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Delay-Dependent Stability Criteria for Stochastic Differential Delay Equations with Markovian Switching

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Abstract

Recently Mao et al. [18] established a number of useful stability criteria in terms of M-matrices for nonlinear stochastic differential delay equations with Markovian switching, and the criteria there are independent of time delay. Such criteria are in general good for large delay but might not be good enough for small delay. When the time lag is sufficiently small, it is useful to obtain delay-dependent stability criteria and this is the aim of this paper.

Key words: Delay equation, generalized Itô's formula, Brownian motion, stability, Markov chain, M-matrix.

AMS (MOS) subject classification: 60H20, 34D08.

1 Introduction

Stochastic modelling has come to play an important role in many branches of science and industry. An area of particular interest has been the automatic control of stochastic systems, with consequent emphasis being placed on the analysis of stability in stochastic models, and the key books in this area are Arnold [1], Elworthy [4], Khasminskii [6], Kolmanovskii & Myshkis [7], Kolmanovskii & Shaikhet [9], Ladde & Lakshmikantham [13], Mao [14, 15, 16] and Mohammed [20]. Recently, the stability of stochastic systems with Markovian switching has received a great deal of attention. For example, Ji & Chizeck [5] and Mariton [19] studied the stability of a jump linear equation

$$\dot{x}(t) = A(r(t))x(t), \tag{1}$$

where $r(t)$ is a Markov chain taking values in $S = \{1, 2, \cdots, N\}$. Basak et al. [2] discussed the stability of a semi-linear stochastic differential equation with Markovian switching of the form

$$dx(t) = A(r(t))x(t)dt + g(x(t), r(t))dw(t), \tag{2}$$

while Mao [17] investigated the stability of a nonlinear equation

$$dx(t) = f(x(t), t, r(t))dt + g(x(t), t, r(t))dw(t). \tag{3}$$

Shaikhet [21] took the time delay into account and considered the stability of a semi-linear stochastic differential delay equation with Markovian switching, while Mao et al. [18] investigated the stability of a nonlinear delay equation

$$dx(t) = f(x(t), x(t-\tau), t, r(t))dt + g(x(t), x(t-\tau), t, r(t))dw(t). \tag{4}$$

The criteria obtained by Mao et al. [18] are all independent of time delay. Such criteria are in general good for large delay but might not be good enough for small delay. When the time lag is sufficiently small, we may write equation (4) as

$$\begin{aligned} dx(t) &= f(x(t), x(t), t, r(t))dt \\ &+ [f(x(t), x(t-\tau), t, r(t)) - f(x(t), x(t), t, r(t))]dt \\ &+ g(x(t), x(t-\tau), t, r(t))dw(t), \end{aligned} \tag{5}$$

and regard equation (4) as the perturbed system of the nonlinear jump equation (without delay)

$$\dot{x}(t) = f(x(t), x(t), t, r(t)), \tag{6}$$

where $[f(x(t), x(t-\tau), t, r(t)) - f(x(t), x(t), t, r(t))]dt$ and $g(x(t), x(t-\tau), t, r(t))dw(t)$ represent the perturbations due to time delay and noise, respectively. If we impose that equation (6) is stable and the stochastic perturbation $g(x(t), x(t-\tau), t, r(t))dw(t)$ is sufficiently small, we may then find the time lag τ sufficiently small for the delay perturbation $[f(x(t), x(t-\tau), t, r(t)) - f(x(t), x(t), t, r(t))]dt$ to be so small that the perturbed equation (4) remains stable. Such stability criteria, which are of course delay-dependent, will be extremely useful for systems with small time delay, and the main aim of this paper is to establish such criteria.

It should be pointed out that one may also write equation (4) as

$$\begin{aligned} dx(t) &= f(x(t), x(t), t, r(t))dt + g(x(t), x(t), t, r(t))dw(t) \\ &+ [f(x(t), x(t-\tau), t, r(t)) - f(x(t), x(t), t, r(t))]dt \\ &+ [g(x(t), x(t-\tau), t, r(t)) - g(x(t), x(t), t, r(t)]dw(t), \end{aligned} \tag{7}$$

and regard equation (4) as the perturbed system of the nonlinear jump stochastic differential equation

$$dx(t) = f(x(t), x(t), t, r(t))dt + g(x(t), x(t), t, r(t))dw(t). \tag{8}$$

Under condition that equation (8) is stable, one may show that if the time lag τ is sufficiently small, than the perturbation

$$[f(x(t), x(t-\tau), t, r(t)) - f(x(t), x(t), t, r(t))]dt + [g(x(t), x(t-\tau), t, r(t)) - g(x(t), x(t), t, r(t)]dw(t)$$

will be sufficiently small for the perturbed equation (4) to remain stable. This different approach will give us alternative delay-dependent stability criteria. In this paper we will concentrate on the approach above but the techniques developed here will certainly be applicable to this different approach.

2 Delay Equations with Markovian Switching

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $w(t) = (w_1(t), \cdots, w_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let τ_1 and τ_2 be both positive numbers and set $\tau = \tau_1 \vee \tau_2$. Let $C([-\tau, 0]; R^n)$ denote the family of continuous functions φ from $[-\tau, 0]$ to R^n with the norm $||\varphi|| = \sup_{-\tau\leq s\leq 0} |\varphi(s)|$, where $|\cdot|$ is the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^T A)}$ while its operator norm is denoted by $||A|| = \sup\{|Ax| : |x| = 1\}$ (without any confusion with $||\varphi||$). If A is a symmetric matrix, denote by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ its largest and smallest eigenvalue, respectively. Denote by $C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$ the family of all bounded, $\mathcal{F}_0$-measurable, $C([-\tau, 0]; R^n)$-valued random variables. If $x(t)$ is a continuous R^n-valued stochastic process on $t \in [-\tau, \infty)$, we let $x_t = \{x(t+s) : -\tau \leq s \leq 0\}$ for $t \geq 0$ which is regarded as a $C([-\tau, 0]; R^n)$-valued stochastic process.

Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while

$$\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. It is known that almost every sample path of $r(t)$ is a right-continuous step function with a finite number of simple jumps in any finite subinterval of R_+.

Consider a stochastic differential delay equation with Markovian switching of the form

$$dx(t) = f(x(t), x(t-\tau_1), t, r(t))dt + g(x(t), x(t-\tau_2), t, r(t))dw(t) \tag{9}$$

on $t \geq 0$ with initial data $x_0 = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$, where

$$f : R^n \times R^n \times R_+ \times S \to R^n \quad \text{and} \quad g : R^n \times R^n \times R_+ \times S \to R^{n\times m}.$$

Comparing (9) with (4), we note that here the time lags τ_1 and τ_2 in the shift coefficient f and the diffusion coefficient g may differ. The stability criteria obtained in this paper will be independent of τ_2 but require τ_1 be sufficiently small. The theory developed in this paper can cope with more complicated situation of time lags, e.g. the equation of the form

$$dx(t) = f(x(t), x(t-\tau_1), t, r(t))dt + \sum_{k=1}^{m} g_k(x(t), x(t-\tau_{1+k}), t, r(t))dw_k(t),$$

but we will concentrate on equation (9) to avoid notation becoming too complicated.

As a standing hypothesis, we assume that both f and g satisfy the local Lipschitz condition and the linear growth condition. In the same way as in Mao et al. [18] we can show that under this hypothesis, equation (9) has a unique continuous solution on $t \geq -\tau$, which is denoted by $x(t;\xi)$ in this paper. Moreover,

$$E\Big[\sup_{-\tau\leq s\leq t} |x(s;\xi)|^2\Big] < \infty \quad \text{on } t \geq 0. \tag{10}$$

Let $C^{2,1}(R^n \times R_+ \times S; R_+)$ denote the family of all nonnegative functions $V(x,t,i)$ on $R^n \times R_+ \times S$ which are continuously twice differentiable in x and once differentiable in t. If $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$, define an operator LV from $R^n \times R^n \times R^n \times R_+ \times S$ to R by

$$\begin{aligned} LV(x,y,z,t,i) &= V_t(x,t,i) + V_x(x,t,i)f(x,y,t,i) \\ &+\frac{1}{2}\text{trace}\big[g^T(x,z,t,i)V_{xx}(x,t,i)g(x,z,t,i)\big] + \sum_{j=1}^{N} \gamma_{ij} V(x,t,j), \end{aligned} \tag{11}$$

where

$$V_t(x,t,i) = \frac{\partial V(x,t,i)}{\partial t}, \qquad V_x(x,t,i) = \Big(\frac{\partial V(x,t,i)}{\partial x_1}, \cdots, \frac{\partial V(x,t,i)}{\partial x_n}\Big),$$

$$V_{xx}(x,t,i) = \Big(\frac{\partial^2 V(x,t,i)}{\partial x_i \partial x_j}\Big)_{n\times n}.$$

For the convenience of the reader we cite the generalized Itô formula (cf. Mao et al. [18] or Skorohod [22]): If $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$, then for any stopping times $0 \leq \rho_1 < \rho_2 < \infty$,

$$\begin{aligned} EV(x(\rho_2), \rho_2, r(\rho_2)) &= EV(x(\rho_1), \rho_1, r(\rho_1)) \\ + \quad E&\int_{\rho_1}^{\rho_2} LV(x(s), x(s-\tau_1), x(s-\tau_2), s, r(s))ds \end{aligned} \tag{12}$$

as long as the expectations of the integrals exist. Let us point out that in the sequel whenever we apply this generalized formula the expectations of integrals involved do exist due to property (10).

3 Exponential Stability

In this section we shall discuss the exponential stability in mean square for equation (9). We impose the following hypotheses:

(H1) For every $i \in S$, there are constants $\alpha_i \in R$ and $\beta_i, \gamma_i \geq 0$ such that

$$2x^T f(x,x,t,i)| \leq \alpha_i |x|^2$$

and

$$|g(x,z,t,i)|^2 \leq \beta_i |x|^2 + \delta_i |z|^2$$

for all $x, z \in R^n$ and $t \geq 0$.

(H2) There are three nonnegative constants K_1, K_2 and K_3 such that

$$|f(x,x,t,i) - f(x,y,t,i)|^2 \leq K_1 |x-y|^2$$

and

$$|f(x,y,t,i)|^2 \leq K_2 |x|^2 + K_3 |y|^2$$

for all $x, y \in R^n$, $t \geq 0$ and $i \in S$.

It is easy to see from these hypotheses that $f(0,0,t,i) \equiv 0$ and $g(0,0,t,i) \equiv 0$ so equation (9) admits a trivial solution $x(t;0) \equiv 0$. This trivial solution is said to be exponentially stable in mean square if

$$\limsup_{t\to\infty} \frac{1}{t} \log(E|x(t;\xi)|^2) < 0 \tag{13}$$

for any initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$.

Theorem 3.1 *Let hypotheses (H1) and (H2) hold. Set*

$$\check{q} = \max_{1\leq i\leq N} q_i, \quad \check{\beta} = \max_{1\leq i\leq N} \beta_i, \quad \check{\delta} = \max_{1\leq i\leq N} \delta_i. \tag{14}$$

Assume that there are positive constants $q_1, q_2, \cdots, q_N$ and θ such that

$$\lambda_1 > \lambda_2 \tag{15}$$

where

$$\lambda_1 = \min_{1\leq i\leq N} \Big(-[\alpha_i + \beta_i + \theta] q_i - \sum_{j=1}^N \gamma_{ij} q_j \Big), \quad \lambda_2 = \max_{1\leq i\leq N} \delta_i q_i. \tag{16}$$

Let

$$\tau^* = \frac{1}{2(K_2+K_3)} \Big(\sqrt{(\check{\beta}+\check{\delta})^2 + 2\theta(\lambda_1-\lambda_2)(K_2+K_3)/\check{q}K_1} - \check{\beta} - \check{\delta} \Big). \tag{17}$$

If the time lag $\tau_1 < \tau^*$ though the time lag τ_2 is arbitrary, then the trivial solution of equation (9) is exponentially stable in mean square.

Proof. Fix any initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$ and write $x(t;\xi) = x(t)$. We divide the whole proof into three steps.

Step 1. Define, for some $\varepsilon > 0$ sufficiently small,

$$V(x,t,i) = q_i e^{\varepsilon t} |x|^2 \quad \text{for } (x,t,i) \in R^n \times R_+ \times S.$$

Clearly, $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$. Moreover, the operator LV from $R^n \times R^n \times R^n \times R_+ \times S$ to R defined by (11) becomes

$$LV(x,y,z,t,i) = e^{\varepsilon t} \Big[\varepsilon q_i |x|^2 + 2q_i x^T f(x,y,t,i) + q_i |g(x,z,t,i)|^2 + \sum_{j=1}^N \gamma_{ij} q_j |x|^2 \Big]. \tag{18}$$

Using hypotheses (H1) and (H2) we derive

$$\begin{aligned}
& 2q_i x^T f(x,y,t,i) \\
= \; & 2q_i x^T f(x,x,t,i) + 2q_i x^T [f(x,y,t,i) - f(x,x,t,i)] \\
\leq \; & \alpha_i q_i |x|^2 + \theta q_i |x|^2 + \theta^{-1} q_i |f(x,y,t,i) - f(x,x,t,i)|^2 \\
\leq \; & (\alpha_i + \theta) q_i |x|^2 + \theta^{-1} \check{q} K_1 |x-y|^2
\end{aligned} \tag{19}$$

and

$$q_i|g(x,z,t,i)|^2 \le \beta_i q_i|x|^2 + \delta_i q_i|z|^2 \le \beta_i q_i|x|^2 + \lambda_2|z|^2,$$

where the elementary inequality $2ab \le \theta a^2 + \theta^{-1}b^2$ has been used. Substituting these into (18) yields that

$$LV(x,y,z,t,i) \le e^{\varepsilon t}\big[-(\lambda_1 - \varepsilon\check{q})|x|^2 + \lambda_2|z|^2 + \theta^{-1}\check{q}K_1|x-y|^2\big]. \tag{20}$$

Noting

$$EV(x(0),0,r(0)) \le \check{q}E|x(0)|^2 \le \check{q}E||\xi||^2 := C_1$$

we obtain, by the generalized Itô formula, that

$$\begin{aligned} e^{\varepsilon t}EV(x(t),t,r(t)) &\le C_1 - (\lambda_1 - \varepsilon\check{q})\int_0^t e^{\varepsilon s}E|x(s)|^2 ds \\ &+ \lambda_2\int_0^t e^{\varepsilon s}E|x(s-\tau_2)|^2 ds \\ &+ \theta^{-1}\check{q}K_1\int_0^t e^{\varepsilon s}E|x(s)-x(s-\tau_1)|^2 ds. \end{aligned} \tag{21}$$

Step 2. In this step we shall always let $t \ge \tau$. Compute

$$\begin{aligned} \int_0^t e^{\varepsilon s}E|x(s-\tau_2)|^2 ds &= \int_0^{\tau_2} e^{\varepsilon s}E|x(s-\tau_2)|^2 ds + e^{\varepsilon\tau_2}\int_{\tau_2}^t e^{\varepsilon(s-\tau_2)}E|x(s-\tau_2)|^2 ds \\ &\le E||\xi||^2\int_0^{\tau_2} e^{\varepsilon s}ds + e^{\varepsilon\tau_2}\int_0^{t-\tau_2} e^{\varepsilon s}E|x(s)|^2 ds \\ &\le \tau_2 e^{\varepsilon\tau_2}E||\xi||^2 + e^{\varepsilon\tau_2}\int_0^t e^{\varepsilon s}E|x(s)|^2 ds. \end{aligned} \tag{22}$$

Moreover, it is easy to show from equation (9) that

$$\begin{aligned} E|x(t)-x(t-\tau_1)|^2 &\le 2\tau_1 E\int_{t-\tau_1}^t |f(x(s),x(s-\tau_1),s,r(s))|^2 ds \\ &+ 2E\int_{t-\tau_1}^t |g(x(s),x(s-\tau_2),s,r(s))|^2 ds. \end{aligned}$$

By hypotheses (H1) and (H2), we have

$$\begin{aligned} &E|x(t)-x(t-\tau_1)|^2 \le 2(\tau_1 K_2 + \check{\beta})\int_{t-\tau_1}^t E|x(s)|^2 ds \\ &+ 2\tau_1 K_3\int_{t-\tau_1}^t E|x(s-\tau_1)|^2 ds + 2\check{\delta}\int_{t-\tau_1}^t E|x(s-\tau_2)|^2 ds. \end{aligned}$$

We further compute

$$\begin{aligned} &\int_0^t e^{\varepsilon s}E|x(s)-x(s-\tau_1)|^2 ds \\ = &\int_0^{\tau_1} e^{\varepsilon s}E|x(s)-x(s-\tau_1)|^2 ds + \int_{\tau_1}^t e^{\varepsilon s}E|x(s)-x(s-\tau_1)|^2 ds \\ \le &\ 2\tau_1 e^{\varepsilon\tau_1}(E||\xi||^2 + E||x_\tau||^2) + 2(\tau_1 K_2 + \check{\beta})\int_{\tau_1}^t e^{\varepsilon s}\Big(\int_{s-\tau_1}^s E|x(u)|^2 du\Big)ds \\ + &\ 2\tau_1 K_3\int_{\tau_1}^t e^{\varepsilon s}\Big(\int_{s-\tau_1}^s E|x(u-\tau_1)|^2 du\Big)ds \\ + &\ 2\check{\delta}\int_{\tau_1}^t e^{\varepsilon s}\Big(\int_{s-\tau_1}^s E|x(u-\tau_2)|^2 du\Big)ds. \end{aligned} \tag{23}$$

But, by changing the order of integrations, we can show that

$$\int_{\tau_1}^t e^{\varepsilon s}\Big(\int_{s-\tau_1}^s E|x(u)|^2 du\Big)ds \le \int_0^t E|x(u)|^2\Big(\int_u^{u+\tau_1} e^{\varepsilon s}ds\Big)du$$

$$\le \tau_1 e^{\varepsilon\tau_1}\int_0^t e^{\varepsilon u}E|x(u)|^2du,$$

and

$$\int_{\tau_1}^t e^{\varepsilon s}\Big(\int_{s-\tau_1}^s E|x(u-\tau_1)|^2du\Big)ds \le \int_0^t E|x(u-\tau_1)|^2\Big(\int_u^{u+\tau_1} e^{\varepsilon s}ds\Big)du$$
$$\le \tau_1 e^{\varepsilon\tau_1}\int_0^t e^{\varepsilon u}E|x(u-\tau_1)|^2du \le \tau_1^2 e^{2\varepsilon\tau_1}E||\xi||^2 + \tau_1 e^{2\varepsilon\tau_1}\int_0^t e^{\varepsilon u}E|x(u)|^2du,$$

and

$$\int_{\tau_1}^t e^{\varepsilon s}\Big(\int_{s-\tau_1}^s E|x(u-\tau_2)|^2du\Big)ds \le \int_0^t E|x(u-\tau_2)|^2\Big(\int_u^{u+\tau_1} e^{\varepsilon s}ds\Big)du$$
$$\le \tau_1 e^{\varepsilon\tau_1}\int_0^t e^{\varepsilon u}E|x(u-\tau_2)|^2du \le \tau_1\tau_2 e^{\varepsilon(\tau_1+\tau_2)}E||\xi||^2 + \tau_1 e^{\varepsilon(\tau_1+\tau_2)}\int_0^t e^{\varepsilon u}E|x(u)|^2du.$$

Substituting these into (23) gives

$$\begin{aligned}&\int_0^t e^{\varepsilon s}E|x(s)-x(s-\tau)|^2ds\\ \le\ & C_2 + 2\tau_1 e^{\varepsilon\tau_1}\Big[\tau_1 K_2 + \check\beta + \tau_1 K_3 e^{\varepsilon\tau_1} + \check\delta e^{\varepsilon\tau_2}\Big]\int_0^t e^{\varepsilon u}E|x(u)|^2du,\end{aligned} \tag{24}$$

where

$$C_2 := 2\tau_1 e^{\varepsilon\tau_1}(E||\xi||^2 + E||x_\tau||^2) + 2\tau_1 e^{\varepsilon\tau_1}E||\xi||^2\Big[K_3\tau_1^2 e^{\varepsilon\tau_1} + \check\delta\tau_2 e^{\varepsilon\tau_2}\Big].$$

Step 3. Substituting (22) and (24) into (21) yields

$$e^{\varepsilon t}EV(x(t),t,r(t)) \le C_3 - \lambda(\varepsilon,\tau_1)\int_0^t e^{\varepsilon s}E|x(s)|^2ds \tag{25}$$

for all $t\ge\tau$, where

$$C_3 = C_1 + \lambda_2\tau_2 e^{\varepsilon\tau_2}E||\xi||^2 + \theta^{-1}\check q K_1 C_2$$

and

$$\lambda(\varepsilon,\tau_1) = \lambda_1 - \varepsilon\check q - \lambda_2 e^{\varepsilon\tau_2} - 2\tau_1\theta^{-1}\check q K_1 e^{\varepsilon\tau_1}\Big[\tau_1 K_2 + \check\beta + \tau_1 K_3 e^{\varepsilon\tau_1} + \check\delta e^{\varepsilon\tau_2}\Big]. \tag{26}$$

Note that

$$\lambda(0,\tau_1) = \lambda_1 - \lambda_2 - 2\tau_1\theta^{-1}\check q K_1\Big[\tau_1(K_2+K_3) + \check\beta + \check\delta\Big]$$

whence, by (17),

$$\lambda(0,\tau^*) = 0.$$

By condition $\tau_1<\tau^*$, we see easily that $\lambda(0,\tau_1)>0$. We can therefore find an $\varepsilon>0$ sufficiently small for

$$\lambda(\varepsilon,\tau_1)\ge 0.$$

It then follows from (25) that

$$e^{\varepsilon t}EV(x(t),t,r(t)) \le C_3,\quad t\ge\tau.$$

Noting

$$EV(x(t),t,r(t)) \ge \hat q E|x(t)|^2,$$

where $\hat q = \min_{1\le i\le N} q_i > 0$, we obtain

$$E|x(t)|^2 \le \frac{C_3}{\hat q}e^{-\varepsilon t},\quad t\ge\tau.$$

Consequently

$$\limsup_{t\to\infty}\frac{1}{t}\log(E|x(t)|^2) \le -\varepsilon < 0.$$

In other words, the trivial solution of equation (9) is exponentially stable in mean square. The proof is therefore complete.

4 Criteria in Terms of M-matrices

The use of Theorem 3.1 is very much dependent of the existence of the $N+1$ positive numbers $q_1, q_2, \cdots, q_N$ and θ. In this section we shall establish some criteria, which can be verified easily, for the existence of such $N+1$ numbers and hence for the exponential stability in mean square.

These criteria will be described in terms of M-matrices. For the convenience of the reader, let us cite some useful results on M-matrices. For more detailed information please see Berman & Plemmons [3]. We will need a few more notations. Let B be a vector or matrix. By $B \geq 0$ we mean each element of B is nonnegative. By $B > 0$ we mean $B \geq 0$ and at least one element of B is positive. By $B \gg 0$ we mean all elements of B are positive. Let B_1 and B_2 be two vectors or matrices with same dimensions. We write $B_1 \geq B_2$, $B_1 > B_2$ and $B_1 \gg B_2$ if and only if $B_1 - B_2 \geq 0$, $B_1 - B_2 > 0$ and $B_1 - B_2 \gg 0$, respectively. Moreover, we also adopt here the traditional notation by letting

$$Z^{N\times N} = \{A = (a_{ij})_{N\times N} : a_{ij} \leq 0, \ i \neq j\}.$$

Definition 4.1 *A square matrix $A = (a_{ij})_{N\times N}$ is called a nonsingular M-matrix if A can be expressed in the form $A = sI - B$ with some $B \geq 0$ and $s > \rho(B)$, where I is the identity matrix and $\rho(B)$ the spectral radius of B.*

It is easy to see that a nonsingular M-matrix A has nonpositive off-diagonal and positive diagonal entries, that is

$$a_{ii} > 0 \text{ while } a_{ij} \leq 0, \ i \neq j.$$

In particular, $A \in Z^{N\times N}$. There are many conditions which are equivalent to the statement that A is a nonsingular M-matrix and we now cite some of them for the use of this paper.

Lemma 4.2 *If $A \in Z^{N\times N}$, then the following statements are equivalent:*

(1) A is a nonsingular M-matrix.

(2) A is semipositive; that is, there exists a column vector $x \gg 0$ in R^N such that $Ax \gg 0$.

(3) A is inverse-positive; that is, A^{-1} exists and $A^{-1} \geq 0$.

(4) All the leading principal minors of A are positive; that is

$$\begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} > 0 \quad \textit{for every } k = 1, 2, \cdots, N.$$

For any parameter $\theta \geq 0$, let us define a matrix

$$\mathcal{A}_\theta = \text{diag}(-(\alpha_1 + \beta_1 + \theta), \cdots, -(\alpha_N + \beta_N + \theta)) - \Gamma. \tag{27}$$

Clearly, $\mathcal{A}_\theta \in Z^{N\times N}$. Let us introduce $\vec{q} = (q_1, q_2, \cdots, q_N)^T$ and set

$$(b_1, b_2, \cdots, b_N) := \mathcal{A}_\theta \vec{q}.$$

Then λ_1 defined by (16) becomes $\lambda_1 = \min_{1\leq i\leq N} b_i$. By condition (15), we know $\lambda_1 > 0$ so

$$\mathcal{A}_\theta \vec{q} \gg 0.$$

By Lemma 4.2, we observe that $\mathcal{A}_\theta$ is a nonsingular M-matrix. Furthermore,

$$0 \ll \mathcal{A}_\theta \vec{q} = \mathcal{A}_0 \vec{q} - \theta \vec{q} \ll \mathcal{A}_0 \vec{q},$$

whence $\mathcal{A}_0$ is a nonsingular M-matrix. In other words, the existence of positive numbers $q_1, q_2, \cdots, q_N$ and θ such that $\lambda_1 > 0$ implies that $\mathcal{A}_0$ is a nonsingular M-matrix. We shall now show that the converse holds too.

Lemma 4.3 *There are positive numbers $q_1, q_2, \cdots, q_N$ and θ such that λ_1 defined by (16) is positive if and only if*

$$\mathcal{A}_0 = \text{diag}(-(\alpha_1 + \beta_1), \cdots, -(\alpha_N + \beta_N)) - \Gamma$$

is a nonsingular M-matrix.

Proof. The necessity has been shown above so we need only to show the sufficiency. If $\mathcal{A}_0$ is a nonsingular M-matrix, then, by lemma 4.2, there is a vector $\vec{q} \gg 0$ such that $\mathcal{A}_0\vec{q} \gg 0$. One can then find a number $\theta > 0$ sufficiently small for

$$(b_1, b_2, \cdots, b_N) := \mathcal{A}_\theta \vec{q} = \mathcal{A}_0\vec{q} - \theta\vec{q} \gg 0.$$

Thus $\lambda_1 = \min_{1\le i\le N} b_i > 0$. This completes the proof.

The following is a sufficient stability criterion in terms of M-matrix.

Theorem 4.4 *Let hypotheses (H1) and (H2) hold. Assume that $\mathcal{A}_0$ is a nonsingular M-matrix and*

$$\mathcal{A}_0^{-1}\vec{1} \ll (\delta_1^{-1}, \delta_2^{-1}, \cdots, \delta_N^{-1})^T, \tag{28}$$

where $\vec{1} = (1, 1, \cdots, 1)^T$ and we, as usual, set $\delta_i^{-1} = \infty$ if $\delta_i = 0$. Then there is a positive constant τ^* such that the trivial solution of equation (9) is exponentially stable in mean square provided $\tau_1 < \tau^*$. (The proof below provides a way to compute τ^*.)

Proof. By Lemma 4.2, there is a column vector $x = (x_1, x_2, \cdots, x_N)^T \gg 0$ such that

$$(b_1, b_2, \cdots, b_N)^T := \mathcal{A}_0 x \gg 0.$$

Note

$$\mathcal{A}_\theta = \mathcal{A}_0 - \theta I,$$

where I is the $N \times N$ identity matrix. Hence, for any $0 < \theta < \min_{1\le i\le N}(b_i/x_i)$,

$$\mathcal{A}_\theta x = \mathcal{A}_0 x - \theta x = (b_1 - \theta x_1, b_2 - \theta x_2, \cdots, b_N - \theta x_N)^T \gg 0,$$

which implies, by Lemma 4.2, that $\mathcal{A}_\theta$ is a nonsingular M-matrix. Note that for all $\theta > 0$ sufficiently small,

$$\mathcal{A}_\theta^{-1} = (\mathcal{A}_0 - \theta I)^{-1} = \mathcal{A}_0^{-1} \sum_{k=0}^{\infty} (\theta \mathcal{A}_0^{-1})^k.$$

Thus $\mathcal{A}_\theta^{-1} \to \mathcal{A}_0^{-1}$ as $\theta \to 0$. By (28), we can then find an $\theta > 0$ sufficiently small such that $\mathcal{A}_\theta$ is a nonsingular M-matrix and

$$\mathcal{A}_\theta^{-1}\vec{1} \ll (\delta_1^{-1}, \delta_2^{-1}, \cdots, \delta_N^{-1})^T. \tag{29}$$

Let

$$\vec{q} = (q_1, q_2, \cdots, q_N)^T = \mathcal{A}_\theta^{-1}\vec{1}.$$

By Lemma 4.2, $\mathcal{A}_\theta^{-1} > 0$ so each row of $\mathcal{A}_\theta^{-1}$ has all nonnegative elements and has at least one positive element, that is, each row > 0. Therefore, $\vec{q} \gg 0$. Now, $\mathcal{A}_\theta \vec{q} = \vec{1}$ so, by (16), $\lambda_1 = 1$. Moreover, by (29),

$$q_i < \delta_i^{-1}, \quad 1 \le i \le N.$$

Hence, by (16), $\lambda_2 < 1$. In other words, condition (15) of Theorem 3.1 is satisfied. We can therefore compute the positive number τ^* by (17) and conclude by Theorem 3.1 that the trivial solution of equation (9) is exponentially stable in mean square provided $\tau_1 < \tau^*$. The proof is complete.

5 Linear Delay Equations

In this section we shall consider the n-dimensional linear stochastic differential delay equation with Markovian switching of the form

$$\begin{aligned} dx(t) &= [A(r(t))x(t) + B(r(t))x(t-\tau_1)]dt \\ &+ \sum_{k=1}^{m} [C_k(r(t))x(t) + D_k(r(t))x(t-\tau_2)]dw_k(t) \end{aligned} \tag{30}$$

on $t \ge 0$ with initial data $x_0 = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$. We shall simply write

$$A(i) = A_i, \quad B(i) = B_i, \quad C_k(i) = C_{ki}, \quad D_k(i) = D_{ki},$$

which are all $n \times n$ matrices. If we define

$$f(x,y,t,i) = A_i x + B_i y \quad \text{and} \quad g(x,z,t,i) = (C_{1i}x + D_{1i}z, \cdots, C_{ki}x + D_{ki}z),$$

then equation (30) can be written as equation (9). Moreover, hypotheses (H1) and (H2) hold and the parameters there are specified as follows

$$\alpha_i = \lambda_{\min}(A_i + B_i + A_i^T + B_i^T), \qquad \beta_i = 2\sum_{k=1}^{m} ||C_{ki}||^2, \qquad \delta_i = 2\sum_{k=1}^{m} ||D_{ki}||^2, \tag{31}$$

and

$$K_1 = \max_{1\le i\le N} ||B_i||^2, \qquad K_2 = 2\max_{1\le i\le N} ||A_i||^2, \qquad K_3 = 2K_1. \tag{32}$$

Corollary 5.1 *Assume that*

$$\mathcal{A}_0 = \mathrm{diag}(-(\alpha_1+\beta_1), \cdots, -(\alpha_N+\beta_N)) - \Gamma$$

is a nonsingular M-matrix and

$$\mathcal{A}_0^{-1}\vec{1} \ll (\delta_1^{-1}, \delta_2^{-1}, \cdots, \delta_N^{-1})^T,$$

where α_i, β_i and δ_i are defined by (31). Then there is a positive constant τ^* such that the trivial solution of equation (30) is exponentially stable in mean square provided $\tau_1 < \tau^*$. This corollary follows from Theorem 4.4 directly.

6 Asymptotic Mean Square Stability

Consider the equation

$$dx(t) = [A(r(t))x(t) + Bx(t-h)]dt + g(t, x_t, r(t))dw(t), \tag{33}$$

In this section we will suppose that $x_t = x(t+s)$, $s \le 0$, and for every positive definite matrix P_i there exists matrix $G_i(s)$ such that

$$Tr[g^T(t,\varphi,i)P_i g(t,\varphi,i)] \le \int_0^\infty \varphi^T(-s)dG_i(s)\varphi(-s), \tag{34}$$

$$dG_i(s) \ge 0, \qquad dG(s) = \max_{i\in S} dG_i(s), \qquad G = \int_0^\infty dG(s).$$

Integral here is understood as a Stieltjes integral, the notations $dG_i(s) \ge 0$ and $dG(s) = \max_{i\in S} dG_i(s)$ are understood as $x^T dG_i(s)x \ge 0$, $x^T dG(s)x = \max_{i\in S} x^T dG_i(s)x$ for arbitrary $x \in R^n$.

Theorem 6.1 *Let $\|B\|h < 1$ and $\max_{i\in S} R_i < 0$, where*

$$R_i = (A(i)+B)^T P_i + P_i(A(i)+B) + \Lambda_i + G + \tau(\rho_i + \beta)I, \tag{35}$$

$$\Lambda_i = \sum_{j\ne i} \gamma_{ij}(P_j - P_i), \quad \rho_i = \|(A(i)+B)^T P_i B + \Lambda_i B\|, \quad \beta = \max_{i\in S}(\tau\|B^T\Lambda_i B\| + \rho_i). \tag{36}$$

Then the zero solution of equation (33) is asymptotically mean square stable.

Proof. Reduce equation (33) to the form of a stochastic differential neutral type equation

$$\frac{d}{dt}\Big(x(t) + \int_{t-h}^{t} Bx(s)ds\Big) = (A(r(t)) + B)x(t) + g(t, x_t, r(t))\dot{\xi}(t).$$

Using the general method of Lyapunov functionals construction (Kolmanovskii & Shaikhet [10, 11, 12]), let us construct Lyapunov functional for equation (33) in the form $V = V_1 + V_2$, where

$$V_1(t, x_t, i) = \Big(x(t) + \int_{t-h}^{t} Bx(s)ds\Big)^T P_i \Big(x(t) + \int_{t-h}^{t} Bx(s)ds\Big).$$

Calculating LV_1 we obtain

$$LV_1(t,x_t,i) = 2x^T(t)(A_i+B)^T P_i\Big(x(t)+\int_{t-h}^t Bx(s)ds\Big) + Tr[g^T(t,x_t,i)P_i g(t,x_t,i)]+$$

$$+\Big(x(t)+\int_{t-h}^t Bx(s)ds\Big)^T \Lambda_i\Big(x(t)+\int_{t-h}^t Bx(s)ds\Big) =$$

$$= 2x^T(t)(A_i+B)^T P_i x(t) + 2x^T(t)(A_i+B)^T P_i \int_{t-h}^t Bx(s)ds + Tr[g^T(t,x_t,i)P_i g(t,x_t,i)]+$$

$$+x^T(t)\Lambda_i x(t) + 2\Big(\int_{t-h}^t Bx(s)ds\Big)^T \Lambda_i x(t) + \Big(\int_{t-h}^t Bx(s)ds\Big)^T \Lambda_i\Big(\int_{t-h}^t Bx(s)ds\Big) =$$

$$= x^T(t)[(A_i+B)^T P_i + P_i(A_i+B) + \Lambda_i]x(t) + Tr[g^T(t,x_t,i)P_i g(t,x_t,i)]+$$

$$+2x^T(t)[(A_i+B)^T P_i B + \Lambda_i B]\int_{t-h}^t x(s)ds + \int_{t-h}^t x^T(s)dsB^T\Lambda_i B\int_{t-h}^t x(\tau)d\tau.$$

Using (34), (35) we have

$$LV_1(t,x_t,i) \le x^T(t)[(A_i+B)^T P_i + P_i(A_i+B) + \Lambda_i]x(t)+$$

$$+\int_0^\infty x^T(t-s)dG_i(s)x(t-s) + \|B^T\Lambda_i B\|\Big(\int_{t-h}^t |x(s)|ds\Big)^2 + \rho_i\int_{t-h}^t (|x(t)|^2+|x(s)|^2)ds \le$$

$$\le x^T(t)[(A(i)+B)^T P_i + P_i(A(i)+B) + \Lambda_i + h\rho_i I]x(t)+$$

$$+\int_0^\infty x^T(t-s)dG_i(s)x(t-s) + \beta\int_{t-h}^t |x(s)|^2 ds,$$

Putting

$$V_2(t,x_t) = \int_0^\infty \int_{t-s}^t x^T(\tau)dG(s)x(\tau) + \beta\int_{t-h}^t (s-t+h)|x(s)|^2 ds,$$

we obtain

$$LV_2(t,x_t) = x^T(t)Gx(t) + \beta h|x(t)|^2 - \int_0^\infty x^T(t-s)dG(s)x(t-s) - \beta\int_{t-h}^t |x(s)|^2 ds,$$

Therefore, using (35) and that matrices R_i are uniformly with respect to $i \in S$ negative definite, for the functional $V = V_1 + V_2$ we have

$$LV(t,x_t,i) \le x^T(t)R_i x(t) \le -c|x(t)|^2. \tag{37}$$

As it is shown in [8] for asymptotic stability of differential equations of neutral type from conditions (37) and $\|B\|h < 1$ it follows that the zero solution of equation (33) is asymptotically mean square stable. Theorem is proved.

7 Example

Consider the 1-dimensional linear stochastic differential delay equation with Markovian switching of the form

$$dx(t) = [a(r(t))x(t) + b(r(t))x(t-\tau_1)]dt + \sigma(r(t))x(t-\tau_2)dw(t) \tag{38}$$

on $t \ge 0$ with initial data $x_0 = \xi \in C^b_{\mathcal{F}_0}([-\tau,0];R)$, where $w(t)$ is a 1-dimensional Brownian motion and

$$a(i) = a_i, \quad b(i) = b_i, \quad \sigma(i) = \sigma_i,$$

are all real numbers. If we define

$$f(x,y,t,i) = a_i x + b_i y \quad \text{and} \quad g(x,z,t,i) = \sigma_i z,$$

then equation (38) can be written as equation (9). Moreover, hypotheses (H1) and (H2) hold and the parameters there are given by

$$\alpha_i = 2(\alpha_i + b_i), \qquad \beta_i = 0, \qquad \delta_i = \sigma_i^2, \tag{39}$$

and

$$K_1 = \max_{1\le i\le N} b_i^2, \qquad K_2 = 2\max_{1\le i\le N} a_i^2, \qquad K_3 = 2K_1. \tag{40}$$

If

$$\mathcal{A}_0 = \mathrm{diag}(-2(a_1+b_1), \cdots, -2(a_N+b_N)) - \Gamma \tag{41}$$

is a nonsingular M-matrix and

$$\mathcal{A}_0^{-1}\vec{1} \ll (\sigma_1^{-2}, \sigma_2^{-2}, \cdots, \sigma_N^{-2})^T, \tag{42}$$

then, by Theorem 4.4, there is a positive constant τ^* such that the trivial solution of equation (38) is exponentially stable in mean square provided $\tau_1 < \tau^*$.

Let us now consider an even simpler case of $N = 2$ i.e. $S = \{1, 2\}$. In this case,

$$\Gamma = \begin{bmatrix} -\gamma_{12} & \gamma_{12} \\ \gamma_{21} & -\gamma_{21} \end{bmatrix}$$

and

$$\mathcal{A}_0 = \begin{bmatrix} -2(a_1+b_1)+\gamma_{12} & -\gamma_{12} \\ -\gamma_{21} & -2(a_2+b_2)+\gamma_{21} \end{bmatrix}.$$

By Lemma 4.2, this $\mathcal{A}_0$ is a nonsingular M-matrix if and only if

$$-2(a_1+b_1)+\gamma_{12} > 0 \tag{43}$$

and

$$\Delta := [-2(a_1+b_1)+\gamma_{12}][-2(a_2+b_2)+\gamma_{21}] - \gamma_{12}\gamma_{21} > 0. \tag{44}$$

Note

$$\mathcal{A}_0^{-1} = \frac{1}{\Delta}\begin{bmatrix} -2(a_2+b_2)+\gamma_{21} & \gamma_{12} \\ \gamma_{21} & -2(a_1+b_1)+\gamma_{12} \end{bmatrix}.$$

So (42) becomes

$$-2(a_2+b_2)+\gamma_{12}+\gamma_{21} < \frac{\Delta}{\sigma_1^2} \quad \text{and} \quad -2(a_1+b_1)+\gamma_{12}+\gamma_{21} < \frac{\Delta}{\sigma_2^2}. \tag{45}$$

It is not difficult to see that (43)-(45) are equivalent to that

$$0 < -2(a_2+b_2)+\gamma_{21} < \frac{\Delta}{\sigma_1^2} - \gamma_{12} \quad \text{and} \quad 0 < -2(a_1+b_1)+\gamma_{12} < \frac{\Delta}{\sigma_2^2} - \gamma_{21}. \tag{46}$$

In other words, we have shown that conditions (41) and (42) hold if and only if (46) holds. We can therefore conclude that in the case of $N = 2$, if (46) holds, then there is a positive constant τ^* such that the trivial solution of equation (38) is exponentially stable in mean square provided $\tau_1 < \tau^*$.

To show how to compute τ^*, let us furthermore specify the system parameters

$$a_1 = -1, \quad a_2 = 1, \quad b_1 = b_2 = -1, \quad \sigma_1 = \sigma_2 = 0.1, \quad \gamma_{12} = 1, \quad \gamma_{21} = 5. \tag{47}$$

It is easy to verify (46). Note that

$$\mathcal{A}_\theta = \mathcal{A}_0 - \theta I = \begin{bmatrix} 5-\theta & -1 \\ -5 & 5-\theta \end{bmatrix}$$

is a nonsingular M-matrix as long as $0 < \theta < 5 - \sqrt{5}$. In view of the proof of Theorem 4.4 we shall look for an θ such that $\mathcal{A}_\theta^{-1}\vec{1} \ll (\sigma_1^{-2}, \sigma_2^{-2})^T$, namely

$$\frac{1}{(5-\theta)^2 - 5}\begin{bmatrix} 6-\theta \\ 10-\theta \end{bmatrix} \ll \begin{bmatrix} 100 \\ 100 \end{bmatrix}.$$

This holds as long as $\theta < 2.7477$. From the proof of Theorem 4.4, we let $\vec{q} = (q_1, q_2)^T = \mathcal{A}_\theta^{-1}\vec{1}$, that is

$$q_1 = \frac{6-\theta}{(5-\theta)^2 - 5} \quad \text{and} \quad q_2 = \frac{10-\theta}{(5-\theta)^2-5}.$$

By (16), $\lambda_1 = 1$ while

$$\lambda_2 = \max\{q_1\sigma_1^2,\ q_2\sigma_2^2\} = \frac{0.01(10-\theta)}{(5-\theta)^2-5}.$$

Consequently, by (14),

$$\check{q} = \frac{10-\theta}{(5-\theta)^2-5}, \quad \check{\beta} = 0, \quad \check{\delta} = 0.01$$

while, by (40),

$$K_1 = 1, \quad K_2 = 2, \quad K_3 = 2.$$

In view of formula (17), to make τ^* as large as possible we should chose $\theta \in (0,\ 2.7477)$ to let

$$\frac{\theta(\lambda_1-\lambda_2)}{\check{q}} = \frac{\theta[(5-\theta)^2 - 5 - 0.01(10-\theta)]}{10-\theta}$$

as big as possible. It is not difficult to show that the best choice is $\theta = 1.32$ and the corresponding

$$\frac{\theta(\lambda_1-\lambda_2)}{\check{q}} = 1.2858.$$

By (17),

$$\tau^* = \frac{1}{8}\Big(\sqrt{0.01^2 + 8\times 1.2858} - 0.01\Big) = 0.3996.$$

We can therefore conclude that given (47) and $N = 2$, the trivial solution of equation (38) is exponentially stable in mean square provided $\tau_1 < 0.3996$.

It is interesting to point out that there is a much simpler way to chose θ which gives a reasonably good result for τ^*. Note that $\theta(\lambda_1-\lambda_2)/\check{q} = 0$ at both $\theta = 0$ and 2.7477. It is therefore reasonable to chose $\theta = 2.7477/2 = 1.37385$. This leads $\theta(\lambda_1-\lambda_2)/\check{q} = 1.2841$ and hence

$$\tau^* = \frac{1}{8}\Big(\sqrt{0.01^2 + 8\times 1.2841} - 0.01\Big) = 0.3994,$$

which is very closed to 0.3996 above.

Let us obtain maximum value of τ_1 using conditions of asymptotic mean square stability (35), (36). Consider equation (33) and suppose that Markov chain $r(t)$ has two states, $S = \{1,2\}$, $a(1) = a_1$, $a(2) = a_2$, $b(r(t)) = b$, $\sigma(r(t)) = \sigma$. In this case the stability conditions (35), (36) have the form

$$2(a_i+b)p_i + \lambda_i + p_2\sigma^2 + \tau_1(\rho_i+\beta) < 0, \quad i = 1,2, \tag{48}$$

where $p_2 > p_1$,

$$\lambda_i = \gamma_{ij}(p_j - p_i), \quad j \neq i, \qquad \rho_i = |(a_i+b)p_i + \lambda_i||b|, \tag{49}$$

$$\beta_i = \tau_1 b^2|\lambda_i| + \rho_i, \qquad \beta = \max_{i\in S}\beta_i. \tag{50}$$

In corresponding with (47) put

$$a_1 = -1, \quad a_2 = 1, \quad b = -1, \quad \sigma = 0.1, \quad \gamma_{12} = 1, \quad \gamma_{21} = 5. \tag{51}$$

Choose also $p_2 = \alpha p_1$, $1.5 < \alpha < 3$. From here and (49), (50) it follows

$$\rho_1 = (3-\alpha)p_1, \quad \rho_2 = 5(\alpha-1)p_1,$$

$$\beta = \beta_2 = 5(\alpha-1)(\tau_1+1)p_1 > \beta_1 = [\tau_1(\alpha-1) + 3 - \alpha]p_1.$$

Therefore condition (48) takes the form

$$5(\alpha-1)\tau_1(\tau_1+1) + \tau_1(3-\alpha) + 1.01\alpha < 5,$$

$$5(\alpha-1)\tau_1(\tau_1+1) + 5\tau_1(\alpha-1) + 0.01\alpha < 5(\alpha-1).$$

Choosing $\alpha = 1.89$ it is easy to get that both of inequalities hold if $\tau_1 < 0.4127$. It means that if $\tau_1 < 0.4127$ then by conditions (51) the zero solution of equation (33) is asymptotic mean square stable.

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