$. The first from these inequalities is impossible if $2|c| < 1$. Suppose that $2|c| \geq 1$. Then

$$\frac{\sigma^2 + |b|}{2 - |c|} < a \leq \left(2 - \frac{1}{|c|}\right)|b|. \tag{3.4}$$

It is easy to see, that these inequalities are incompatible. Really, from (3.4) the impossible inequality $\sigma^2|c| + 2|b|(1-|c|)^2 < 0$ follows. Thus, condition (3.2) is impossible.

Consider condition (3.3). Suppose that $bc \geq 0$. From here and $a > 0$ we have $bc = |bc|$, $|ac+b| = a|c| + |b|$ and condition (3.3) takes the form

$$2|bc| < a|c| + |b|, \qquad a > |b| + \frac{p}{1 - |c|}.$$

If $2|c| < 1$ then the first inequality holds for all a and b. If $2|c| \geq 1$ then the second inequality implies the first one. So, if $bc \geq 0$ then from condition (3.3) we have

$$bc \geq 0, \qquad a > |b| + \frac{p}{1 - |c|}. \tag{3.5}$$

Let $bc < 0$. Then the first inequality (3.3) holds and condition (3.3) takes the form

$$bc < 0, \qquad p + |ac + b| - bc < a. \tag{3.6}$$

Since $bc < 0$ then $|ac + b| = |a|c| - |b||$. So, if $a|c| \geq |b|$ then from (3.6) we have

$$\frac{p}{1 - |c|} - a < |b| \leq a|c|. \tag{3.7}$$

If $a|c| < |b|$ then

$$a|c| < |b| < a - \frac{p}{1 + |c|}. \tag{3.8}$$

Combining (3.7) and (3.8), we obtain

$$bc < 0, \qquad \frac{p}{1 - |c|} - a < |b| < a - \frac{p}{1 + |c|}. \tag{3.9}$$

Note that the system

$$|b| = \frac{p}{1 - |c|} - a, \qquad |b| = a - \frac{p}{1 + |c|},$$

by $bc < 0$ has the solution

$$a = \frac{p}{1 - c^2}, \qquad b = -\frac{pc}{1 - c^2}. \tag{3.10}$$

So, combining (3.5), (3.9), (3.10), we obtain the stability condition in the form

$$a > \begin{cases} \frac{p}{1-c} + b, b > -\frac{pc}{1-c^2}, \\[2mm] \frac{p}{1+c} - b, b \leq -\frac{pc}{1-c^2}. \end{cases} \tag{3.11}$$

Thus, if condition (3.11) holds, then the trivial solution of equation (3.1) is asymptotically mean square stable.

The stability regions for equation (3.1), given by stability conditions (3.11), are shown on Fig.3.1 for $c = -0.5$, $h = 1$ and different values of p: 1) $p = 0$, 2) $p = 0.5$, 3) $p = 1$, 4) $p = 1.5$. On Fig.3.2 the stability regions are shown for $c = 0.5$ and the same values of other parameters.

3.1.2 *Second way of Lyapunov functional construction.*

To get another stability condition represent equation (3.1) in the form

$$\dot{z}(x_t) = -(a + b)x(t) - \sigma x(t - \tau)\dot{\xi}(t),$$

$$z(z_t) = x(t) + cx(t - h) - b \int_{t-h}^{t} x(s)ds.$$

Using condition (1.2) it is necessary to suppose that

$$|c| + |b|h < 1. \tag{3.12}$$

Suppose also that $a+b > 0$. Then the function $v = y^2$ is a Lyapunov function for the auxiliary equation $\dot{y}(t) = -(a+b)y(t)$, since $\dot{v} = -2(a+b)y^2$. Thus, the trivial solution of the auxiliary equation is asymptotically stable. Put $V_1 = z^2(x_t)$. Then

$$LV_1 = -2(a+b)x(t)z(x_t) + \sigma^2 x^2(t-\tau) = -2(a+b)x^2(t)-$$

$$- 2(a+b)cx(t)x(t-h) + 2(a+b)b \int_{t-h}^{t} x(t)x(s)ds + \sigma^2 x^2(t-\tau)$$

$$\leq (a+b)(-2+|c|+|b|h)x^2(t) + \sigma^2 x^2(t-\tau)+$$

$$+ (a+b) \left(|c|x^2(t-h) + |b| \int_{t-h}^{t} x^2(s)ds \right).$$

Let

$$V_2 = (a+b) \int_{t-h}^{t} [|c| + |b|(s-t+h)]x^2(s)ds + \sigma^2 \int_{t-\tau}^{t} x^2(s)ds.$$

Then for the functional $V = V_1 + V_2$ we obtain

$$LV \leq [-2(a+b)(1-|c|-|b|h) + \sigma^2]x^2(t).$$

Thus, using (3.12) we obtain that the stability condition has the form

$$p < (a+b)(1-|c|-|b|h), \qquad p = \frac{\sigma^2}{2}$$

or

$$a > \frac{p}{1-|c|-|b|h} - b, \qquad |b| < \frac{1-|c|}{h}. \tag{3.13}$$

The stability regions for equation (3.1), given by stability condition (3.13), are shown on Fig.3.3 for $|c| = 0.5$, $h = 0.2$ and different values of p: 1) $p = 0.2$, 2) $p = 0.6$, 3) $p = 1$, 4) $p = 1.4$ and on Fig.3.4 for $|c| = 0.5$, $p = 0.4$ and different values of h: 1) $h = 0.1$, 2) $h = 0.15$, 3) $h = 0.2$, 4) $h = 0.25$.

It is easy to check, that for $b \leq 0$ condition (3.11) is better than (3.13). So, condition (3.13) is better to use for $b > 0$ only in the form

$$a > \frac{p}{1-|c|-bh} - b, \qquad 0 < b < \frac{1-|c|}{h}. \tag{3.14}$$

The stability regions for equation (3.1), given by stability conditions (3.11) and (3.14) together, are shown on Fig.3.5 for $c = -0.6$, $p = 0.4$ and different values of h: 1) $h = 0.05$, 2) $h = 0.1$, 3) $h = 0.15$, 4) $h = 0.2$. On Fig.3.6 the stability regions are shown for $c = 0.6$ and the same values of other parameters.

3.1.3 Some particular cases.

1) Note that for $h \to 0$ condition (3.14) takes the form

$$a > \frac{p}{1-|c|} - b, \qquad b > 0. \tag{3.15}$$

On the other hand for $h = 0$ we have $LV = [-2(a+b)(1+c) + \sigma^2]x^2(t)$. So, for $h = 0$ the necessary and sufficient condition of asymptotic mean square stability has the form

$$a > \frac{p}{1+c} - b. \tag{3.16}$$

For $b > 0$ and $c > 0$ condition (3.15) is essentially worse than (3.16). But for $b > 0$ and $c \leq 0$ condition (3.15) coincides with (3.16). The second condition of (3.11) coincides with condition (3.16) as well.

2) For $c = 0$ the necessary and sufficient condition of asymptotic mean square stability of the trivial solution of equation (3.1) has the form

$$Q > p, \tag{3.17}$$

where

$$Q = \begin{cases} \frac{a+b\cos(qh)}{1+bq^{-1}\sin(qh)}, & q = \sqrt{b^2 - a^2}, \quad |a| \le |b|, \\[3mm] \frac{a+b\cosh(qh)}{1+bq^{-1}\sinh(qh)}, & q = \sqrt{a^2 - b^2}, \quad |a| > |b|. \end{cases} \tag{3.18}$$

On Fig.3.7 the stability regions, given by sufficient conditions (3.11), (3.14) and necessary and sufficient conditions (3.17), (3.18), are shown for different values of p: 1) $p = 0$, 2) $p = 0.5$, 3) $p = 1$, 4) $p = 1.5$, 5) $p = 2$. On Fig.3.8 the same stability regions are shown in another scale. We can see that sufficient conditions (3.11), (3.14) give us the stability region, which is enough close to exact one.

3) For $p = 0$ from (3.11), (3.14) it follows

$$a > \begin{cases} b, b \ge \frac{1-|c|}{h}, \\[3mm] -b, b < \frac{1-|c|}{h}. \end{cases} \tag{3.19}$$

On Fig.3.9 the exact stability regions are shown for 1) $c = 0.5$, 2) $c = -0.5$ and stability region, given by sufficient conditions (3.19) for 3) $|c| = 0.5$. On Fig.3.10 the similar regions are shown for 1) $c = 0.85$, 2) $c = -0.85$, 3) $|c| = 0.85$.

3.2   *Linear differential equation with two delays in deterministic part*

Consider the equation

$$\dot{x}(t) + ax(t - h_1) + bx(t - h_2) + \sigma x(t - \tau)\dot{\xi}(t) = 0, \qquad h_1 \le h_2. \tag{3.20}$$

Using different representations of this equation we will obtain different conditions of asymptotic mean square stability for the trivial solution of equation (3.20).

3.2.1   *The first way of Lyapunov functional construction.*

Represent equation (3.20) in the form

$$\dot{z}(x_t) = -ax(t) - bx(t - h_2) - \sigma x(t - \tau)\dot{\xi}(t),$$

$$z(x_t) = x(t) - a\int_{t-h_1}^{t} x(s)ds.$$

Putting $V_1 = z^2(x_t)$ we obtain

$$LV_1 = 2z(x_t)(-ax(t) - bx(t - h_2)) + \sigma^2 x^2(t - \tau)$$

$$= -2ax^2(t) - 2bx(t)x(t - h_2) +$$

$$+ 2a^2 \int_{t-h_1}^{t} x(t)x(s)ds + 2ab \int_{t-h_1}^{t} x(t - h_2)x(s)ds + \sigma^2 x^2(t - \tau)$$

$$\le -2ax^2(t) + |b|(x^2(t) + x^2(t - h_2)) + a^2 \left(x^2(t)h_1 + \int_{t-h_1}^{t} x^2(s)ds \right) +$$

$$+ |ab| \left(x^2(t - h_2)h_1 + \int_{t-h_1}^{t} x^2(s)ds \right) + \sigma^2 x^2(t - \tau)$$

$$= (-2a + a^2h_1 + |b|)x^2(t) + (a^2 + |ab|) \int_{t-h_1}^{t} x^2(s)ds +$$

$$+ (|b| + |ab|h_1)x^2(t - h_2) + \sigma^2 x^2(t - \tau).$$

Putting

$$V_2 = (a^2 + |ab|) \int_{t-h_1}^t (s - t + h_1)x^2(s)ds+$$

$$+ (|b| + |ab|h_1) \int_{t-h_2}^t x^2(s)ds + \sigma^2 \int_{t-\tau}^t x^2(s)ds,$$

for $V = V_1 + V_2$ we obtain

$$LV \le 2(-a + a^2h_1 + |b| + |ab|h_1 + p)x^2(t), \qquad p = \frac{\sigma^2}{2}.$$

From here and (1.2) it follows that if

$$|b| < \frac{a(1 - ah_1) - p}{1 + ah_1}, \qquad 0 < a < \frac{1}{h_1}, \tag{3.21}$$

then the trivial solution of equation (3.20) is asymptotically mean square stable.

Remark 3.1. Representing equation (3.20) in the form

$$\dot{z}(x_t) = -bx(t) - ax(t - h_1) - \sigma x(t - \tau)\dot{\xi}(t),$$

$$z(x_t) = x(t) - b \int_{t-h_2}^t x(s)ds$$

and using a symmetry we obtain another sufficient condition of asymptotic mean square stability of the trivial solution of equation (3.20)

$$|a| < \frac{b(1 - bh_2) - p}{1 + bh_2}, \qquad 0 < b < \frac{1}{h_2}. \tag{3.22}$$

3.2.2 *The second way of Lyapunov functional construction.*

Represent equation (3.20) in the form

$$\dot{z}(x_t) = -(a + b)x(t) + bx(t - h_1) - bx(t - h_2) - \sigma x(t - \tau)\dot{\xi}(t),$$

$$z(x_t) = x(t) - (a + b) \int_{t-h_1}^t x(s)ds.$$

Putting $V_1 = z^2(x_t)$ we obtain

$$LV_1 = 2z(x_t)[-(a + b)x(t) + bx(t - h_1) - bx(t - h_2)] + \sigma^2 x^2(t - \tau)$$

$$= -2(a + b)x^2(t) + 2bx(t)x(t - h_1) - 2bx(t)x(t - h_2)+$$

$$+ 2(a + b)^2 \int_{t-h_1}^t x(t)x(s)ds - 2(a + b)b \int_{t-h_1}^t x(t - h_1)x(s)ds+$$

$$+ 2(a + b)b \int_{t-h_1}^t x(t - h_2)x(s)ds + \sigma^2 x^2(t - \tau)$$

$$\le -2(a + b)x^2(t) + |b|(x^2(t) + x^2(t - h_1)) + |b|(x^2(t) + x^2(t - h_2))+$$

$$+ (a + b)^2 \left(x^2(t)h_1 + \int_{t-h_1}^t x^2(s)ds \right) + \sigma^2 x^2(t - \tau)+$$

$$+ |(a + b)b| \left(x^2(t - h_1)h_1 + x^2(t - h_2)h_1 + 2 \int_{t-h_1}^t x^2(s)ds \right)$$

$$= [-2(a + b) + (a + b)^2 h_1 + 2|b|]x^2(t) + \sigma^2 x^2(t - \tau)+$$

$$+ |b|(1 + |a + b|h_1)(x^2(t - h_1) + x^2(t - h_2))+$$

$$+ ((a + b)^2 + 2|b(a + b)|) \int_{t-h_1}^t x^2(s)ds.$$

Putting

$$V_2 = ((a+b)^2 + 2|b(a+b)|) \int_{t-h_1}^t (s - t + h_1)x^2(s)ds +$$

$$+ |b|(1 + |a+b|)h_1) \left(\int_{t-h_1}^t x^2(s)ds + \int_{t-h_2}^t x^2(s)ds \right) + \sigma^2 \int_{t-\tau}^t x^2(s)ds,$$

for $V = V_1 + V_2$ we obtain

$$LV \le 2[-(a+b) + (a+b)^2 h_1 + 2|b|(1 + (a+b)h_1) + p]x^2(t).$$

From here and (1.2) it follows that if

$$2|b| < \frac{(a+b)(1 - (a+b)h_1) - p}{1 + (a+b)h_1}, \qquad 0 < a + b < \frac{1}{h_1}, \tag{3.23}$$

then the trivial solution of equation (3.20) is asymptotically mean square stable.

Remark 3.2. Similarly to Remark 3.1 using a symmetry we obtain another sufficient condition of asymptotic mean square stability of the trivial solution of equation (3.20)

$$2|a| < \frac{(a+b)(1 - (a+b)h_2) - p}{1 + (a+b)h_2}, \qquad 0 < a + b < \frac{1}{h_2}. \tag{3.24}$$

3.2.3 The Third way of Lyapunov functional construction.

Represent equation (3.20) in the form

$$\dot{z}(x_t) = -(a+b)x(t) - \sigma x(t - \tau)\dot{\xi}(t),$$

$$z(x_t) = x(t) - a \int_{t-h_1}^t x(s)ds - b \int_{t-h_2}^t x(s)ds. \tag{3.25}$$

Putting $V_1 = z^2(x_t)$ we obtain

$$LV_1 = 2z(x_t)(-(a+b))x(t) + \sigma^2 x^2(t-\tau) =$$

$$= -2(a+b)x^2(t) + 2a(a+b) \int_{t-h_1}^t x(t)x(s)ds +$$

$$+ 2b(a+b) \int_{t-h_2}^t x(t)x(s)ds + \sigma^2 x^2(t-\tau). \tag{3.26}$$

Therefore,

$$LV_1 \le -2(a+b)x^2(t) + |a(a+b)| \left(x^2(t)h_1 + \int_{t-h_1}^t x^2(s)ds \right) +$$

$$+ |b(a+b)| \left(x^2(t)h_2 + \int_{t-h_2}^t x^2(s)ds \right) + \sigma^2 x^2(t-\tau)$$

$$= [-2(a+b) + |a+b|(|a|h_1 + |b|h_2)]x^2(t) + \sigma^2 x^2(t-\tau) +$$

$$+ |a+b| \left(|a| \int_{t-h_1}^t x^2(s)ds + |b| \int_{t-h_2}^t x^2(s)ds \right).$$

Putting

$$V_2 = |a(a+b)| \int_{t-h_1}^t (s - t + h_1)x^2(s)ds +$$

$$+ |b(a+b)| \int_{t-h_2}^t (s - t + h_2)x^2(s)ds + \sigma^2 \int_{t-\tau}^t x^2(s)ds,$$

for $V = V_1 + V_2$ we obtain

$$LV \leq 2[-(a+b) + |a+b|(|a|h_1 + |b|h_2) + p]x^2(t).$$

From here and (1.2) it follows that if

$$a + b > \frac{p}{1 - |a|h_1 - |b|h_2}, \qquad |a|h_1 + |b|h_2 < 1, \qquad (3.27)$$

then the trivial solution of equation (3.20) is asymptotically mean square stable.

Remark 3.3. Note that condition (3.27) follows from condition (3.22), condition (3.22) follows from condition (3.24), condition (3.21) follows from condition (3.23). Let us show, for example, that if condition (3.22) holds then condition (3.27) holds too. Really, rewrite (3.22) and (3.27) in the form

$$p < b(1 - bh_2) - |a|(1 + bh_2), \qquad 0 < b < \frac{1}{h_2}$$
$$p < (a+b)(1 - |a|h_1 - |b|h_2), \qquad |a|h_1 + |b|h_2 < 1.$$

It is enough to prove that $b(1-bh_2) - |a|(1+bh_2) \leq (a+b)(1 - |a|h_1 - |b|h_2)$ or $|a|(a+b)h_1 + abh_2 \leq a + |a| + |a|bh_2$. Consider at first the case $a \geq 0$. Then we obtain $(a+b)h_1 < 2$. Using $h_1 \leq h_2$ and $ah_1 + bh_2 < 1$ we have $(a+b)h_1 \leq ah_1 + bh_2 < 1 < 2$. Let now $a < 0$. Then it is necessary to prove that $(a+b)h_1 \leq 2bh_2$. But it follows from $(a+b)h_1 < bh_1 \leq bh_2 \leq 2bh_2$.

3.2.4 *The fourth way of Lyapunov functional construction.*

Let us show that using the same representations of the initial equation but different ways of LV_1 estimation we can get different stability conditions.

Represent equation (3.20) in form (3.25) and put $V_1 = z^2(x_t)$ again. Using (3.26) and condition $a + b > 0$, let us estimate LV_1 by the following way

$$LV_1 = -2(a+b)x^2(t) + 2(a+b)^2 \int_{t-h_1}^t x(t)x(s)ds+$$

$$+ 2(a+b)b \int_{t-h_2}^{t-h_1} x(t)x(s)ds + \sigma^2 x^2(t-\tau)$$

$$\leq -2(a+b)x^2(t) + (a+b)^2 \left(h_1 x^2(t) + \int_{t-h_1}^t x^2(s)ds \right) +$$

$$+ (a+b)|b| \left((h_2 - h_1)x^2(t) + \int_{t-h_2}^{t-h_1} x^2(s)ds \right) + \sigma^2 x^2(t-\tau)$$

$$= [-2(a+b) + (a+b)^2 h_1 + (a+b)|b|(h_2 - h_1)]x^2(t) + \sigma^2 x^2(t-\tau)+$$

$$+ (a+b)^2 \int_{t-h_1}^t x^2(s)ds + (a+b)|b| \int_{t-h_2}^{t-h_1} x^2(s)ds.$$

Putting

$$V_2 = (a+b)^2 \int_{t-h_1}^t (s-t+h_1)x^2(s)ds + (a+b)|b| \int_{t-h_2}^{t-h_1} (s-t+h_2)x^2(s)ds+$$

$$+ (a+b)|b|(h_2 - h_1) \int_{t-h_1}^t x^2(s)ds + \sigma^2 \int_{t-\tau}^t x^2(s)ds,$$

for the functional $V = V_1 + V_2$ we obtain

$$LV \leq 2[-(a+b) + (a+b)^2 h_1 + (a+b)|b|(h_2 - h_1) + p]x^2(t), \qquad p = \frac{\sigma^2}{2}.$$

From here and (1.2) it follows that if

$$a + b > \frac{p}{1 - (a+b)h_1 - |b|(h_2 - h_1)}, \qquad |a|h_1 + |b|h_2 < 1, \qquad (3.28)$$

then the trivial solution of equation (3.20) is asymptotically mean square stable.

It is easy to see that condition (3.28) coincides with (3.27) for $p = 0$. If $p > 0$ then condition (3.28) coincides with (3.27) for $a \geq 0$, $b \geq 0$, but in the case $ab < 0$ condition (3.28) is weaker than (3.27). Using *Remark 3.3* we can conclude that stability conditions (3.21) and (3.28) together are better than all other.

Put $p = 0$, $h_2 = 1$. On Fig.3.11 the stability regions, given by conditions (3.21)-(3.24), (3.27) (with numbers 1-5 correspondingly), are shown for $h_1 = 0.1$. On Fig.3.12 the similar picture is shown for $h_1 = 0.2$ with addition of the bound of the stability region, given by the necessary and sufficient condition of asymptotic stability. On Fig.3.13 we can see how the picture on Fig.3.12 is changed for $h_1 = 0.25$. On Fig.3.14 stability region, given by condition (3.21), is shown for $h_1 = 0.01$ together with the bound of the stability region, given by the necessary and sufficient condition of asymptotic stability.

Consider now the case $p > 0$, $h_2 = 1$. On Fig.3.15 the stability regions, given by conditions (3.21)-(3.24), (3.27), (3.28) (with numbers 1-6 correspondingly), are shown for $p = 0.1$, $h_1 = 0.1$. On Fig.3.16 these stability regions are shown for $p = 0.1$, $h_1 = 0.5$ and on Fig.3.17 for $p = 0.25$, $h_1 = 0.5$.

3.2.5   *One generalization for equation with n delays*

Consider the equation with n delays in deterministic part and nonlinear stochastic part

$$\dot{x}(t) + \sum_{i=1}^{n} a_i x(t - h_i) + \sigma(t, x_t)\dot{\xi}(t) = 0. \qquad (3.29)$$

Here it is supposed that

$$h_1 \leq h_2 \leq ... \leq h_n,$$
$$|\sigma(t, \varphi)| \leq \int_0^\infty |\varphi(-s)| dR(s), \qquad R = \int_0^\infty dR(s). \qquad (3.30)$$

Represent equation (3.29) in the form

$$\dot{z}(x_t) = -S_1 x(t) - \sigma x(t - \tau)\dot{\xi}(t),$$
$$z(x_t) = x(t) - \sum_{i=1}^{n} a_i \int_{t-h_i}^{t} x(s)ds,$$
$$S_j = \sum_{i=j}^{n} a_i, \qquad j = 1, ..., n. \qquad (3.31)$$

Using the functional $V = V_1 + V_2$, where $V_1 = z^2(x_t)$,

$$V_2 = S_1 \sum_{i=1}^{n} |a_i| \int_{t-h_i}^{t} (s - t + h_i)x^2(s)ds + R \int_0^\infty \int_{t-s}^{t} x^2(\theta)d\theta dR(s),$$

similar to (3.27) we obtain a sufficient condition of asymptotic mean square stability for the trivial solution of equation (3.29) in the form

$$S_1 > p \left(1 - \sum_{i=1}^{n} |a_i|h_i \right)^{-1}, \qquad \sum_{i=1}^{n} |a_i|h_i < 1, \qquad p = \frac{1}{2}R^2.$$

This condition is a generalization of condition (3.27).

Let us obtain a generalization of condition (3.28). Using representation (3.31), consider the functional $V = V_1 + V_2$, where $V_1 = z^2(x_t)$. Then using (3.30), (3.31), we get

$$LV_1 = -2S_1 x^2(t) + 2S_1 x(t) \sum_{i=1}^{n} a_i \int_{t-h_i}^{t} x(s)ds + \sigma^2(t, x_t)$$

$$= -2S_1 x^2(t) + 2S_1 x(t) \sum_{i=1}^{n} a_i \left(\int_{t-h_1}^{t} x(s)ds + \sum_{j=1}^{i-1} \int_{t-h_{j+1}}^{t-h_j} x(s)ds \right) + \sigma^2(t, x_t)$$

$$\leq -2S_1 x^2(t) + S_1^2 \left(h_1 x^2(t) + \int_{t-h_1}^{t} x^2(s)ds \right) +$$

$$+ S_1 \sum_{j=1}^{n-1} |S_{j+1}| \left((h_{j+1} - h_j) x^2(t) + \int_{t-h_{j+1}}^{t-h_j} x(s)ds \right) + R \int_{0}^{\infty} x^2(t-s)dR(s)$$

$$= \left[-2S_1 + S_1^2 h_1 + S_1 \sum_{j=1}^{n-1} |S_{j+1}|(h_{j+1} - h_j) \right] x^2(t) +$$

$$+ S_1^2 \int_{t-h_1}^{t} x^2(s)ds + S_1 \sum_{j=1}^{n-1} |S_{j+1}| \int_{t-h_{j+1}}^{t-h_j} x(s)ds + R \int_{0}^{\infty} x^2(t-s)dR(s).$$

Putting

$$V_2 = S_1^2 \int_{t-h_1}^{t} (s - t + h_1) x^2(s)ds + R \int_{0}^{\infty} \int_{t-s}^{t} x^2(\theta)d\theta dR(s) +$$

$$+ S_1 \sum_{j=1}^{n-1} |S_{j+1}| \left[\int_{t-h_{j+1}}^{t-h_j} (s - t + h_{j+1}) x^2(s)ds + (h_{j+1} - h_j) \int_{t-h_j}^{t} x^2(s)ds \right],$$

for the functional $V = V_1 + V_2$ we obtain

$$LV \leq 2 \left[-S_1 + S_1^2 h_1 + S_1 \sum_{j=1}^{n-1} |S_{j+1}|(h_{j+1} - h_j) + p \right] x^2(t).$$

From here and (1.2) we obtain a generalization of condition (3.28) in the form

$$S_1 > p \left(1 - S_1 h_1 - \sum_{j=1}^{n-1} |S_{j+1}|(h_{j+1} - h_j) \right)^{-1}, \qquad \sum_{i=1}^{n} |a_i| h_i < 1.$$

To prove that this condition is weaker than previous one it is enough to show that

$$S_1 h_1 + \sum_{j=1}^{n-1} |S_{j+1}|(h_{j+1} - h_j) \leq \sum_{i=1}^{n} |a_i| h_i.$$

Rewrite this inequality in the form

$$\sum_{j=0}^{n-1} |S_{j+1}| h_{j+1} \leq \sum_{j=1}^{n-1} |S_{j+1}| h_j + \sum_{i=1}^{n} |a_i| h_i,$$

that is equivalent to

$$\sum_{j=1}^{n} |S_j| h_j \leq \sum_{j=1}^{n-1} |S_{j+1}| h_j + \sum_{j=1}^{n} |a_j| h_j.$$

Now it is enough to note that $|S_j| \leq |S_{j+1}| + |a_j|$, $j = 1, ..., n-1$ and $S_n = a_n$.

3.3 *Nonlinear scalar equation*

Consider the nonlinear equation of neutral type

$$dz(x_t) = (b - ax(t))z(x_t)(dt + \sigma d\xi(t)),$$
$$z(x_t) = x(t) - cx(t - h), \tag{3.32}$$

where $a > 0$, $b > 0$, $h \geq 0$, $|c| < 1$, σ are some constants. Let us investigate mean–square stability of the solution $x_0(t) \equiv x_0 = a^{-1}b$ with respect to perturbations of the initial function ϕ, satisfying the following inequality

$$z(0) = \phi(0) - c\phi(-h) > 0. \tag{3.33}$$

Put $h = 0$. Then system (3.32) takes the form

$$dx(t) = -a(x(t) - x_0)x(t)(dt + \sigma d\xi(t)). \tag{3.34}$$

Let us show that if

$$0 < pb < 1, \qquad p = \frac{\sigma^2}{2}, \tag{3.35}$$

then the function

$$v(x(t)) = x(t) - x_0 - x_0 \ln \frac{x(t)}{x_0} \tag{3.36}$$

is a Lyapunov function for auxiliary equation (3.34).

First of all note that $v(x_0) = 0$ and since $x - 1 - \ln x \geq 0$ for $x > 0$ then

$$v(x(t)) = x_0 \left(\frac{x(t)}{x_0} - 1 - \ln \frac{x(t)}{x_0} \right) \geq 0.$$

Using $ax_0 = b$ we obtain

$$Lv(x(t)) = -a(x(t) - x_0)x(t) \left(1 - \frac{x_0}{x(t)} \right) + pa^2(x(t) - x_0)^2 x^2(t) \frac{x_0}{x^2(t)}$$
$$= -a(1 - bp)(x(t) - x_0)^2.$$

So, by condition (3.35) function (3.36) is a Lyapunov function for solution x_0 of auxiliary equation (3.34).

Using the procedure of Lyapunov functionals construction we will construct Lyapunov functional V for equation (3.32) in the form $V = V_1 + V_2$, where

$$V_1 = v(z(x_t)) = z(x_t) - z_0 - z_0 \ln \frac{z(x_t)}{z_0},$$
$$z(x_t) := x(t) - cx(t - h), \qquad z_0 = (1 - c)x_0.$$

Calculating LV_1 for equation (3.32) we obtain

$$LV_1 = -a(x(t) - x_0)z(x_t) \left(1 - \frac{z_0}{z(x_t)} \right) + pa^2(x(t) - x_0)^2 z^2(x_t) \frac{z_0}{z^2(x_t)}$$
$$= -a(x(t) - x_0)(z(x_t) - z_0) + abp(1 - c)(x(t) - x_0)^2$$
$$= -a(x(t) - x_0)(x(t) - cx(t - h) - x_0 + cx_0) + abp(1 - c)(x(t) - x_0)^2$$
$$= -a[1 - bp(1 - c)](x(t) - x_0)^2 + ac(x(t) - x_0)(x(t - h) - x_0)$$
$$\leq -a[1 - bp(1 - c)](x(t) - x_0)^2 + \frac{a|c|}{2}[(x(t) - x_0)^2 + (x(t - h) - x_0)^2]$$
$$= -a \left[1 - bp(1 - c) - \frac{|c|}{2} \right] (x(t) - x_0)^2 + \frac{a|c|}{2}(x(t - h) - x_0)^2.$$

Putting

$$V_2 = \frac{a|c|}{2} \int_{t-h}^{t} (x(s) - x_0)^2 ds$$

for the functional $V = V_1 + V_2$ we have

$$LV \leq -a[1 - bp(1 - c) - |c|](x(t) - x_0)^2.$$

From here it follows that if conditions (3.33) and

$$0 < bp < \begin{cases} 1 + \frac{c}{1-c}, -1 < c < 0, \\ \\ 1, 0 \leq c < 1, \end{cases} \tag{3.37}$$

hold then the solution $x_0(t) \equiv a^{-1}b$ of equation (3.32) is asymptotically mean square stable. Note that condition (3.37) does not depend on $a > 0$ and $h \geq 0$.

On Fig.3.18 the stability regions are shown for different values of p: 1) $p = 0$, 2) $p = 0.2$, 3) $p = 0.4$, 4) $p = 0.6$, 5) $p = 0.8$.

3.4 *Integro-differential equation*

Let us obtain stability conditions for the equation

$$\dot{x}(t) = -ax(t) + \int_0^\infty x(t - s)dK(s) + \sigma(t, x_t)\dot{\xi}(t). \tag{3.38}$$

It is supposed that

$$|\sigma(t, x_t)| \leq \int_0^\infty |x(t - s)|dR(s), \qquad R = \int_0^\infty dR(s). \tag{3.39}$$

3.4.1 *The first way of Lyapunov functional construction.*

Put

$$k_0 = \int_0^\infty |dK(s)|. \tag{3.40}$$

The auxiliary equation (2.2) we will consider in the form $\dot{y}(t) = -ay(t)$, $a > 0$. Put $V = V_1 + V_2$, where $V_1 = x^2$. Calculating LV_1, we obtain

$$LV_1 = 2x(t)\left(-ax(t) + \int_0^\infty x(t - s)dK(s)\right) + \sigma^2(t, x_t). \tag{3.41}$$

From here and (3.39), (3.40) it follows that

$$LV_1 \leq -2ax^2(t) + \int_0^\infty (x^2(t) + x^2(t - s))|dK(s)| + R \int_0^\infty x^2(t - s)dR(s)$$

$$= (-2a + k_0)x^2(t) + \int_0^\infty x^2(t - s)|dK(s)| + R \int_0^\infty x^2(t - s)dR(s).$$

Putting

$$V_2 = \int_0^\infty \int_{t-s}^t x^2(\theta)d\theta|dK(s)| + R \int_0^\infty \int_{t-s}^t x^2(\theta)d\theta dR(s),$$

for the functional $V = V_1 + V_2$ we obtain

$$LV \leq (-2a + 2k_0 + R^2)x^2(t).$$

So, if

$$a > k_0 + p, \qquad p = \frac{1}{2}R^2, \tag{3.42}$$

then the trivial solution of equation (3.38) is asymptotically mean square stable.

Example 3.1. Put

$$dK(s) = f(s)ds, \qquad f(s) = \begin{cases} b(h - s), s \in [0, h], \\ \\ 0, s > h. \end{cases} \tag{3.43}$$

In this case condition (3.42) takes the form

$$a > |b|\frac{h^2}{2} + p. \tag{3.44}$$

On Fig.3.19 stability regions, given by condition (3.39), are shown for $h = 0.9$ and following values of the parameter p: 1) $p = 0$, 2) $p = 0.5$, 3) $p = 1$, 4) $p = 1.5$, 5) $p = 2$.

3.4.2 *The second way of Lyapunov functional construction.*

Put

$$k = \int_0^\infty dK(s), \qquad k_1 = \int_0^\infty s|dK(s)| \tag{3.45}$$

and represent equation (3.38) in the form

$$\dot{z}(x_t) = -(a - k)x(t) + \sigma(t, x_t)\dot{\xi}(t),$$

$$z(x_t) = x(t) + \int_0^\infty \int_{t-s}^t x(\theta)d\theta dK(s).$$

Note that

$$\int_0^\infty \int_{t-s}^t x(\theta)d\theta dK(s) = \int_0^\infty \int_0^s x(t-\theta)d\theta dK(s) = \int_0^\infty x(t-\theta) \int_\theta^\infty dK(s)d\theta$$

and

$$\left| \int_0^\infty \int_\theta^\infty dK(s)d\theta \right| \le \int_0^\infty \int_\theta^\infty |dK(s)|d\theta = \int_0^\infty \int_0^s d\theta |dK(s)| = k_1.$$

Therefore, following (1.2) and using the auxiliary equation (2.2) in the form $\dot{y}(t) = -(a - k)y(t)$, we will suppose that

$$k < a, \qquad k_1 < 1. \tag{3.46}$$

Put $V = V_1 + V_2$, where $V_1 = z^2(x_t)$. Calculating LV_1 and using (3.45), (3.46), we have

$$LV_1 = -2z(x_t)(a - k)x(t) + \sigma^2(t, x_t)$$

$$\le (a - k)\left[-2x^2(t) + \int_0^\infty \int_{t-s}^t (x^2(t) + x^2(\theta))d\theta |dK(s)| \right] + R \int_0^\infty x^2(t-s)dR(s)$$

$$= -(a - k)(2 - k_1)x^2(t) +$$

$$+ (a - k)\int_0^\infty \int_{t-s}^t x^2(\theta)d\theta |dK(s)| + R \int_0^\infty x^2(t-s)dR(s).$$

Putting

$$V_2 = (a - k)\int_0^\infty \int_{t-s}^t (\theta - t + s)x^2(\theta)d\theta |dK(s)| +$$

$$+ R \int_0^\infty \int_{t-s}^t x^2(\theta)d\theta dR(s),$$

for the functional $V = V_1 + V_2$ we obtain

$$LV \leq [-2(a-k)(1-k_1) + R^2]x^2(t).$$

So, if

$$a > k + p(1-k_1)^{-1}, \qquad k_1 < 1, \qquad p = \frac{1}{2}R^2. \tag{3.47}$$

then the trivial solution of equation (3.38) is asymptotically mean square stable.

Example 3.2. In the case (3.43) condition (3.47) takes the form

$$a > b\frac{h^2}{2} + p\left(1 - |b|\frac{h^3}{6}\right)^{-1}, \qquad |b|\frac{h^3}{6} < 1. \tag{3.48}$$

On Fig.3.20 stability regions, given by condition (3.48) are shown for $h = 0.9$ and following values of the parameter p: 1) $p = 0$, 2) $p = 0.1$, 3) $p = 0.5$, 4) $p = 1$, 5) $p = 1.5$, 6) $p = 2$.

3.4.3   *The third way of Lyapunov functional construction.*

Let us show that in some particular cases using a special way of LV_1 estimation we can get new stability conditions.

Consider equation (3.38), (3.43) and suppose that $b \leq 0$. Choosing V_1 in the form $V_1 = x^2$ similar to (3.41) we obtain

$$LV_1 = 2x(t)\left(-ax(t) + b\int_0^h (h-s)x(t-s)ds\right) + \sigma^2(t, x_t)$$

$$\leq -2ax^2(t) + 2bx(t)\int_0^h (h-s)x(t-s)ds + R\int_0^\infty x^2(t-s)dR(s). \tag{3.49}$$

Choosing V_2 in the form

$$V_2 = |b|\int_0^h \left(\int_{t-s}^t x(\theta)d\theta\right)^2 ds + R\int_0^\infty \int_{t-s}^t x^2(\theta)d\theta dR(s),$$

we have

$$LV_2 = 2|b|\int_0^h \int_{t-s}^t x(\theta)d\theta(x(t) - x(t-s))ds +$$

$$+ R^2 x^2(t) - R\int_0^\infty x^2(t-s)dR(s). \tag{3.50}$$

Note that

$$2|b|\int_0^h \int_{t-s}^t x(\theta)d\theta(x(t) - x(t-s))ds$$

$$= 2|b|\int_0^h \int_0^s x(t-\theta)d\theta(x(t) - x(t-s))ds$$

$$= 2|b|x(t)\int_0^h \int_0^s x(t-\theta)d\theta ds - 2|b|\int_0^h x(t-s)\int_0^s x(t-\theta)d\theta ds \tag{3.51}$$

and

$$2|b|x(t)\int_0^h \int_0^s x(t-\theta)d\theta ds = 2|b|x(t)\int_0^h \int_\theta^h ds x(t-\theta)d\theta$$

$$= -2bx(t)\int_0^h (h-s)x(t-s)ds. \tag{3.52}$$

So, from (3.49)-(3.52) for the functional $V = V_1 + V_2$ it follows

$$LV \leq (-2a + R^2)x^2(t) - 2|b| \int_0^h x(t-s) \int_0^s x(t-\theta)d\theta ds. \tag{3.53}$$

Changing the order of integration, we have

$$\int_0^h x(t-s) \int_0^s x(t-\theta)d\theta ds = \int_0^h x(t-\theta) \int_\theta^h x(t-s)ds d\theta$$

$$= \int_0^h x(t-s) \int_s^h x(t-\theta)d\theta ds.$$

Therefore,

$$2|b| \int_0^h x(t-s) \int_0^s x(t-\theta)d\theta ds = |b| \int_0^h x(t-s) \int_0^s x(t-\theta)d\theta ds +$$

$$+ |b| \int_0^h x(t-s) \int_s^h x(t-\theta)d\theta ds = |b| \left(\int_0^h x(t-s)ds \right)^2 \geq 0.$$

From here and (3.53) it follows

$$LV \leq -2(a-p)x^2(t), \qquad p = \frac{1}{2}R^2.$$

So, if

$$a > p, \qquad b \leq 0, \tag{3.54}$$

then the trivial solution of equation (3.38), (3.43) is asymptotically mean square stable.

Using (3.44), (3.48), (3.54), we can get sufficient condition of asymptotic mean square stability of the trivial solution of equation (3.38), (3.43) in the following form. If $ph < 3$ then

$$a > \begin{cases} b\frac{h^2}{2} + p, b \geq 0, \\ b\frac{h^2}{2} + p(1 + b\frac{h^3}{6})^{-1}, -2(3-ph)h^{-3} \leq b < 0, \\ p, b < -2(3-ph)h^{-3}. \end{cases} \tag{3.55}$$

If $ph \geq 3$ then

$$a > \begin{cases} b\frac{h^2}{2} + p, b \geq 0, \\ p, b < 0. \end{cases} \tag{3.56}$$

On Fig.3.21 stability regions are shown, given by conditions (3.55), (3.56) for $h = 0.9$ and following values of the parameter p: 1) $p = 0$, 2) $p = 0.1$, 3) $p = 0.3$, 4) $p = 0.7$, 5) $p = 1.2$, 6) $p = 2$, 7) $p = 2.5$, 8) $ph = 3$.
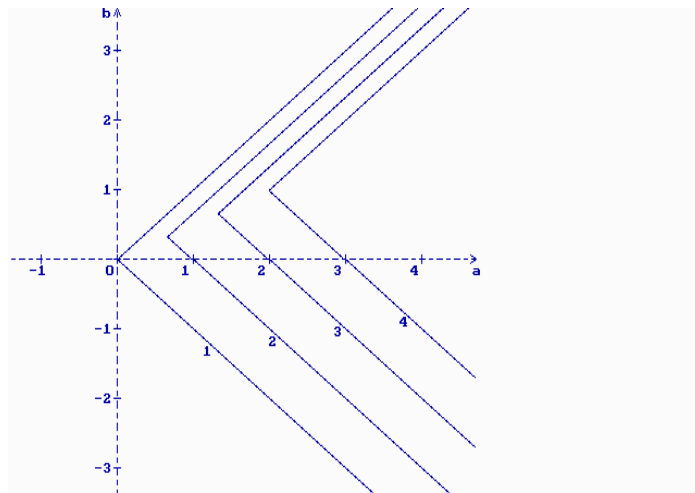
Figure 3.1
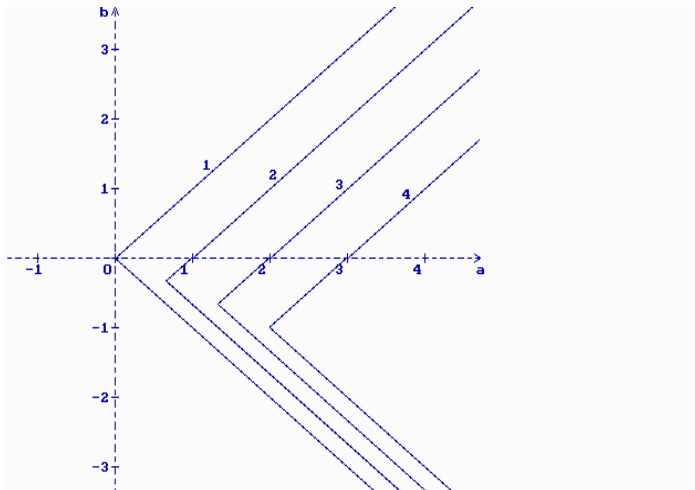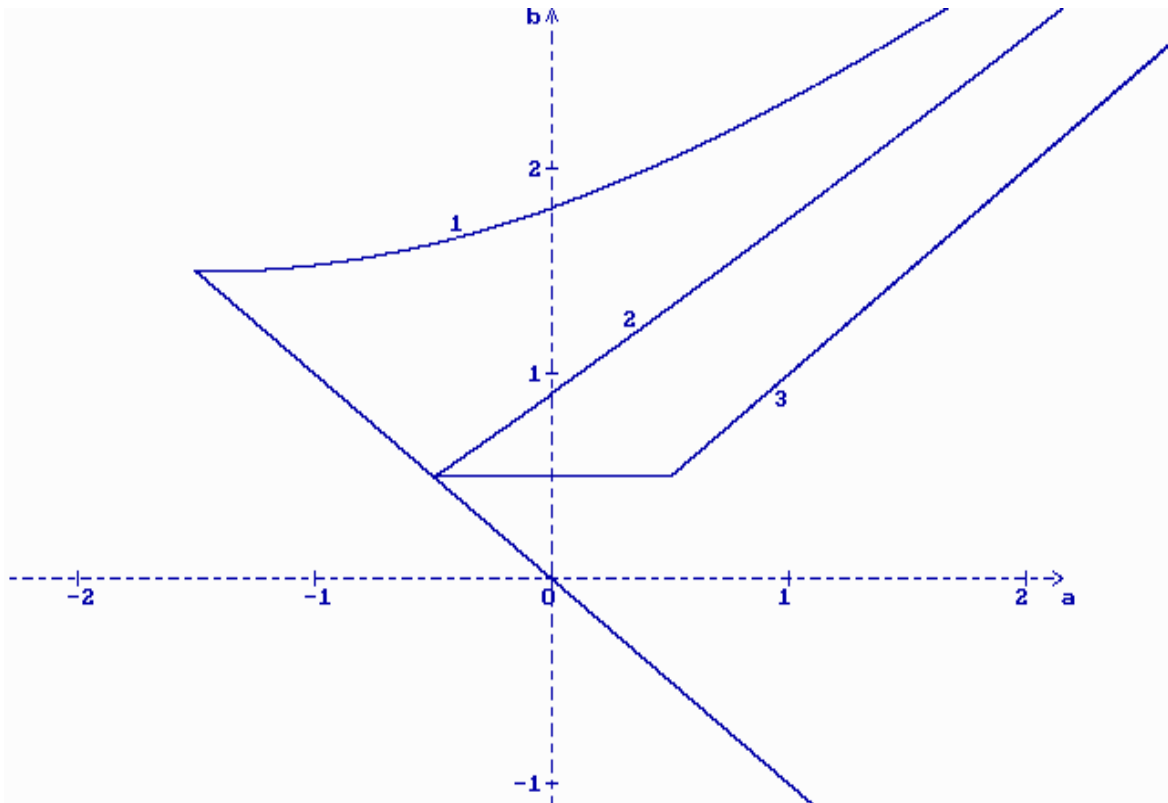

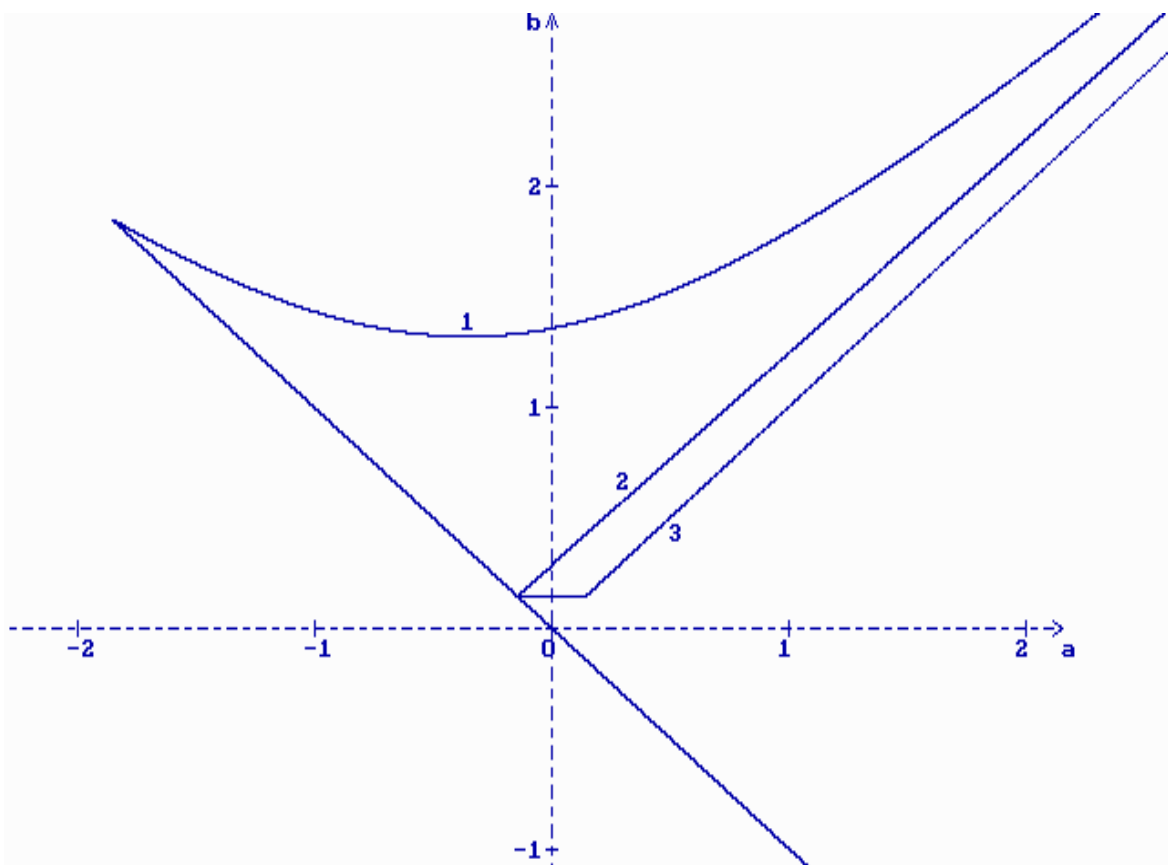Figure 3.2
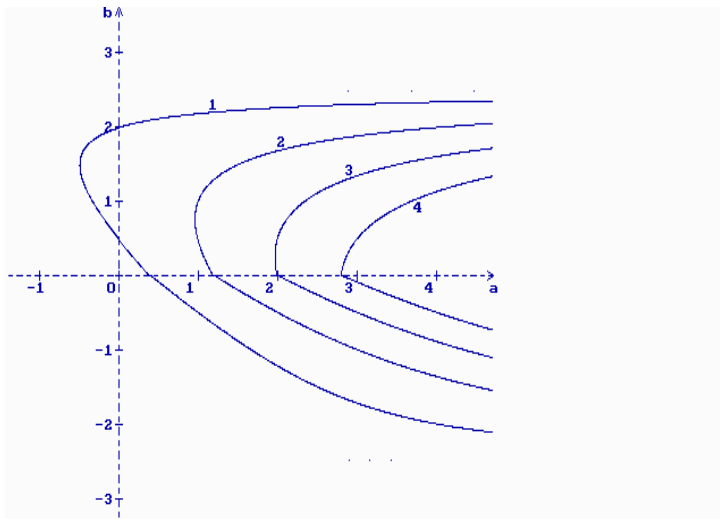
Figure 3.3
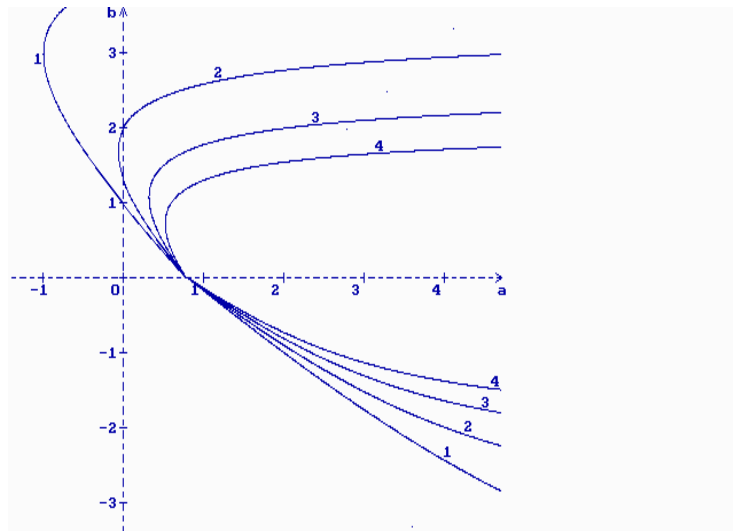
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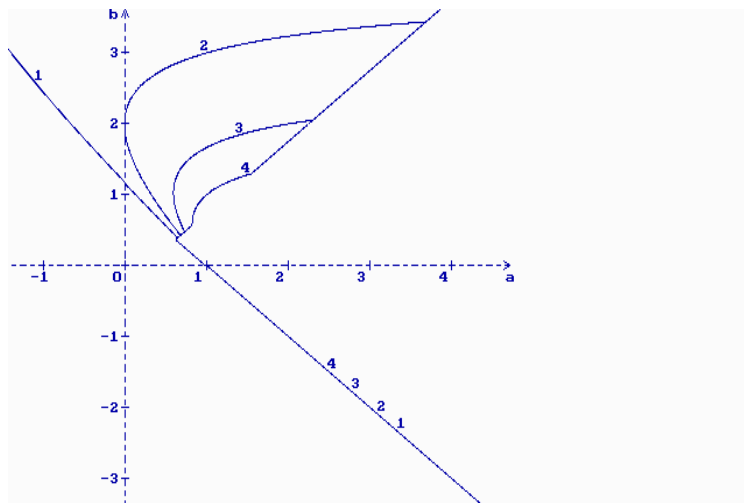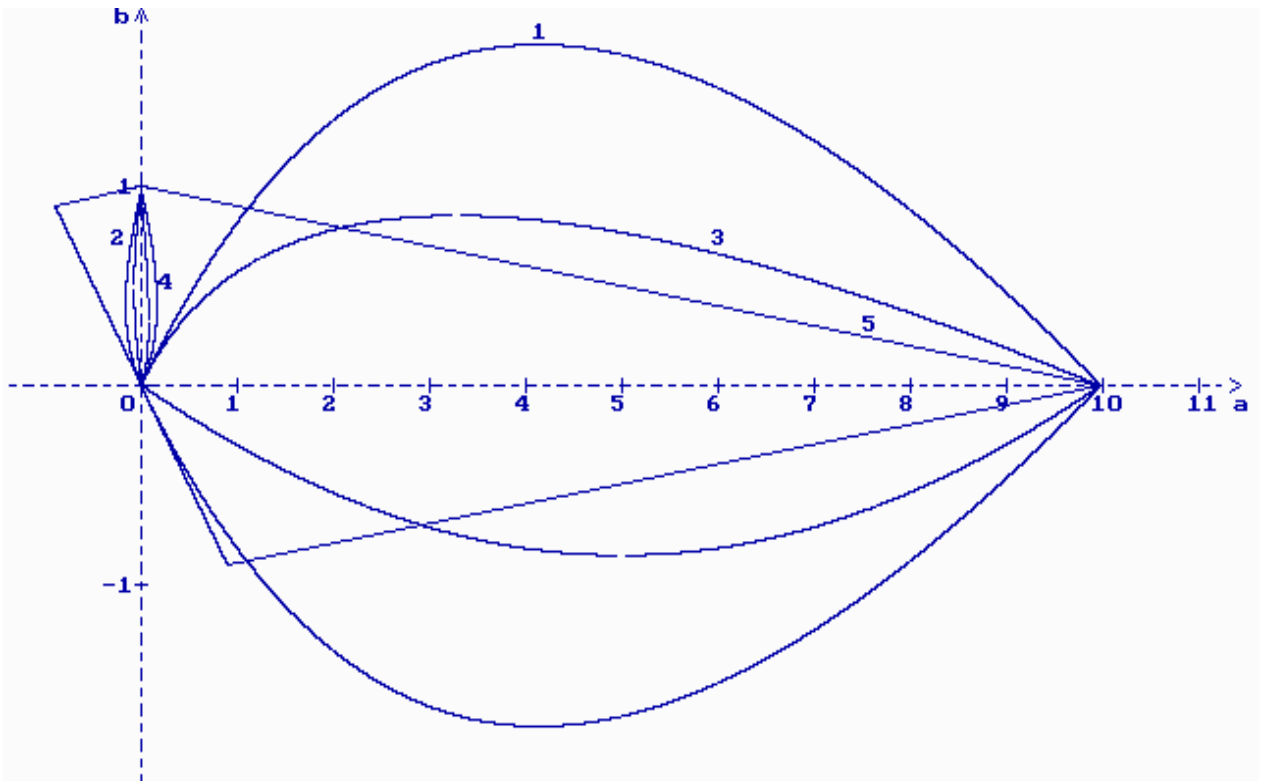
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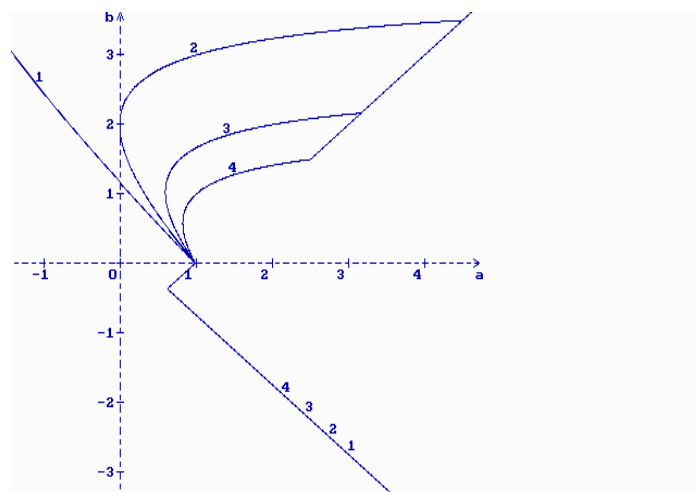
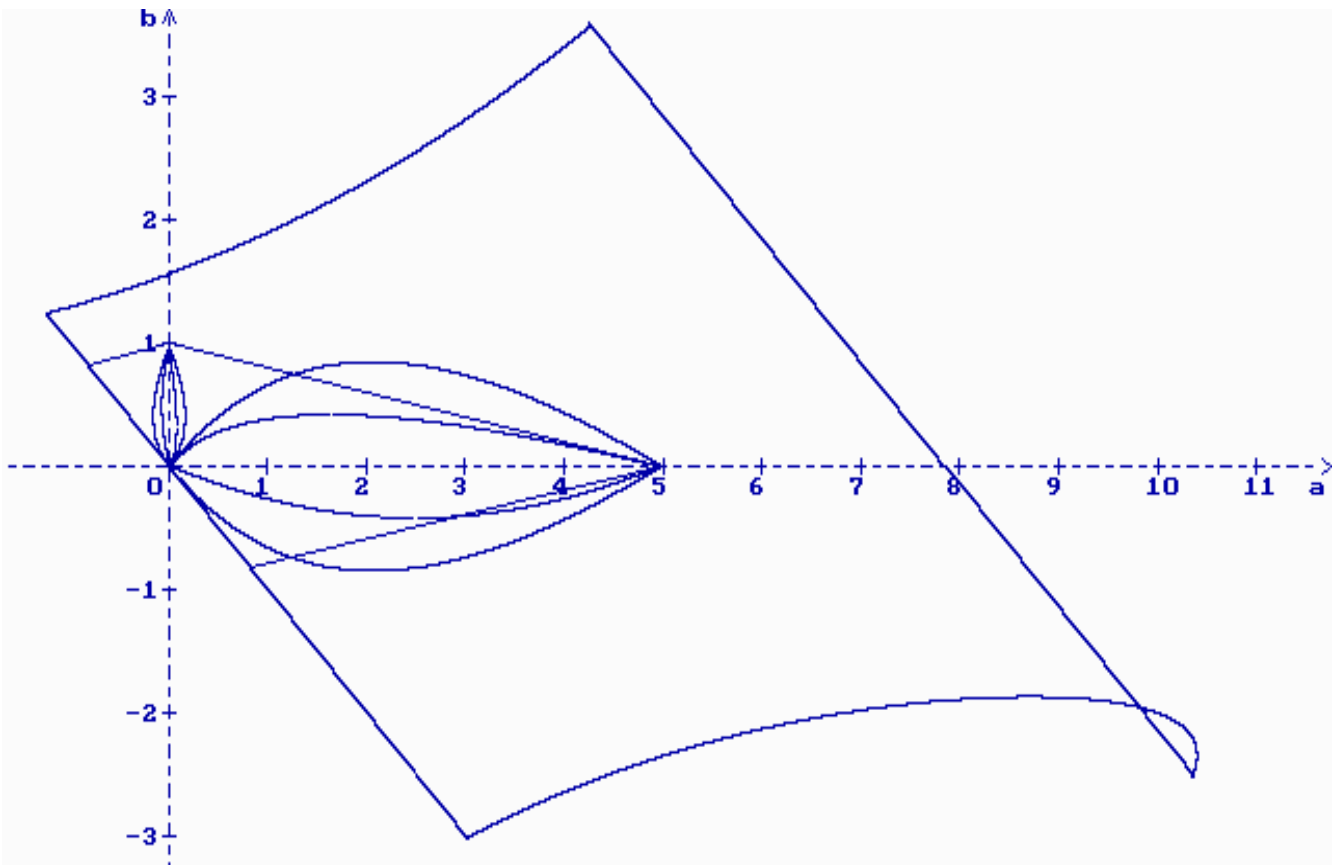Figure 3.6
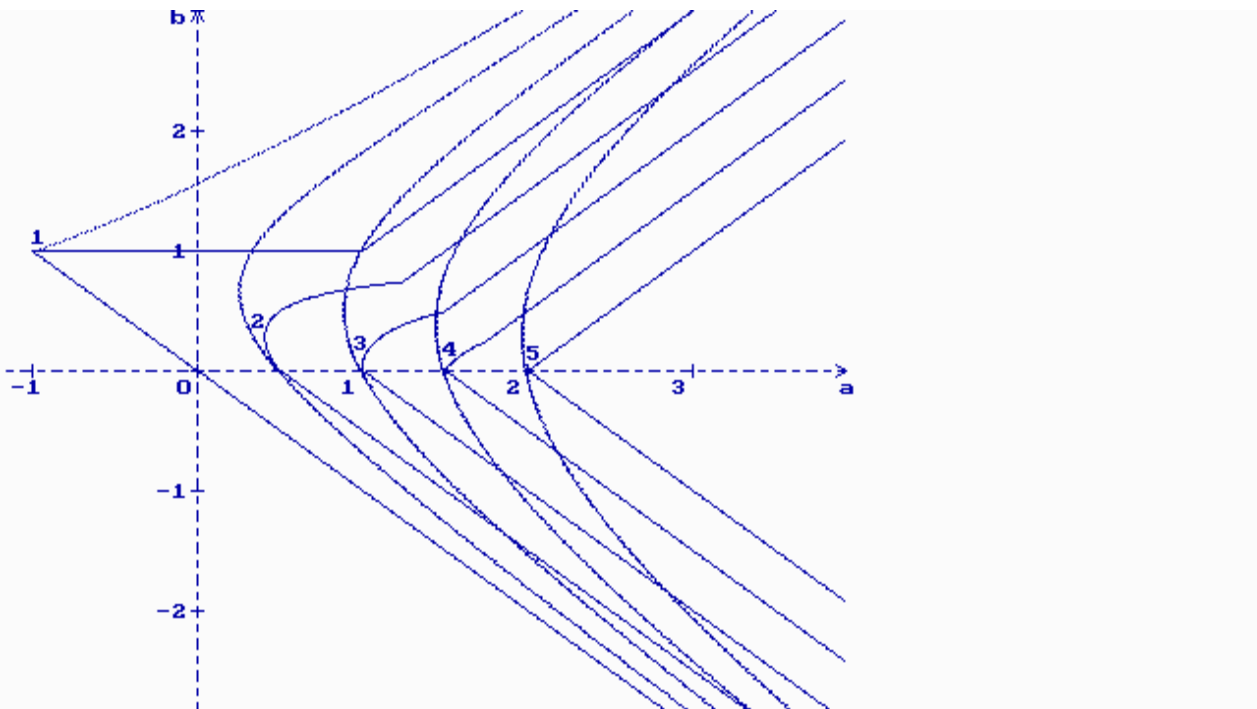
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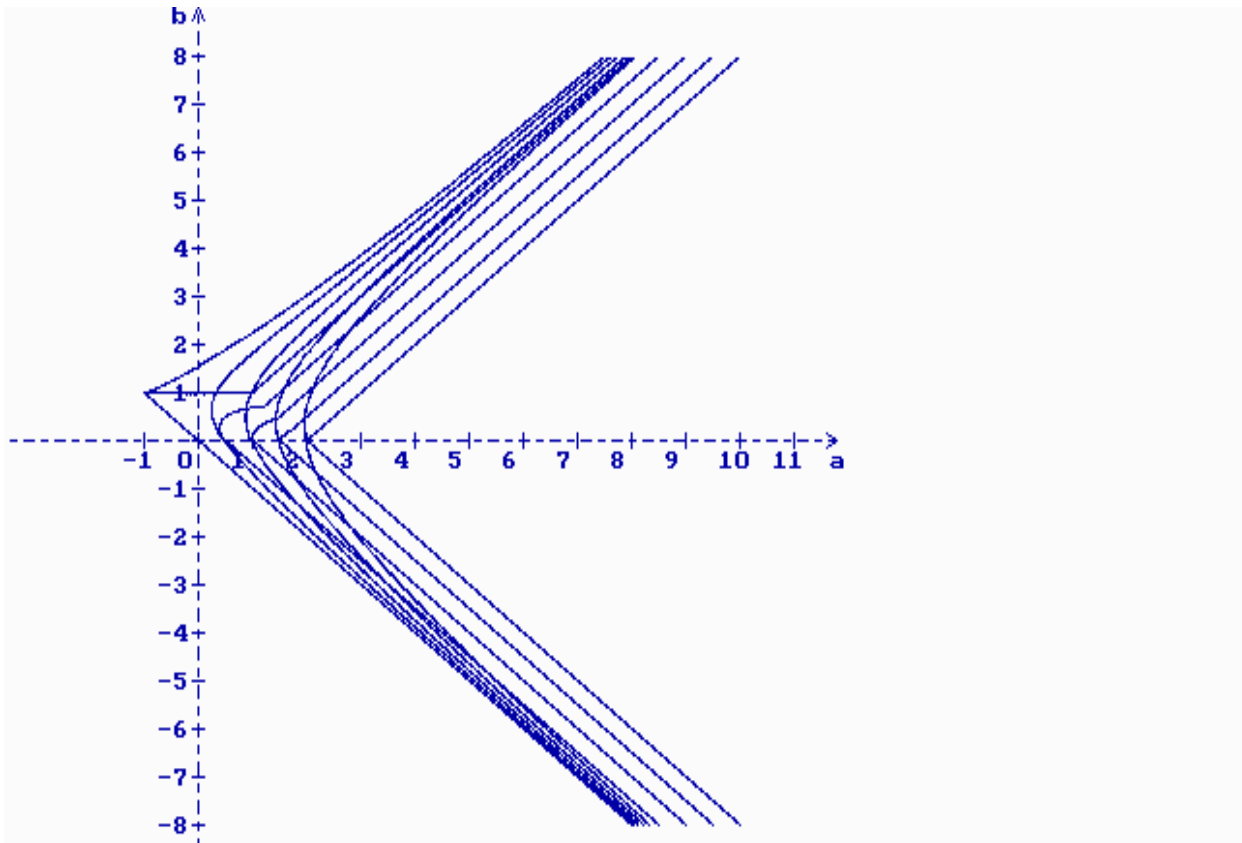
Figure 3.8

Figure 3.9

Figure 3.10

Figure 3.11

Figure 3.12

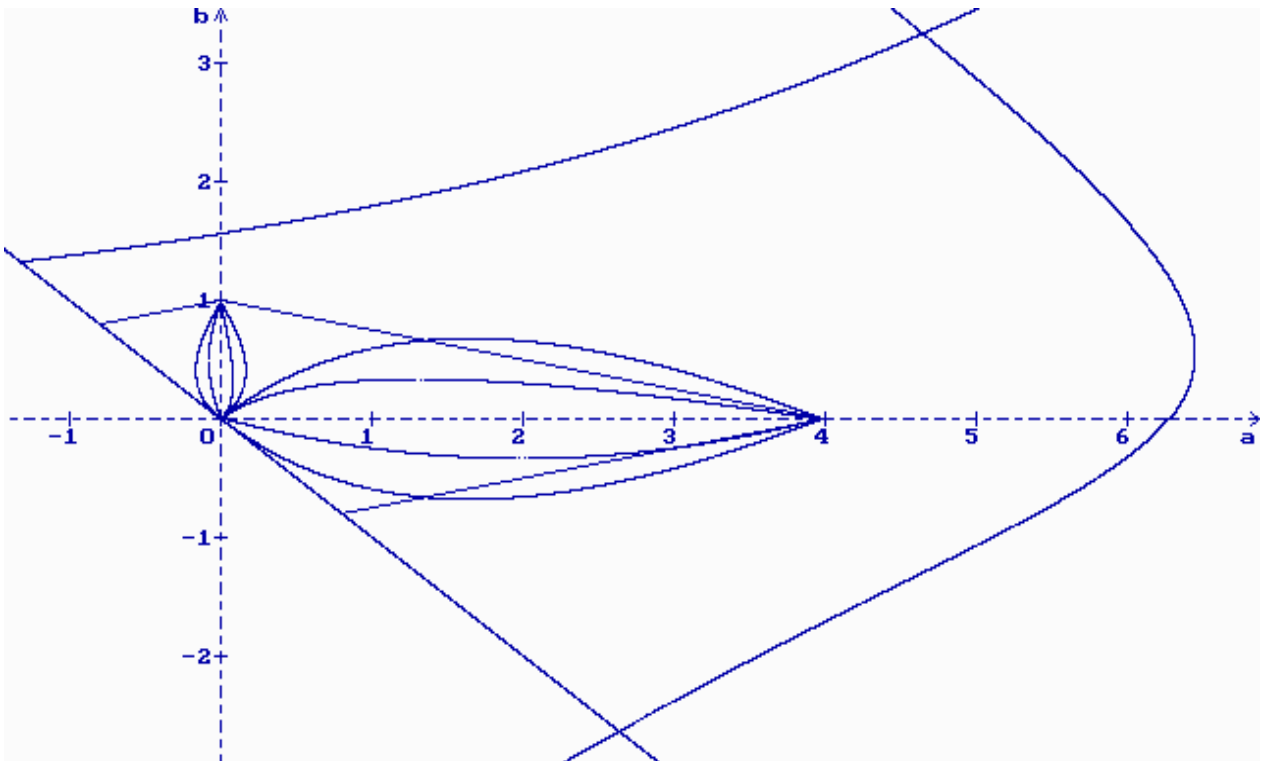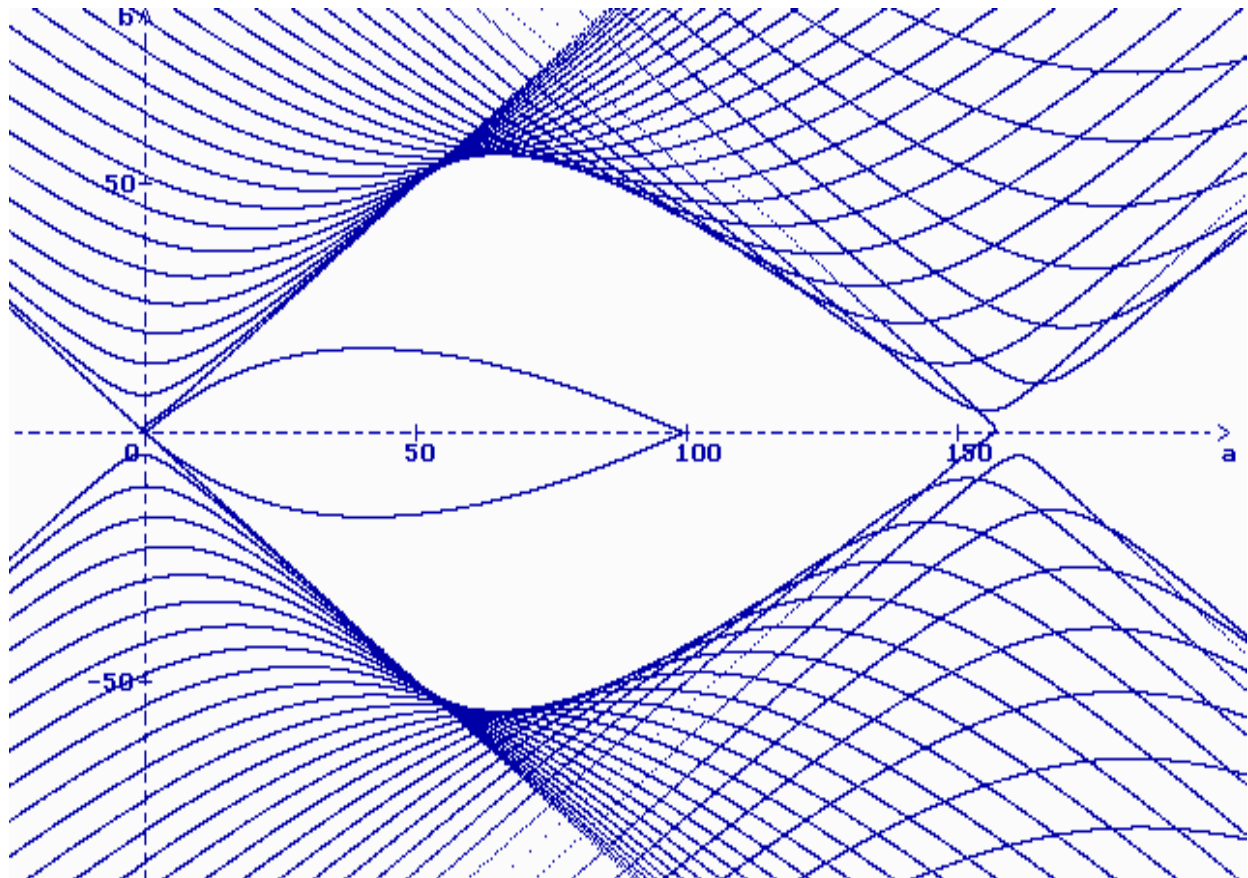Figure 3.13

Figure 3.14
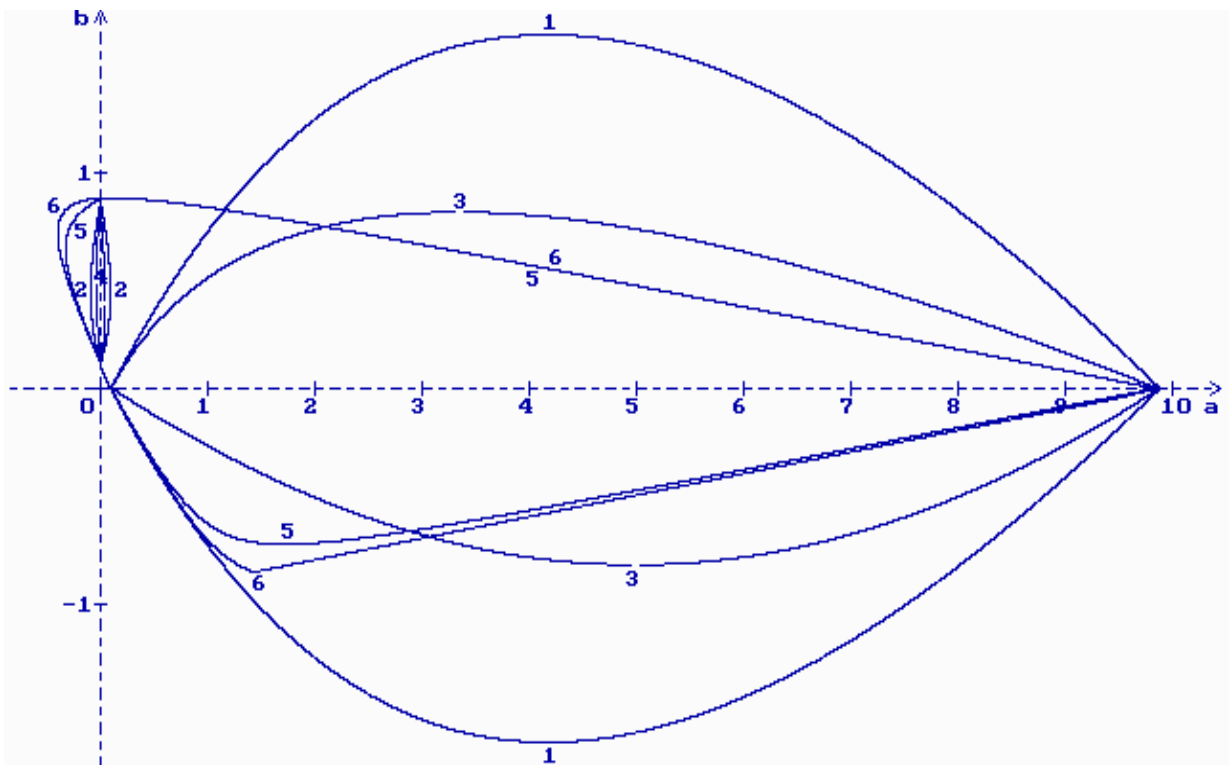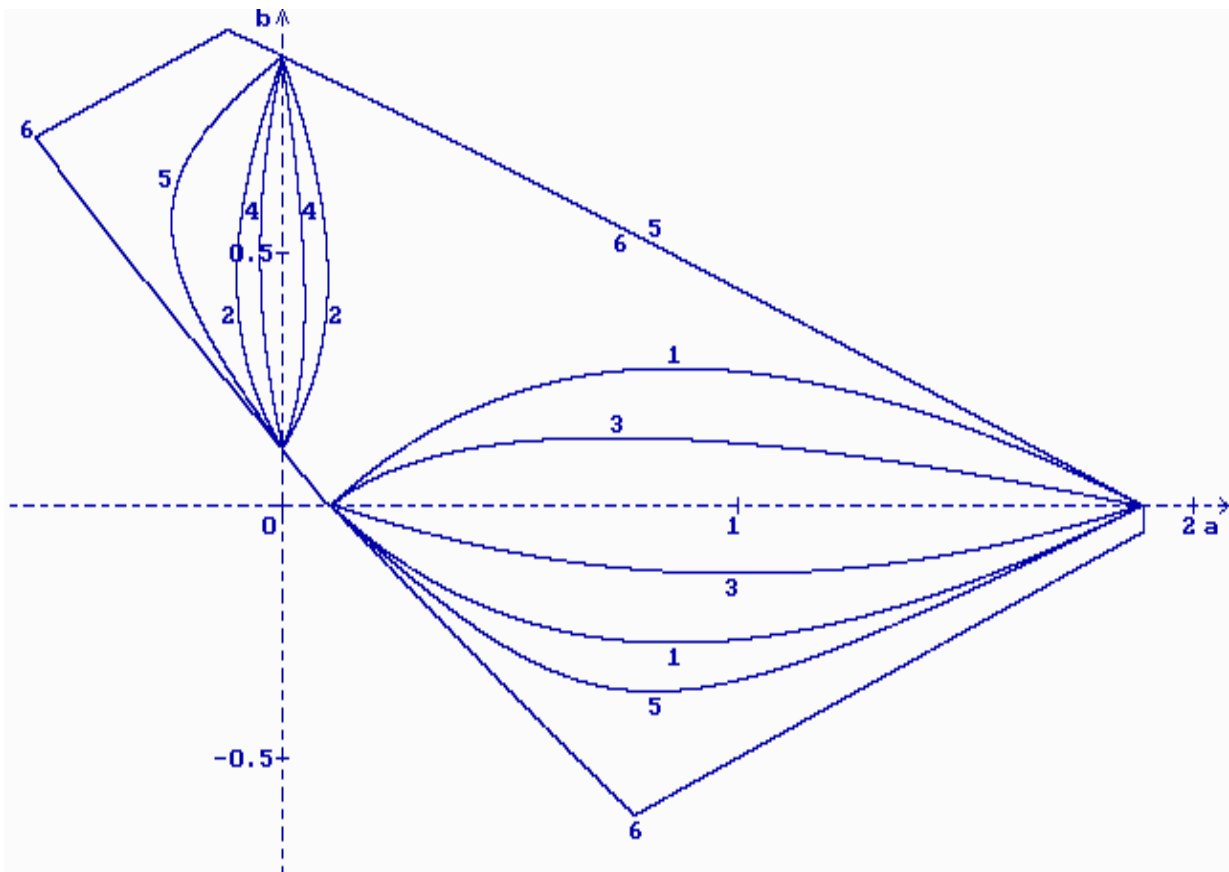
Figure 3.15

Figure 3.16
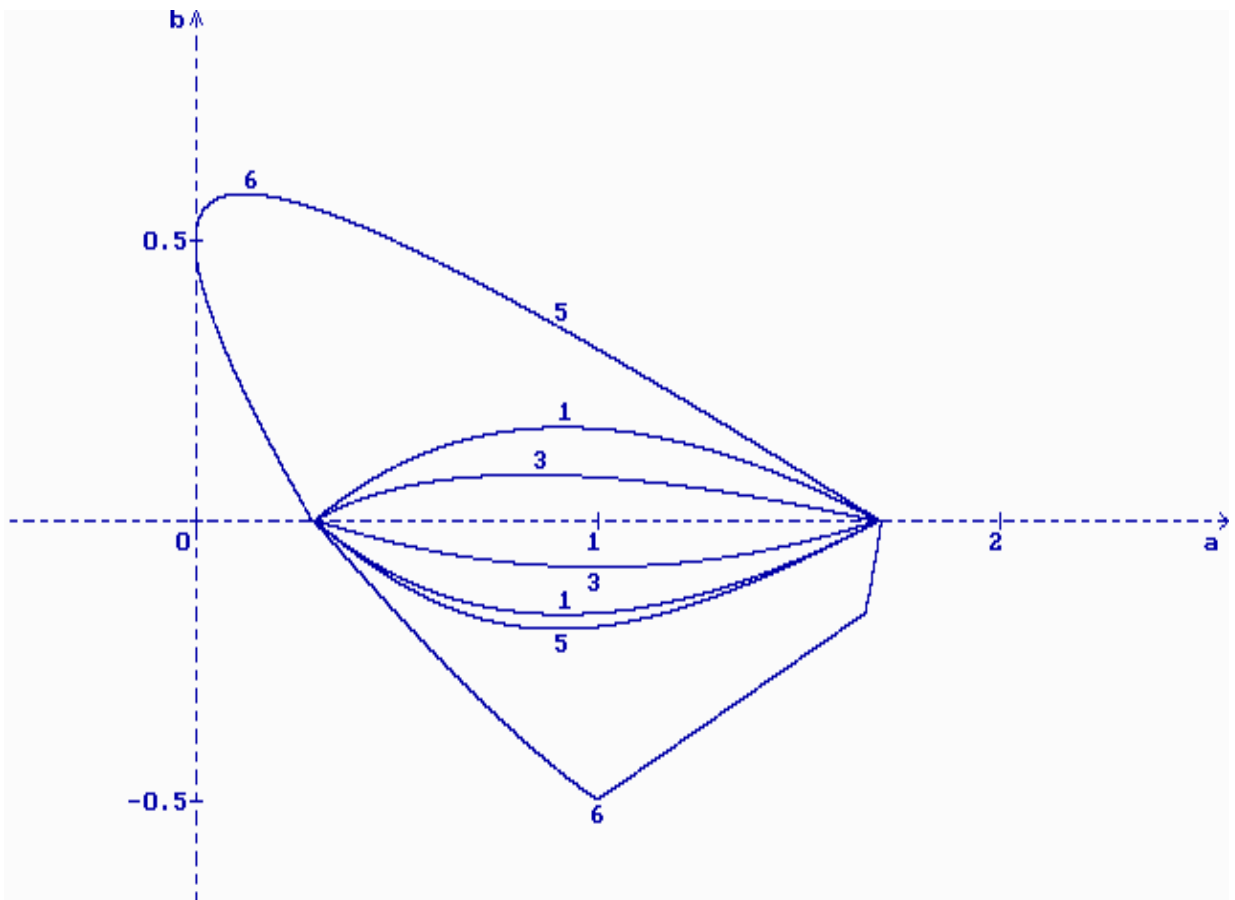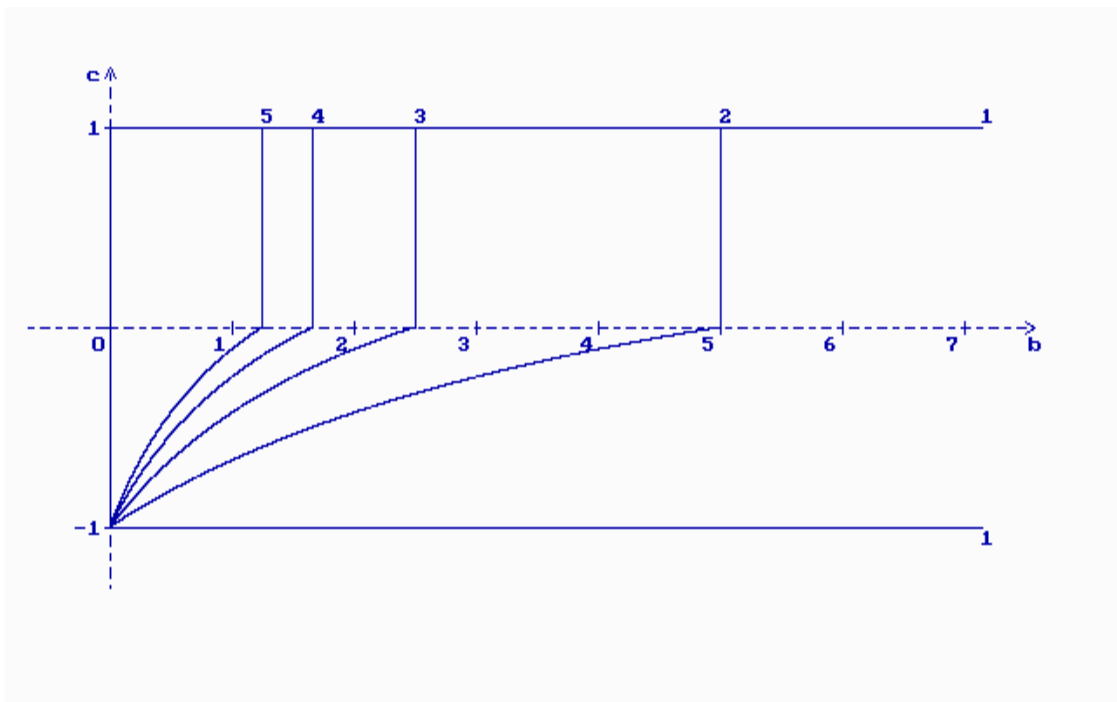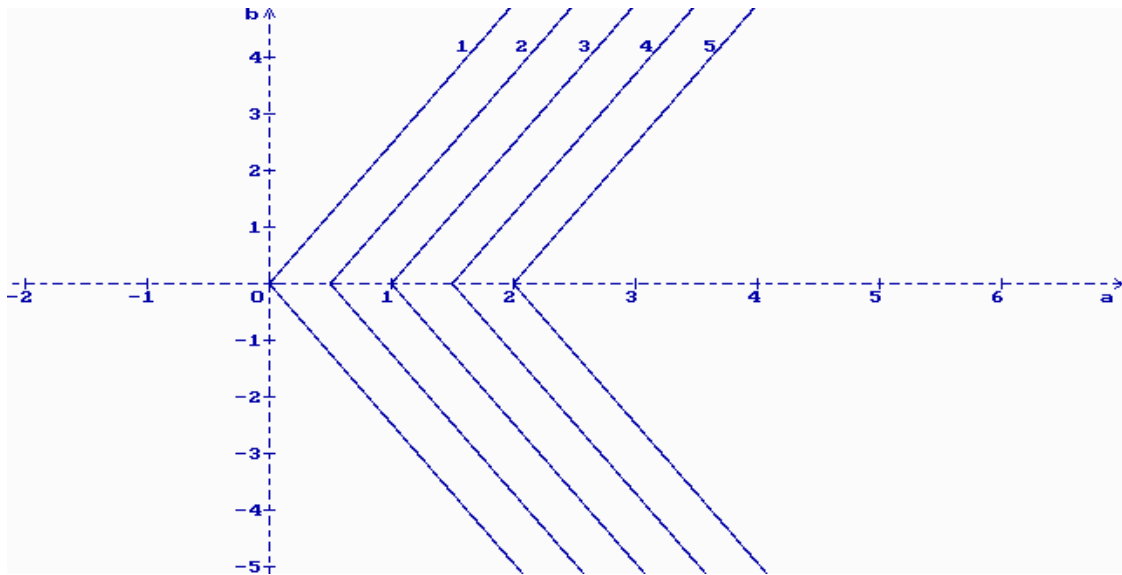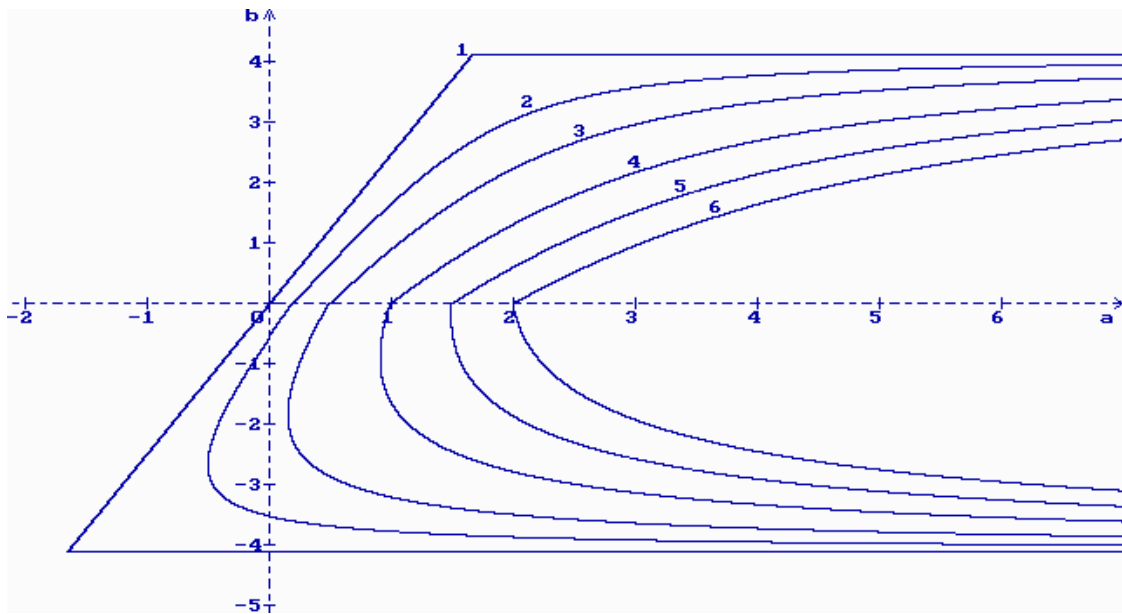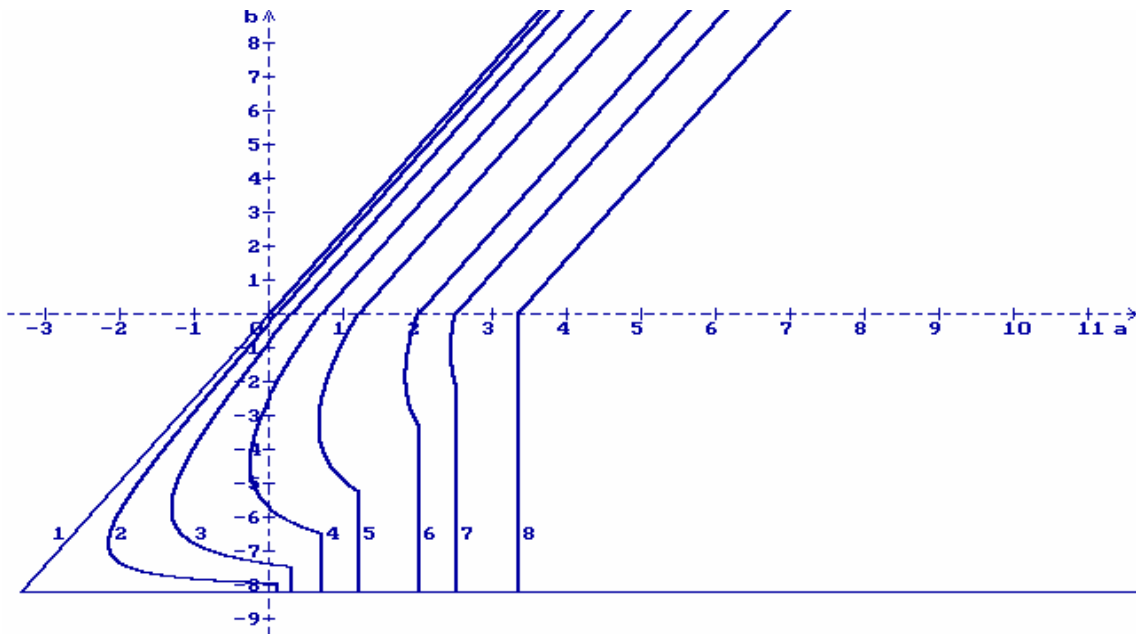
Figure 3.17

Figure 3.18

Figure 3.19

Figure 3.20

Figure 3.21

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