Neural, Parallel & Scientific Computations 10 (2002) 199-208

NUMERICAL SIMULATION AND STABILITY OF STOCHASTIC SYSTEMS WITH MARKOVIAN SWITCHING

Leonid SHAIKHET
Department of Mathematics, Informatics and Computing,
Donetsk State Academy of Management,
Chelyuskintsev 163-a, Donetsk 83015, Ukraine

ABSTRACT: A numerical procedure for investigation of stability of stochastic systems with Markovian switching is proposed. The procedure can be used in the cases when analitical conditions of stability are absent. Some examples of the proposed procedure using are considered. Results of calculations are presented by a quantity of figures.

AMS (MOS) Subject Classification. 50A10, 50B20

Introduction

Investigation of systems with Markovian switching (or in more general sense: systems with stochastic structure) was begun more than forty years ago in the work [1] which was very conceptually rich one and the last time it received a great deal of attention again [2-8]. Taking into account that it is enough difficult in each case to get analitical conditions of stability it is very interesting to have numerical methods of stability investigation. One of such methods is considered here.

1. Linear equation of first order

Consider the scalar differential equation with delay

$$\begin{aligned} \dot{x}(t) &= \eta(t)x(t) + bx(t-h), \\ x(s) &= \varphi(s), \quad s \in [-h, 0]. \end{aligned} \tag{1.1}$$

Here $\eta(t)$ is a Markov chain with two states $\{a_1, a_2\}$, an initial distribution

$$P_i = \mathbf{P}\{\eta(0) = a_i\}, \quad i = 1, 2, \tag{1.2}$$

and probabilities of transition $P_{ij}(\Delta)$, which have the form

$$\begin{gathered} P_{ij}(\Delta) = \mathbf{P}\{\eta(t+\Delta) = a_j/\eta(t) = a_i\} = \lambda_{ij}\Delta + o(\Delta), \\ i, j = 1, 2, \quad i \neq j. \end{gathered} \tag{1.3}$$

It is supposed also that

$$a_2 < 0, \qquad a_1 > |a_2| > b > 0, \qquad \lambda_{12} > \lambda_{21} > 0. \tag{1.4}$$

Statement 1.1. *Let $bh < 1$ and there exist positive numbers p_1, p_2, such that*

$$2(a_i + b)p_i + \lambda_{ij}(p_j - p_i) + h(\rho_i + \beta) < 0, \tag{1.5}$$

Received December 13, 2001

where

$$\rho_i = b|(a_i + b)p_i + \lambda_{ij}(p_j - p_i)|, \quad \beta = \max(\beta_1, \beta_2),$$
$$\beta_i = hb^2\lambda_{ij}|p_j - p_i| + \rho_i, \quad i, j = 1, 2, \quad j \neq i. \tag{1.6}$$

Then the trivial solution of equation (1.1) is asymptotically mean square stable.

Statement 1.1 (in more general case) is proven in [8].

Let us transform condition (1.5) to more visual form. From (1.4) it follows that for $p_1 \leq p_2$ and $i = 1$ condition (1.5) is impossible. So, put $p_1 > p_2$. Then from (1.6) it follows

$$\begin{aligned}
\rho_1 &< b((a_1 + b)p_1 + \lambda_{12}(p_1 - p_2)),\\
\rho_2 &< b(|a_2 + b|p_2 + \lambda_{21}(p_1 - p_2)),\\
\beta_1 < \overline{\beta}_1 &= b((a_1 + b)p_1 + \lambda_{12}(1 + bh)(p_1 - p_2)),\\
\beta_2 < \overline{\beta}_2 &= b(|a_2 + b|p_2 + \lambda_{21}(1 + bh)(p_1 - p_2)).
\end{aligned} \tag{1.7}$$

Using (1.4) we obtain $|a_2 + b| = -a_2 - b = |a_2| - b < a_1 + b$. Since, besides, $p_1 > p_2$ and $\lambda_{12} > \lambda_{21}$ then $\overline{\beta}_1 > \overline{\beta}_2$. Thus, condition (1.5) follows from

$$2(a_i + b)p_i + \lambda_{ij}(p_j - p_i) + h(\rho_i + \overline{\beta}_1) < 0. \tag{1.8}$$

In addition, using (1.7) we obtain that (1.8) follows from

$$\begin{gathered}
2(a_1 + b)p_1 + \lambda_{12}(p_2 - p_1)+\\
+hb[2(a_1 + b)p_1 + \lambda_{12}(2 + bh)(p_1 - p_2)] < 0,\\
2(a_2 + b)p_2 + \lambda_{21}(p_1 - p_2)+\\
+hb[(|a_2| - b)p_2 + (a_1 + b)p_1 + (\lambda_{12}(1 + bh) + \lambda_{21})(p_1 - p_2)] < 0.
\end{gathered}$$

Putting $\gamma = \frac{p_2}{p_1} \in (0, 1)$ we can rewrite these inequalities in the form

$$\begin{gathered}
2(a_1 + b) + \lambda_{12}(\gamma - 1)+\\
+hb[2(a_1 + b) + \lambda_{12}(2 + bh)(1 - \gamma)] < 0,\\
2(a_2 + b)\gamma + \lambda_{21}(1 - \gamma)+\\
+hb[(|a_2| - b)\gamma + a_1 + b + (\lambda_{12}(1 + bh) + \lambda_{21})(1 - \gamma)] < 0,
\end{gathered} \tag{1.9}$$

Suppose now that

$$bh < \sqrt{2} - 1. \tag{1.10}$$

Then from (1.9) we obtain

$$\frac{A + (a_1 + b)bh}{A + (2 - bh)(|a_2| - b)} < \gamma < 1 - \frac{2(a_1 + b)(1 + bh)}{B}, \tag{1.11}$$

where

$$A = (\lambda_{21} + \lambda_{12}bh)(1 + bh), \quad B = \lambda_{12}(1 - 2bh - b^2h^2). \tag{1.12}$$

Note that from (1.10) it follows that $B > 0$. At last using (1.4) and (1.11) we obtain

$$|a_2| < a_1 < \frac{B(2 - bh)(|a_2| - b)}{Bbh + 2(1 + bh)[A + (2 - bh)(|a_2| - b)]} - b. \tag{1.13}$$

Thus, by condition (1.10) if conditions (1.13) hold then there exists $\gamma \in (0, 1)$ such that conditions (1.11) hold too and therefore the conditions of Statement 1.1 hold. It means that by condition (1.10) conditions (1.13), (1.12) are sufficient conditions of asymptotic mean square stability of the trivial solution of equation (1.1).

On Fig.1.1 the stability regions, given by conditions (1.13), (1.12), are shown for $\lambda_{12} = 28$, $\lambda_{21} = 0.05$, $b = 1$ and different values of parameter h: 1) $h = 0.01$, 2) $h = 0.02$, 3) $h = 0.03$, 4) $h = 0.04$.

On Fig.1.2 the stability region is shown, given by conditions (1.13), (1.12), for $\lambda_{12} = 15$, $\lambda_{21} = 1$, $b = 0.2$, $h = 0.2$. Putting $\lambda_{12} = 5$ and using the same values of other parameters, we obtain the stability region, which is shown on Fig.1.3. We can see that in the case $\lambda_{12} = 15$ (Fig.1.2) the point

$$(a_1, a_2) = (1, -0.5) \tag{1.14}$$

belongs to the stability region and therefore in this point the trivial solution of equation (1.1) is asymptotically mean square stable. From the other hand in the case $\lambda_{12} = 5$ (Fig.1.3) the point (1.14) does not belong to the stability region. Since conditions (1.13), (1.12) are sufficient conditions only then for $\lambda_{12} = 5$ in the point (1.14) the trivial solution of equation (1.1) can be either stable or unstable.

Let us investigate a stability of the trivial solution of equation (1.1) in point (1.14) using the following numerical method. Consider difference analogue of equation (1.1) in the form

$$x_{i+1} = (1 + \eta_i\Delta)x_i + bx_{i-m}\Delta,$$

where $x_i = x(t_i)$, $\eta_i = \eta(t_i)$, $t_i = i\Delta$, $h = m\Delta$, $\Delta > 0$.

Simulation of the Markov chain η_i, $i = 0, 1, \ldots$, can be reduced to a more simple problem: simulation of a sequence of independent random variables ζ_i, which are uniformly distributed on [0,1]. Really, using (1.2) we have

$$P_1 = \mathbf{P}\{\eta_0 = a_1\} = \mathbf{P}\{\zeta_0 < P_1\}, \quad P_2 = \mathbf{P}\{\eta_0 = a_2\} = \mathbf{P}\{\zeta_0 > P_1\}. \tag{1.15}$$

So, if as a result of simulation ζ_0 we obtain $\zeta_0 < P_1$ then we put $\eta_0 = a_1$, if we obtain $\zeta_0 > P_1$ then we put $\eta_0 = a_2$.

Further, if $\eta_{i-1} = a_1$, $i > 0$, then from (1.3) for small enough $\Delta > 0$ we have

$$\begin{aligned} \mathbf{P}\{\eta_i = a_2/\eta_{i-1} = a_1\} &= \mathbf{P}\{\zeta_i < \lambda_{12}\Delta\} = \lambda_{12}\Delta, \\ \mathbf{P}\{\eta_i = a_1/\eta_{i-1} = a_1\} &= \mathbf{P}\{\zeta_i > \lambda_{12}\Delta\} = 1 - \lambda_{12}\Delta. \end{aligned} \tag{1.16}$$

Therefore, we obtain the following algorithm: if as a result of simulation ζ_i we have $\zeta_i > \lambda_{12}\Delta$ then we put $\eta_i = a_1$ else we put $\eta_i = a_2$. Analogously, if $\eta_{i-1} = a_2$, $i > 0$, then from (1.3) for small enough $\Delta > 0$ we have

$$\begin{aligned} \mathbf{P}\{\eta_i = a_1/\eta_{i-1} = a_2\} &= \mathbf{P}\{\zeta_i < \lambda_{21}\Delta\} = \lambda_{21}\Delta, \\ \mathbf{P}\{\eta_i = a_2/\eta_{i-1} = a_2\} &= \mathbf{P}\{\zeta_i > \lambda_{21}\Delta\} = 1 - \lambda_{21}\Delta. \end{aligned} \tag{1.17}$$

Therefore, if as a result of simulation ζ_i we have $\zeta_i < \lambda_{21}\Delta$ then we put $\eta_i = a_1$ else we put $\eta_i = a_2$.

On Fig.1.4 one of the possible trajectories of the Markov chain $\eta(t)$ and one of the corresponding trajectories of the solution of equation (1.1) are shown for the following values of parameters: $a_1 = 1$, $a_2 = -0.5$, $b = 0.2$, $h = 0.2$, $x(s) = 3.5$ for $s \leq 0$, $P_1 = P_2 = 0.5$, $\lambda_{12} = 15$, $\lambda_{21} = 1$, $\Delta = 0.01$. On Fig.1.5 hundred trajectories of the solution of equation (1.1) are shown for these values of parameters. We can see that all trajectories converge to zero in the whole accordance with the properties of stabiity.

Putting $\lambda_{12} = 5$ for the same values of the other parameters we obtain (Fig.1.6 and Fig.1.7) another situation: the trajectories of the solution fill by itself whole admissible space (between the solutions of equation (1.1) with $\eta(t) \equiv a_1$ and $\eta(t) \equiv a_2$) that says us about instability of the trivial solution of equation (1.1).

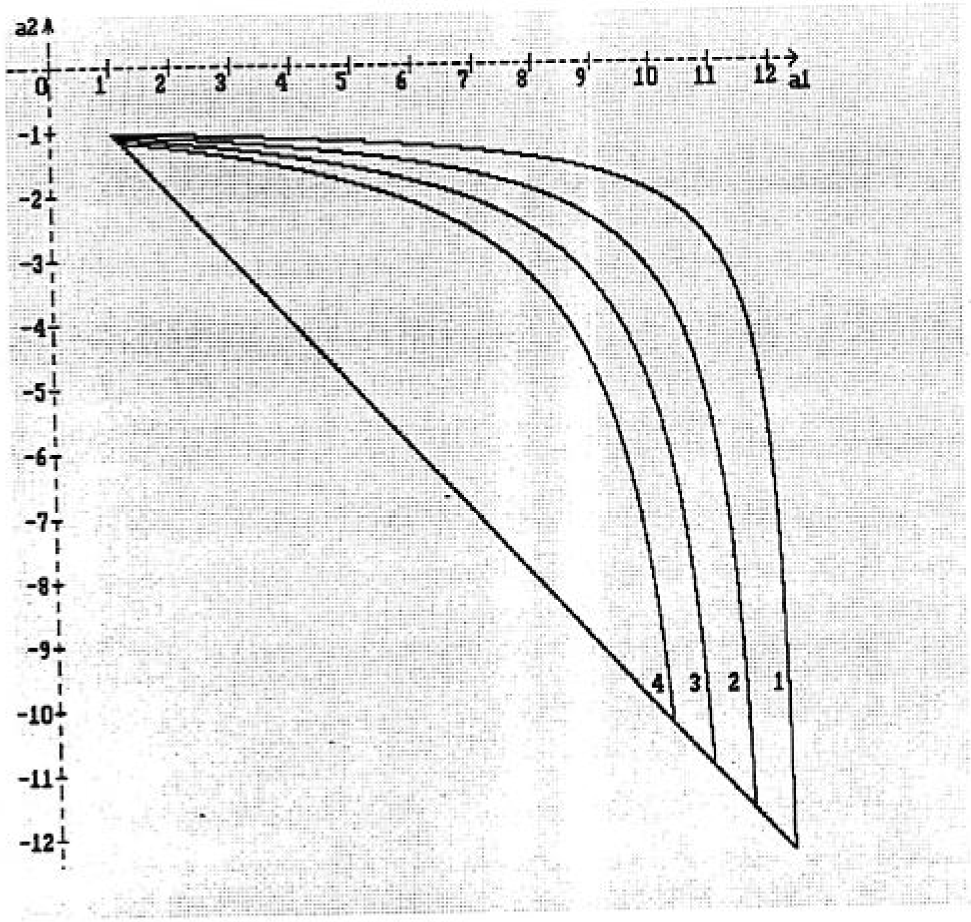

Fig.1.1

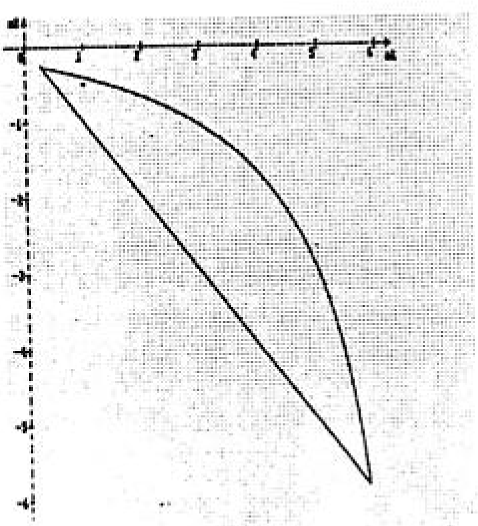

Fig.1.2

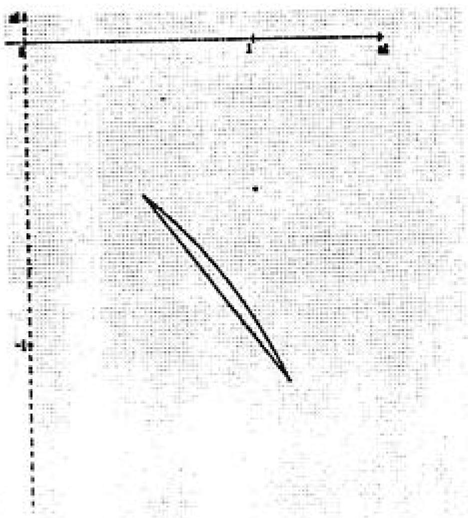

Fig.1.3

2. Mathematical pendulum

Consider the controllable system

$$\ddot{x}(t) + \eta(t)x(t) = u(t), \tag{2.1}$$

where $\eta(t)$ is a Markov chain with two states $\{a_1, a_2\}$, such that $a_1 > 0$ and $a_2 < 0$, an initial distribution (1.2) and probabilities of transition (1.3). It is easy to see that for $u(t) \equiv 0$ the trivial solution of equation (2.1) is stable but not asymptotically stable in the case $\eta(t) \equiv a_1 > 0$ (mathematical pendulum) and unstable in the case $\eta(t) \equiv a_2 < 0$ (inverted mathematical pendulum).

The problem is to stabilize the trivial solution of equation (2.1) using control

$$u(t) = b_1 x(t - h_1) + b_2 x(t - h_2). \tag{2.2}$$

The initial condition for system (2.1), (2.2) has the form $x(s) = \varphi(s)$, $s \in [-h, 0]$, $h = \max(h_1, h_2)$.

The problem of stabilization of inverted pendulum is considered in [9,10] where the following statement is proven.

Statement 2.1. *Let*

$$k_0 = b_1 + b_2 < a_2, \qquad k_1 = b_1 h_1 + b_2 h_2 > 0,$$
$$k_2 = |b_1| h_1^2 + |b_2| h_2^2 < \frac{4}{1 + \sqrt{1 + \left(\frac{1 + a_2 - k_0}{k_1}\right)^2}}. \tag{2.3}$$

Then the trivial solution of system (2.1), (2.2) (with $\eta(t) \equiv a_2 < 0$) is asymptotically stable.

It is proven also, that for each $a_2 < 0$ there exist such numbers b_1, b_2, h_1, h_2, that conditions (2.3) hold and therefore the trivial solution of system (2.1), (2.2) (with $\eta(t) \equiv a_2 < 0$) is asymptotically stable.

Let us investigate a stability of the trivial solution of system (2.1), (2.2) using a numerical method and numerical simulation of the Markov chain $\eta(t)$ as in the previous example. In this connection consider difference analogue of system (2.1), (2.2) in the form

$$x_{i+1} = (2 - \Delta^2 \eta_i) x_i - x_{i-1} + \Delta^2 (b_1 x_{i-m_1} + b_2 x_{i-m_2}),$$
$$x_i = x(t_i), \quad \eta_i = \eta(t_i), \quad t_i = i\Delta, \quad h_1 = m_1 \Delta, \quad h_2 = m_2 \Delta, \quad \Delta > 0.$$

Put $a_1 = 1$, $a_2 = -1$, $b_1 = 1$, $b_2 = -2.1$, $h_1 = 0.8$, $h_2 = 0.3$, $x(s) = 3.5$, $s \leq 0$, $\Delta = 0.01$. If $\eta(t) \equiv a_1$ then the solution of system (2.1), (2.2) goes to $\pm\infty$. The trajectories of $\eta(t)$ and $x(t)$ in this case are shown on Fig.2.1. If $\eta(t) \equiv a_2$ then conditions (2.3) hold and the solution of system (2.1), (2.2) converges to zero (Fig.2.2).

Put $P_1 = P_2 = 0.5$, $\lambda_{21} = 15$, $\lambda_{12} = 1$. One of the possible trajectories of the Markov chain $\eta(t)$ and one of the corresponding trajectories of the solution $x(t)$ of system (2.1), (2.2) are shown on Fig.2.3. Hundred trajectories of the solution at the same time are shown on Fig.2.4. We can see that in this case the trivial solution of system (2.1), (2.2) is unstable.

If $\lambda_{21} = 1$, $\lambda_{12} = 15$ then the trivial solution of system (2.1), (2.2) is asymptotically stable. One of the possible trajectories of the Markov chain $\eta(t)$ and one of the corresponding trajectories of the solution $x(t)$ of system (2.1), (2.2) are shown on Fig.2.5. Hundred trajectories of the solution at the same time are shown on Fig.2.6.

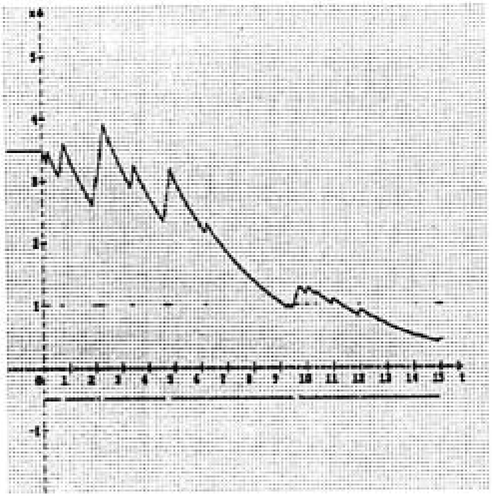

Fig.1.4

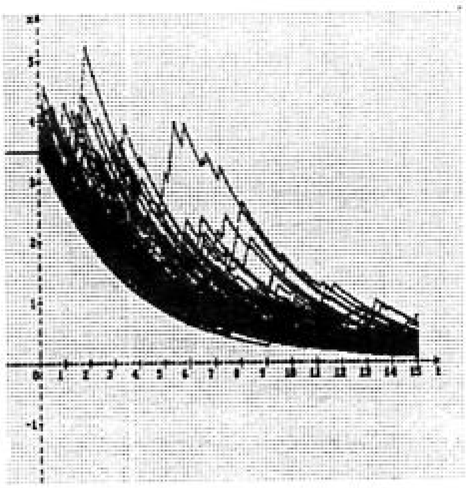

Fig.1.5

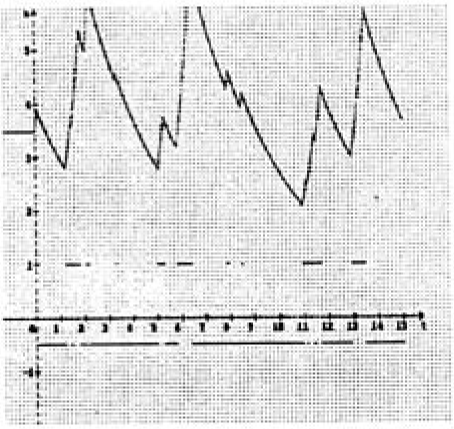

Fig.1.6

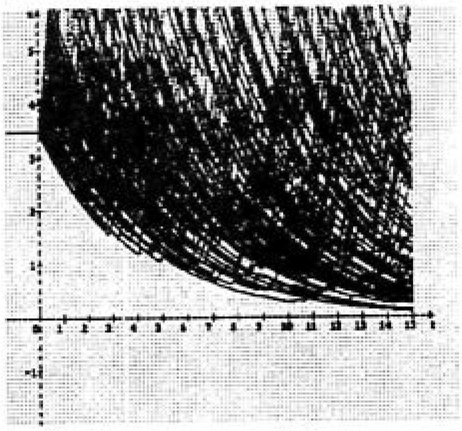

Fig.1.7

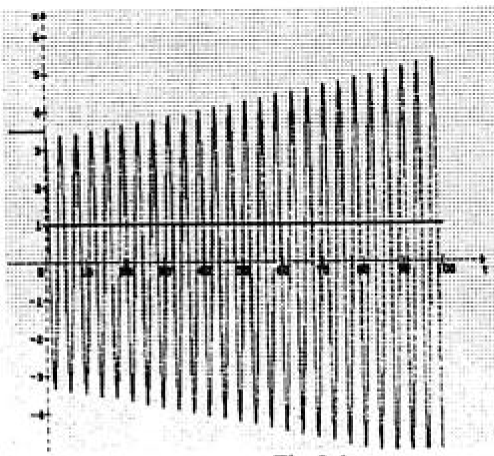

Fig.2.1

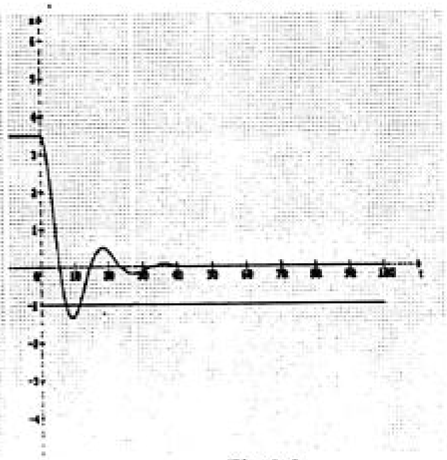

Fig.2.2

3. System with stochastic delay

Consider the differential equation

$$\begin{aligned} &\dot{x}(t) + bx(t - \eta(t)) = 0, \\ &x(s) = \varphi(s), \quad s \in [-h, 0]. \end{aligned} \tag{3.1}$$

with random delay. Here $b > 0$, $\eta(t)$ is a Markov chain with two states $\{a_1, a_2\}$, such that $0 < a_1 < a_2 = h$, an initial distribution (1.2) and probabilities of transition (1.3).

It is well known [11], that if $\eta(t) = h = const$ then the inequality $bh < \frac{\pi}{2}$ is the necessary and sufficient condition of asymptotic stability of the trivial solution of equation (3.1).

Let us investigate a stability of the trivial solution of differential equation (3.1) using the numerical method and the numerical simulation of the Markov chain $\eta(t)$ as in the previous examples. Consider the difference analogue of equation (3.1) in the form

$$\begin{aligned} x_{i+1} &= x_i - \Delta b x_{i-\eta_i}, \\ x_i = x(t_i), \quad \eta_i &= \eta(t_i), \quad t_i = i\Delta, \quad \Delta > 0. \end{aligned}$$

Put $b = 1$, $a_1 = 1$, $a_2 = 2$, $x(s) = const$, $s \leq 0$, $\Delta = 0.001$. On Fig.3.1 several solutions of differential equation (3.1) are shown for $\eta(t) \equiv a_1 = 1$ and different values of initial function. We can see that all solutions converge to zero. On Fig.3.2 it is shown that in the case $\eta(t) \equiv \frac{\pi}{2}$ all solutions are bounded only but do not converge to zero. In the case $\eta(t) \equiv a_2 = 2$ all solutions go to $\pm\infty$ as it is shown on Fig.3.3.

On Fig.3.4 one of the possible trajectories of the Markov chain $\eta(t)$ and one of the corresponding trajectories of the solution of equation (3.1) are shown for $P_1 = P_2 = 0.5$, $\lambda_{12} = 1$, $\lambda_{21} = 3$ and the same as before values of other parameters. On Fig.3.5 hundred trajectories of the solution of equation (3.1) are shown for the same values of the parameters. We can see that all trajectories converge to zero. If $\lambda_{12} = 1$, $\lambda_{21} = 6$, then hundred trajectories of the solution of equation (3.1) converge to zero more quickly (Fig.3.6). Using previous investigations we can conclude that in both these cases the trivial solution of equation (3.1) is asymptotically mean square stable.

On Fig.3.7 one of the possible trajectories of the Markov chain $\eta(t)$ and one of the corresponding trajectories of the solution of equation (3.1) are shown for $P_1 = P_2 = 0.5$, $\lambda_{12} = 3$, $\lambda_{21} = 1$. On Fig.3.8 hundred trajectories of the solution of equation (3.1) are shown for these values of λ_{12} and λ_{21}. We can see that in this case all trajectories of solution go to $\pm\infty$ and therefore we can conclude that the trivial solution of equation (3.1) is unstable.

4. Some generalization

Let us show that the proposed numerical simulation (1.15)-(1.17) of a Markov chain with two states can be generalized for a Markov chain $\eta(t)$ with n states $\{a_1, ..., a_n\}$, an initial distribution $P_i = \mathbf{P}\{\eta(0) = a_i\}$, $i = 1, ..., n$, and probabilities of transition

$$\begin{aligned} P_{ij}(\Delta) = \mathbf{P}\{\eta(t+\Delta) = a_j / \eta(t) = a_i\} = \lambda_{ij}\Delta + o(\Delta), \\ i, j = 1, ..., n, \qquad i \neq j. \end{aligned}$$

Really, put $\eta_k = \eta(t_k)$, $t_k = k\Delta$, $\Delta > 0$. Reduce a simulation of a Markov chain η_k, $k = 0, 1, ...$, to a simulation of a sequence of independent random variables ζ_k, which are uniformly distributed on [0,1]. Note that

$$P_i = \mathbf{P}\{\eta_0 = a_i\} = \mathbf{P}\{S_{i-1} < \zeta_0 < S_i\}, \quad i = 1, ..., n,$$

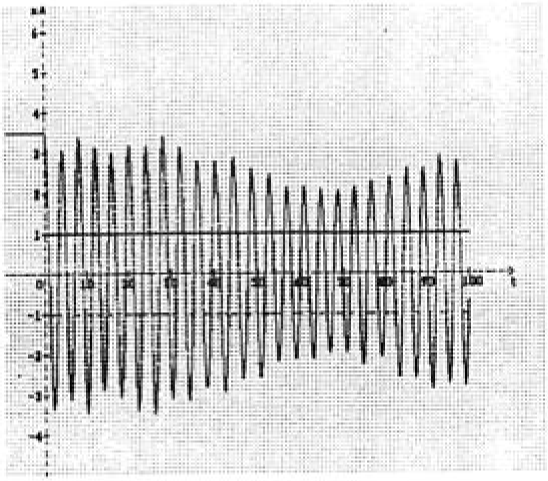

Fig.2.3

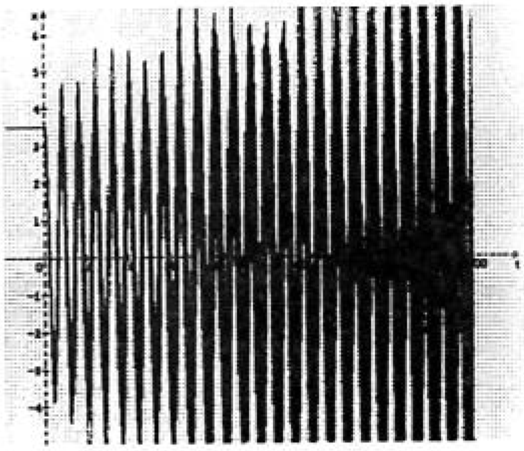

Fig.2.4

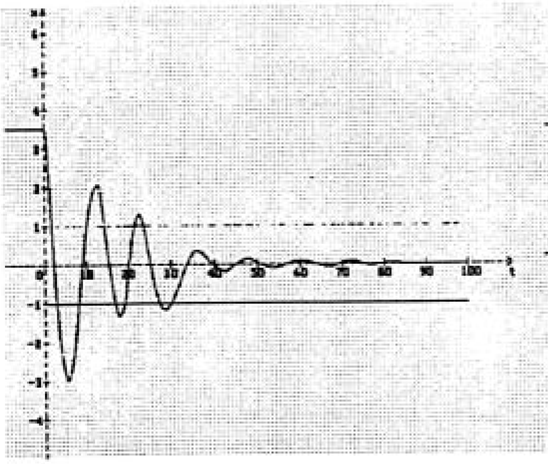

Fig.2.5

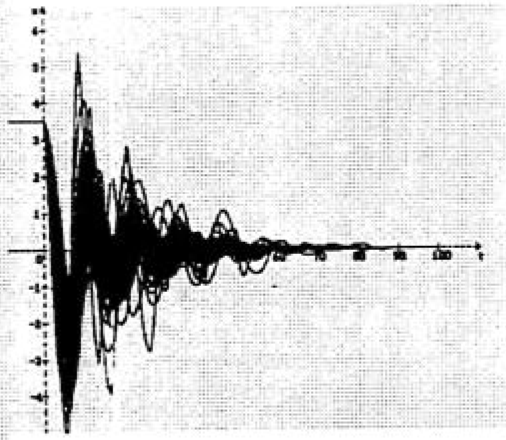

Fig.2.6

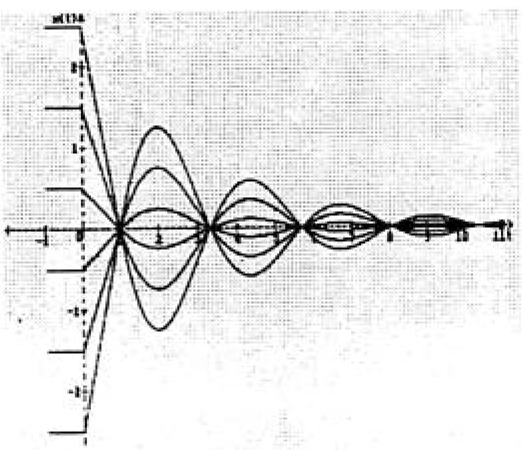

Fig.3.1

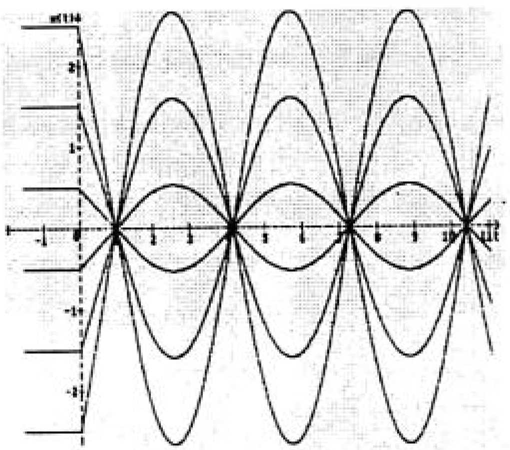

Fig.3.2

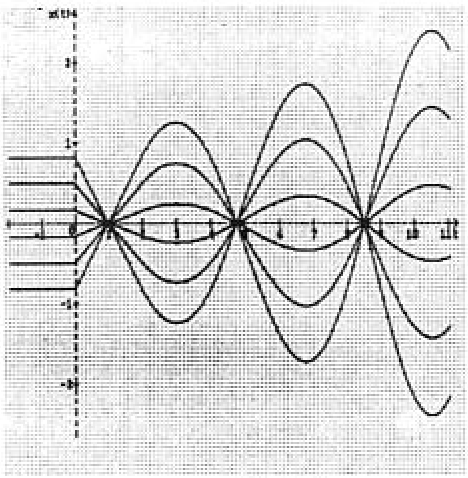

Fig.3.3

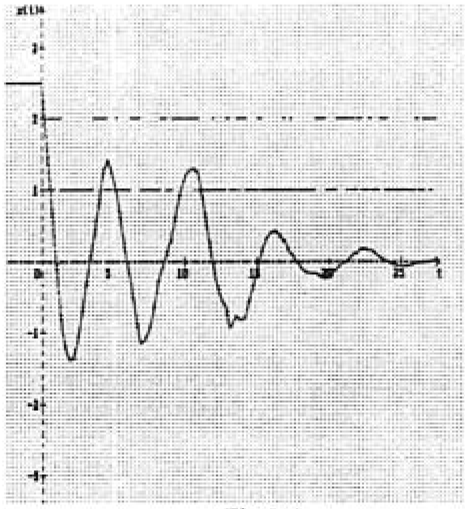

Fig.3.4

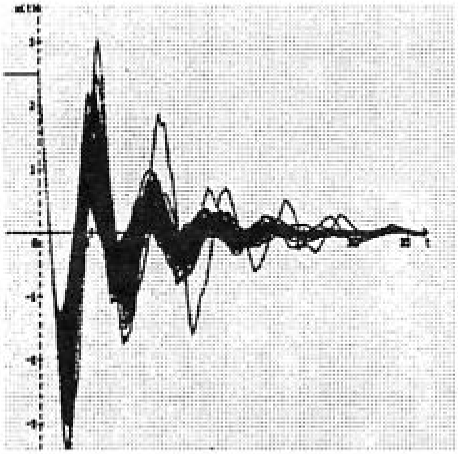

Fig.3.5

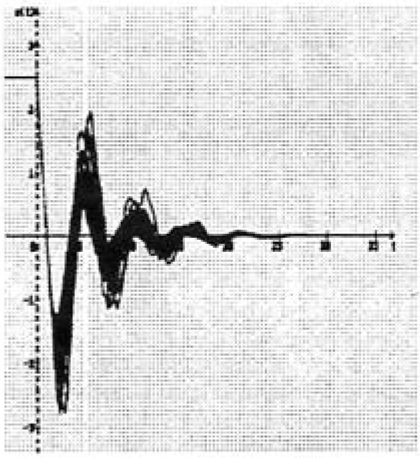

Fig.3.6

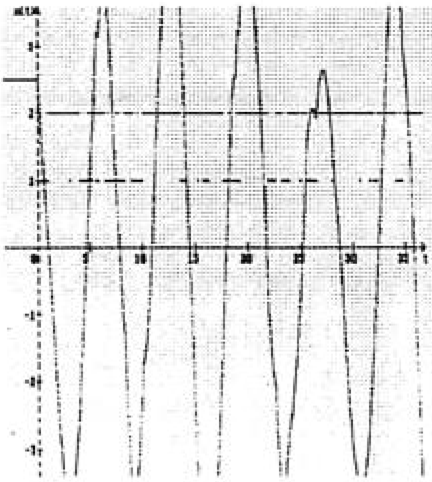

Fig.3.7

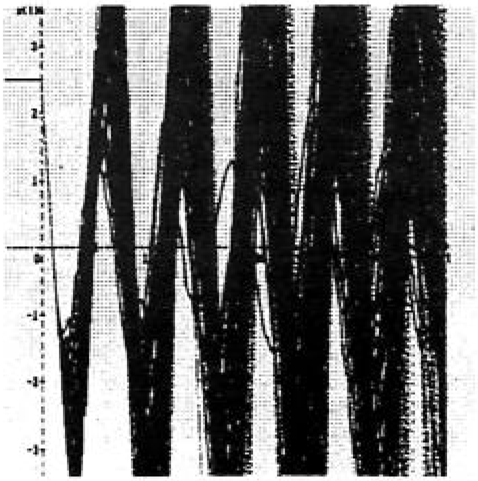

Fig.3.8

where

$$S_0 = 0, \quad S_i = \sum_{j=1}^{i} P_j, \quad i = 1, ..., n-1, \quad S_n = 1.$$

It is easy to see that for each result of simulation ζ_0 there exists a number i such that $S_{i-1} < \zeta_0 < S_i$. So, we put $\eta_0 = a_i$.

Further, put $Q_{i0} = 0$,

$$Q_{ij} = \sum_{l=1}^{j} \lambda_{il}\Delta, \quad 1 \leq j < i,$$

$$Q_{ij} = \sum_{l=1}^{i-1} \lambda_{il}\Delta + \sum_{l=i+1}^{j} \lambda_{il}\Delta, \quad i < j \leq n.$$

Then

$$\mathbf{P}\{\eta_k = a_j/\eta_{k-1} = a_i\} = \mathbf{P}\{Q_{i,j-1} < \zeta_k < Q_{ij}\} = \lambda_{ij}\Delta \quad for \quad j < i \quad or \quad j > i+1,$$

$$\mathbf{P}\{\eta_k = a_j/\eta_{k-1} = a_i\} = \mathbf{P}\{Q_{i,i-1} < \zeta_k < Q_{i,i+1}\} = \lambda_{ij}\Delta \quad for \quad j = i+1,$$

$$\mathbf{P}\{\eta_k = a_i/\eta_{k-1} = a_i\} = \mathbf{P}\{Q_{in} < \zeta_k\} = 1 - Q_{in}.$$

Thus, we obtain the following algorithm. Let $\eta_{k-1} = a_i$, $k > 0$. If as a result of simulation ζ_k there exists a number $j \leq n$ such that $j \neq i$ and

$$Q_{i,j-1} < \zeta_k < Q_{ij} \quad for \quad j < i \quad or \quad j > i+1,$$

$$Q_{i,i-1} < \zeta_k < Q_{i,i+1} \quad for \quad j = i+1$$

then we put $\eta_k = a_j$. If such number j does not exist, i.e. $\zeta_k > Q_{in}$ for $i < n$ or $\zeta_k > Q_{n,n-1}$ for $i = n$, then we put $\eta_k = a_i$.

References

1. Kac, I.Ya. and Krasovskii, N.N., *About stability of systems with stochastic parameters*, Prikladnaya matematika i mekhanika **24**, **N.5** (1960), pp. 809-823 (in Russian).
2. Kac, I.Ya., *Method of Lyapunov functions in problems of stability and stabilization of systems of stochastic structure*, Ekaterinburg, 1998 (in Russian).
3. Ji, Y. and Chizeck, H.J., *Controllability, stabilizability and continuous-time Markovian jump linear quadratic control*, IEEE Trans. Automat. Control **35** (1990), pp. 777-788.
4. Mariton, M., *Jump Linear Systems in Automatic Control*, Marcel Dekker, 1990.
5. Shaikhet L., *Stability of Stochastic Hereditary Systems with Markov Switching*, Theory of stochastic processes **2(18)**, **N.3-4** (1996), pp. 180-184.
6. Mao, X., *Stability of stochastic differential equations with Markovian switching*, Stochastic Processes and their Applications **79** (1999), pp. 45-67.
7. Mao, X., Matasov, A. and Piunovskiy, A.B., *Stochastic differential delay equations with Markovian switching*, Bernoulli **6(1)** (2000), pp. 73-90.
8. Mao X. and Shaikhet L., *Delay-Dependent Stability Criteria for Stochastic Differential Delay Equations with Markovian Switching*, Stability and Control: Theory and Application **3**, **N.2** (2000), pp.88-101.
9. Borne P., Kolmanovskii V. and Shaikhet L., *Stabilization of inverted pendulum by control with delay*, Dynamic Systems and Applications **9**, **N.4** (2000), pp. 501-515.
10. Borne P., Kolmanovskii V. and Shaikhet L., *Steady-state solutions of nonlinear model of inverted pendulum*, Proceedings of The Third Ukrainian-Scandinavian Conference in Probability Theory and Mathematical Statistics, June 8-12, 1999, Kyiv, Ukraine. Theory of Stochastic Processes **5(21)**, **N.3-4** (1999), pp. 203-209.
11. Kolmanovskii V.B. and Myshkis A.D., *Introduction to the Theory and Applications of Functional Differential Equations*, Kluwer Academic Publishers, Dordrecht, 1999.