

MATRIX RICCATI EQUATIONS AND
STABILITY OF STOCHASTIC LINEAR SYSTEMS
WITH NONINCREASING DELAYS

V.B. KOLMANOVSKII * AND L.E. SHAIKHET †

Abstract. Many real processes can be modeled by stochastic differential equations with aftereffect [1]-[3]. Stability conditions for such systems can be obtained by construction of appropriate Lyapunov functionals using special procedure of Lyapunov functionals construction [4]-[14]. In this paper asymptotic mean square stability of stochastic linear differential equations with discrete and distributed delays is considered. Stability conditions are formulated in terms of existence of positive definite solutions of matrix Riccati equations. The method of different Riccati equations construction is proposed.

Key Words. Asymptotic mean square stability, stochastic linear equations with delays, matrix Riccati equations.

AMS(MOS) subject classification. 93K25, 94L27,...

1. Introduction. Consider the stochastic linear differential equation

$$(1) \quad \dot{x}(t) = Ax(t) + Cx(t)\dot{\xi}(t).$$

Here A and C are constant $(n \times n)$ -matrices, $x(t) \in \mathbf{R}^n$, $\xi(t)$ is a scalar Wiener process.

Denote $P > 0$ any symmetric positive definite matrix. Then an appropriate Lyapunov function V for the equation (1) is a quadratic form $V = x'Px$, where the matrix P is a positive solution of the linear matrix equation [15].

$$(2) \quad A'P + PA + C'PC = -Q.$$

The necessary and sufficient conditions of asymptotic mean square stability of the system (1) can be formulated in terms of existence of a positive definite solution P of the matrix equation (2) for any $Q > 0$.

* MIEM, Moscow, Russia

† Donetsk State Academy of Management, Donetsk, Ukraine

But for stochastic linear differential equations with delays, for example,

$$(3) \quad \dot{x}(t) = Ax(t) + Bx(t - h(t)) + Cx(t - \tau(t))\dot{\xi}(t),$$

$$x_0(s) = \varphi(s), \quad s \leq 0.$$

this problem is more complicated.

Below we will obtain the conditions of asymptotic mean square stability for the equation (3) and some other more general systems.

Let $\{\Omega, \sigma, \mathbf{P}\}$ be the probability space, $\{f_t, t \geq 0\}$ be the family of σ -algebras, $f_t \in \sigma$, H be the space of f_0 -adapted functions $\varphi(s) \in R^n$, $s \leq 0$, $\|\varphi\|_0^2 = \sup_{s \leq 0} \mathbf{E}|\varphi(s)|^2$, \mathbf{E} be the mathematical expectation, $\|B\|$ be arbitrary matrix norm of matrix B .

DEFINITION 1. *The zero solution of the equation (3) is called mean square stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that $\mathbf{E}|x(t)|^2 < \epsilon$ for all $t \geq 0$ if $\|\varphi\|_0^2 < \delta$. If, besides, $\lim_{t \rightarrow \infty} \mathbf{E}|x(t)|^2 = 0$, then the zero solution of the equation (3) is called asymptotic mean square stable.*

THEOREM 1. *Let there exists the functional $V(t, \varphi)$, which satisfies the conditions*

$$\mathbf{E}V(0, \varphi) \leq c_1 \|\varphi\|_0^2,$$

$$\mathbf{E}V(t, x_t) \geq c_2 \mathbf{E}|x(t)|^2,$$

$$\mathbf{E}LV(t, x_t) \leq -c_3 \mathbf{E}|x(t)|^2,$$

where $c_i > 0$, $i = 1, 2, 3$, $x_t = x(t + s)$, $s \leq 0$, L is the generator of the equation (3). Then the zero solution of the equation (3) is asymptotic mean square stable.

2. The stochastic equation with one delay in deterministic part and one delay in stochastic part. From Theorem 1 it follows that the construction of stability conditions for the equation (3) is reduced to the construction of appropriate Lyapunov functionals. Constructing different Lyapunov functionals we will obtain different stability conditions. Using the general method of Lyapunov functionals construction [4]-[14], we will construct two different Lyapunov functionals for the equation (3).

It is supposed that delays $h(t)$ and $\tau(t)$ are nonnegative differentiable functions satisfying the conditions:

$$(4) \quad \dot{h}(t) \leq 0, \quad \dot{\tau}(t) \leq 0,$$

$$(5) \quad \alpha = \sup_{t \geq 0} |\dot{h}(t)| < \infty.$$

2.1. We will construct the Lyapunov functional V in the form $V = V_1 + V_2$, where $V_1 = x'Px$. Calculating LV_1 , we get

$$\begin{aligned} LV_1 &= (Ax(t) + Bx(t - h(t)))'Px(t) + x'(t)P(Ax(t) + Bx(t - h(t))) + \\ &\quad + x'(t - \tau(t))C'PCx(t - \tau(t)) = \\ &= x'(t)(A'P + PA)x(t) + x'(t - \tau(t))C'PCx(t - \tau(t)) + \\ &\quad + x'(t - h(t))B'Px(t) + x'(t)PBx(t - h(t)). \end{aligned}$$

Note that for arbitrary vectors a, b and any $R > 0$ we have

$$(6) \quad a'b + b'a = a'Ra + b'R^{-1}b - (Ra - b)'R^{-1}(Ra - b) \leq a'Ra + b'R^{-1}b.$$

Using (6) for $a = x(t - h(t))$ and $b = B'Px(t)$ we have

$$\begin{aligned} &x'(t - h(t))B'Px(t) + x'(t)PBx(t - h(t)) \leq \\ &\leq x'(t - h(t))Rx(t - h(t)) + x'(t)PBR^{-1}B'Px(t). \end{aligned}$$

Then

$$\begin{aligned} LV_1 &\leq x'(t)(A'P + PA + PBR^{-1}B'P)x(t) + \\ &\quad + x'(t - h(t))Rx(t - h(t)) + x'(t - \tau(t))C'PCx(t - \tau(t)). \end{aligned}$$

Choosing the functional V_2 in the form

$$V_2 = \int_{t-h(t)}^t x'(s)Rx(s)ds + \int_{t-\tau(t)}^t x'(s)C'PCx(s)ds,$$

we have

$$\begin{aligned} LV_2 &= x'(t)Rx(t) - (1 - \dot{h}(t))x'(t - h(t))Rx(t - h(t)) + \\ &\quad + x'(t)C'PCx(t) - (1 - \dot{\tau}(t))x'(t - \tau(t))C'PCx(t - \tau(t)). \end{aligned}$$

Using (4) as a result for $V = V_1 + V_2$ we have $LV \leq -x'(t)Qx(t)$, where

$$(7) \quad Q = -[A'P + PA + C'PC + R + PBR^{-1}B'P].$$

Thus, it is proved

THEOREM 2. *Let the condition (4) hold and for some symmetric matrices $Q > 0$ and $R > 0$ there exists a positive definite solution P of the matrix Riccati equation (7). Then the zero solution of the equation (3) is asymptotic mean square stable.*

REMARK 1. *Using (6) for $a = x(t)$ and $b = PBx(t - h(t))$ we obtain*

$$\begin{aligned} &x'(t - h(t))B'Px(t) + x'(t)PBx(t - h(t)) \leq \\ &\leq x'(t - h(t))B'PR^{-1}PBx(t - h(t)) + x'(t)Rx(t). \end{aligned}$$

In this case choosing the functional V_2 in the form

$$V_2 = \int_{t-h(t)}^t x'(s)B'PR^{-1}PBx(s)ds + \int_{t-\tau(t)}^t x'(s)C'PCx(s)ds,$$

we obtain $LV \leq -x'(t)Qx(t)$, where

$$(8) \quad Q = -[A'P + PA + C'PC + R + B'PR^{-1}PB].$$

Thus, in Theorem 2 in place of the equation (7) can be used the equation (8).

EXAMPLE 1. *In the scalar case a positive solution of the equation (7) (or (8)) there exists if and only if*

$$A + |B| + \frac{1}{2}C^2 < 0.$$

EXAMPLE 2. *Consider the two-dimensional system (3), with $h(t) = h(0)$ (i.e. $\alpha = 0$) and*

$$A = \begin{pmatrix} -a_1 & 0 \\ 0 & -a_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 \\ -b_2 & b_1 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}.$$

*Let be $Q = qI$, $R = rI$, $q, r > 0$, I is $(2 * 2)$ -identity matrix. The positive solution ($P_{12} = 0$, $P_{11} > 0$, $P_{22} > 0$) of the Riccati equation (7) in this case there exists if and only if*

$$a_i > \sqrt{b_1^2 + b_2^2} + \frac{1}{2}c_i^2, \quad i = 1, 2.$$

EXAMPLE 3. Consider Example 2 by additional assumptions: $b_1 = 0$, $b_2 = 1$, $2a_1 = 6 + c_1^2$, $2a_2 = \frac{3}{2} + c_2^2$. In this case the condition $a_i > 1 + \frac{1}{2}c_i^2$ is fulfilled for $i = 1$ and isn't fulfilled for $i = 2$. Therefore the Riccati equation (7) doesn't have a positive solution. But the Riccati equation (8) has positive solution $P_{11} = 1$, $P_{22} = 2$, $P_{12} = 0$ for $q = r = 1$.

2.2. Consider now another way of the construction of a Lyapunov functional V leading to another Riccati equation.

Let us represent the equation (3) as a neutral type equation in the form

$$\begin{aligned} \frac{d}{dt}(x(t) + B \int_{t-h(t)}^t x(s)ds) &= (A + B)x(t) + \\ &+ \dot{h}(t)Bx(t - h(t)) + Cx(t - \tau(t))\dot{\xi}(t). \end{aligned}$$

We will construct the Lyapunov functional V in the form $V = V_1 + V_2$ again, but now the functional V_1 we will choose in the form

$$(9) \quad V_1 = (x(t) + B \int_{t-h(t)}^t x(s)ds)'P(x(t) + B \int_{t-h(t)}^t x(s)ds).$$

Calculating LV_1 , we get:

$$\begin{aligned} LV_1 &= (x(t) + B \int_{t-h(t)}^t x(s)ds)'P[(A + B)x(t) + \dot{h}(t)Bx(t - h(t))] + \\ &+ [x'(t)(A + B)' + \dot{h}(t)x'(t - h(t))B']P(x(t) + B \int_{t-h(t)}^t x(s)ds) + \\ &+ x'(t - \tau(t))C'PCx(t - \tau(t)) = \\ &= x'(t)[P(A + B) + (A + B)'P]x(t) + x'(t - \tau(t))C'PCx(t - \tau(t)) + \\ &+ \int_{t-h(t)}^t (x'(t)(A + B)'PBx(s) + x'(s)B'P(A + B)x(t))ds + \\ &+ \dot{h}(t)(x'(t)PBx(t - h(t)) + x'(t - h(t))B'Px(t)) + \\ &+ \dot{h}(t) \int_{t-h(t)}^t (x'(s)B'PBx(t - h(t)) + x'(t - h(t))B'PBx(s))ds. \end{aligned}$$

Using (6) for any $R > 0$, $a = x(s)$, $b = B'P(A + B)x(t)$, for any $R_1 > 0$, $a = x(t)$, $b = PBx(t - h(t))$ and for any $R_2 > 0$, $a = x(s)$, $b = B'PBx(t - h(t))$ we obtain

$$\int_{t-h(t)}^t [x'(t)(A + B)'PBx(s) + x'(s)B'P(A + B)x(t)]ds \leq$$

$$(10) \leq h(t)x'(t)(A + B)'PBR^{-1}B'P(A + B)x(t) + \int_{t-h(t)}^t x'(s)Rx(s)ds,$$

$$\dot{h}(t)(x'(t)PBx(t - h(t)) + x'(t - h(t))B'Px(t)) \leq$$

$$(11) \leq |\dot{h}(t)|(x'(t)R_1x(t) + x'(t - h(t))B'PR_1^{-1}PBx(t - h(t))),$$

$$\dot{h}(t) \int_{t-h(t)}^t (x'(s)B'PBx(t - h(t)) + x'(t - h(t))B'PBx(s))ds \leq$$

$$(12) |\dot{h}(t)| \left[\int_{t-h(t)}^t x'(s)R_2x(s)ds + h(t)x'(t - h(t))B'PBR_2^{-1}B'PBx(t - h(t)) \right].$$

Then by virtue of (4), (5) we have

$$LV_1 \leq x'(t)[(A+B)'P+P(A+B)+h(0)(A+B)'PBR^{-1}B'P(A+B)+\alpha R_1]x(t)+$$

$$+ \int_{t-h(t)}^t x'(s)Rx(s)ds + x'(t - \tau(t))C'PCx(t - \tau(t))+$$

$$+\alpha x'(t - h(t))B'PR_1^{-1}PBx(t - h(t))+$$

$$+\alpha \left(\int_{t-h(t)}^t x'(s)R_2x(s)ds + h(0)x'(t - h(t))B'PBR_2^{-1}B'PBx(t - h(t)) \right).$$

Choosing functional V_2 in the form

$$V_2 = \int_{t-h(t)}^t (s - t + h(t))x'(s)Rx(s)ds + \int_{t-\tau(t)}^t x'(s)C'PCx(s)ds +$$

$$+\alpha \int_{t-h(t)}^t x'(s)B'PR_1^{-1}PBx(s)ds + \alpha \int_{t-h(t)}^t (s - t + h(t))x'(s)R_2x(s)ds +$$

$$+\alpha h(0) \int_{t-h(t)}^t x'(s)B'PBR_2^{-1}B'PBx(s)ds,$$

for $V = V_1 + V_2$ we obtain $LV \leq -x'(t)Qx(t)$, where

$$\begin{aligned} Q = & -[(A + B)'P + P(A + B) + C'PC + \\ & +h(0)(R + (A + B)'PBR^{-1}B'P(A + B)) + \\ (13) \quad & +\alpha(R_1 + B'PR_1^{-1}PB) + \alpha h(0)(R_2 + B'PBR_2^{-1}B'PB)]. \end{aligned}$$

REMARK 2. *Functional (9) is so-called degenerated functional (i.e. non-positive definite). It means that direct application of Theorem 1 is impossible. The appropriate modification of Theorem 1 for degenerated functionals there is in [1]. Using this modification we obtain the following*

THEOREM 3. *Let the conditions (4), (5) and the inequality $h(0)\|B\| < 1$ hold and for some symmetric matrices $Q > 0, R > 0, R_1 > 0, R_2 > 0$ there exists a positive definite solution P of the matrix Riccati equation (13). Then the zero solution of the equation (3) is asymptotic mean square stable.*

REMARK 3. *Using (6) for any $R > 0, a = x(t), b = (A+B)'PBx(s)$, for any $R_1 > 0, a = x(t-h(t)), b = B'Px(t)$ and for any $R_2 > 0, a = x(t-h(t)), b = B'PBx(s)$ in place of the inequalities (10)-(12) we obtain the inequalities*

$$\begin{aligned} & \int_{t-h(t)}^t [x'(t)(A + B)'PBx(s) + x'(s)B'P(A + B)x(t)]ds \leq \\ (14) \quad & \leq h(t)x'(t)Rx(t) + \int_{t-h(t)}^t x'(s)B'P(A + B)R^{-1}(A + B)'PBx(s)ds, \end{aligned}$$

$$\begin{aligned} & \dot{h}(t)(x'(t)PBx(t - h(t)) + x'(t - h(t))B'Px(t)) \leq \\ (15) \quad & \leq |\dot{h}(t)|(x'(t - h(t))R_1x(t - h(t)) + x'(t)PBR_1^{-1}B'Px(t)), \end{aligned}$$

$$\begin{aligned} & \dot{h}(t) \int_{t-h(t)}^t (x'(s)B'PBx(t - h(t)) + x'(t - h(t))B'PBx(s))ds \leq \\ (16) \quad & \leq |\dot{h}(t)|[h(t)x'(t - h(t))R_2x(t - h(t)) + \end{aligned}$$

$$\int_{t-h(t)}^t x'(s)B'PBR_2^{-1}B'PBx(s)ds].$$

Using different combinations of the inequalities (10)-(12), (14)-(16) and choosing a corresponding form of the functional V_2 , in place of the equation (13) in Theorem 3 we can use one from the four different matrix Riccati equations. For example, using in the place of the all inequalities (10)-(12) the inequalities (14)-(16) and choosing the functional V_2 in the form

$$V_2 = \int_{t-h(t)}^t (s-t+h(t))x'(s)B'P(A+B)R^{-1}(A+B)'PBx(s)ds + \int_{t-\tau(t)}^t x'(s)C'PCx(s)ds,$$

we can see that in Theorem 3 in place of the equation (13) can be used the equation

$$\begin{aligned} Q = & -[(A+B)'P + P(A+B) + C'PC + \\ & + h(0)(R + B'P(A+B)R^{-1}(A+B)'PB) + \\ (17) \quad & + \alpha(R_1 + PBR_1^{-1}B'P) + \alpha h(0)(R_2 + B'PBR_2^{-1}B'PB)]. \end{aligned}$$

EXAMPLE 4. In scalar case a positive solution of the equation (13) (or (17)) there exists if and only if

$$(A+B)(1-h(0)|B|) + \alpha|B|(1+h(0)|B|) + \frac{1}{2}C^2 < 0, \quad h(0)|B| < 1.$$

3. The stochastic equation with several delays in deterministic part and one delay in stochastic part. Consider now more general stochastic linear differential equation with delays

$$(18) \quad \dot{x}(t) = Ax(t) + \sum_{i=1}^m B_i x(t-h_i(t)) + Cx(t-\tau(t))\dot{\xi}(t).$$

Here m is a positive integer, delays $h_i(t)$ are nonnegative differentiable functions satisfying the conditions

$$(19) \quad \dot{\tau}(t) \leq 0, \quad \dot{h}_i(t) \leq 0, \quad \alpha_i = \sup_{t \geq 0} |\dot{h}_i(t)| < \infty.$$

For the construction of Lyapunov functionals for the equation (18) we will use now both previous representations of initial equation. Namely, represent the equation (18) in the form

$$\begin{aligned} \frac{d}{dt}(x(t) + \sum_{j=1}^{m_0} B_j \int_{t-h_j(t)}^t x(s) ds) &= (A + B)x(t) + \\ + \sum_{l=1}^{m_0} \dot{h}_l(t) B_l x(t - h_l(t)) &+ \sum_{i=m_0+1}^m B_i x(t - h_i(t)) + Cx(t - \tau(t)) \dot{\xi}(t). \end{aligned}$$

Here

$$B = \sum_{j=1}^{m_0} B_j, \quad 0 \leq m_0 \leq m.$$

We will construct now a Lyapunov functional V in the form $V = V_1 + V_2$, where

$$V_1 = (x(t) + \sum_{j=1}^{m_0} B_j \int_{t-h_j(t)}^t x(s) ds)' P(x(t) + \sum_{j=1}^{m_0} B_j \int_{t-h_j(t)}^t x(s) ds).$$

In this case

$$\begin{aligned} LV_1 &= [(A + B)x(t) + \sum_{i=m_0+1}^m B_i x(t - h_i(t)) + \\ + \sum_{l=1}^{m_0} \dot{h}_l(t) B_l x(t - h_l(t))] &' P(x(t) + \sum_{j=1}^{m_0} B_j \int_{t-h_j(t)}^t x(s) ds) + \\ + (x(t) + \sum_{j=1}^{m_0} B_j \int_{t-h_j(t)}^t &x(s) ds)' P[(A + B)x(t) + \\ + \sum_{i=m_0+1}^m B_i x(t - h_i(t)) &+ \sum_{l=1}^{m_0} \dot{h}_l(t) B_l x(t - h_l(t))] + \\ + x'(t - \tau(t)) C' P C x(t - \tau(t)) &= \\ = x'(t) [(A + B)' P + P(A + B)] x(t) &+ x'(t - \tau(t)) C' P C x(t - \tau(t)) + \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=m_0+1}^m [x'(t-h_i(t))B'_iPx(t) + x'(t)PB_i x(t-h_i(t))] + \\
 & + \sum_{j=1}^{m_0} \int_{t-h_j(t)}^t [x'(t)(A+B)'PB_jx(s) + x'(s)B'_jP(A+B)x(t)]ds + \\
 & + \sum_{i=m_0+1}^m \sum_{j=1}^{m_0} \int_{t-h_j(t)}^t [x'(t-h_i(t))B'_iPB_jx(s) + x'(s)B'_jPB_i x(t-h_i(t))]ds + \\
 & + \sum_{l=1}^{m_0} \dot{h}_l(t)[x'(t-h_l(t))B'_lPx(t) + x'(t)PB_lx(t-h_l(t))] + \\
 & + \sum_{l=1}^{m_0} \sum_{j=1}^{m_0} \dot{h}_l(t) \int_{t-h_j(t)}^t [x'(t-h_l(t))B'_lPB_jx(s) + x'(s)B'_jPB_lx(t-h_l(t))]ds.
 \end{aligned}$$

Using (6) for any $R_i > 0$, $i = m_0 + 1, \dots, m$ and $a = x(t - h_i(t))$, $b = B'_iPx(t)$ we obtain

$$\begin{aligned}
 & x'(t-h_i(t))B'_iPx(t) + x'(t)PB_i x(t-h_i(t)) \leq \\
 (20) \quad & \leq x'(t)PB_iR_i^{-1}B'_iPx(t) + x'(t-h_i(t))R_ix(t-h_i(t)).
 \end{aligned}$$

For any $G_j > 0$, $j = 1, \dots, m_0$, and $a = x(s)$, $b = B'_jP(A+B)x(t)$ we obtain

$$\begin{aligned}
 & x'(t)(A+B)'PB_jx(s) + x'(s)B'_jP(A+B)x(t) \leq \\
 (21) \quad & \leq x'(t)(A+B)'PB_jG_j^{-1}B'_jP(A+B)x(t) + x'(s)G_jx(s).
 \end{aligned}$$

For any $S_{ij} > 0$, $i = m_0 + 1, \dots, m$, $j = 1, \dots, m_0$ and $a = x(s)$, $b = B'_jPB_ix(t-h_i(t))$ we obtain

$$\begin{aligned}
 & x'(t-h_i(t))B'_jPB_jx(s) + x'(s)B'_jPB_ix(t-h_i(t)) \leq \\
 (22) \quad & \leq x'(t-h_i(t))B'_jPB_jS_{ij}^{-1}B'_jPB_ix(t-h_i(t)) + x'(s)S_{ij}x(s).
 \end{aligned}$$

For any $U_l > 0$, $l = 1, \dots, m_0$ and $a = x(t - h_l(t))$, $b = B'_lPx(t)$ using (19) we obtain

$$\dot{h}_l(t)[x'(t-h_l(t))B'_lPx(t) + x'(t)PB_lx(t-h_l(t))] \leq$$

$$(23) \quad \leq \alpha_l[x'(t)PB_lU_l^{-1}B'_lPx(t) + x'(t - h_l(t))U_lx(t - h_l(t))].$$

For any $Z_{lj} > 0$, $l = 1, \dots, m_0$, $j = 1, \dots, m_0$ and $a = x(s)$, $b = B'_jPB_lx(t - h_l(t))$ using (19) we obtain

$$(24) \quad \begin{aligned} & \dot{h}_l(t)[x'(t - h_l(t))B'_lPB_jx(s) + x'(s)B'_jPB_lx(t - h_l(t))] \leq \\ & \leq \alpha_l[x'(t - h_l(t))B'_lPB_jZ_{lj}^{-1}B'_jPB_lx(t - h_l(t)) + x'(s)Z_{lj}x(s)]. \end{aligned}$$

Using the inequalities (20)-(24), we obtain

$$\begin{aligned} LV_1 & \leq x'(t)[(A + B)'P + P(A + B) + \\ & + \sum_{l=1}^{m_0} \alpha_l PB_l U_l^{-1} B'_l P + \sum_{i=m_0+1}^m PB_i R_i^{-1} B'_i P + \\ & + \sum_{j=1}^{m_0} h_j(0)(A + B)'PB_j G_j^{-1} B'_j P(A + B)]x(t) + \\ & + x'(t - \tau(t))C'PCx(t - \tau(t)) + \\ & + \sum_{i=m_0+1}^m x'(t - h_i(t))R_i x(t - h_i(t)) + \sum_{j=1}^{m_0} \int_{t-h_j(t)}^t x'(s)G_j x(s)ds + \\ & + \sum_{i=m_0+1}^m \sum_{j=1}^{m_0} \left[\int_{t-h_j(t)}^t x'(s)S_{ij} x(s)ds + \right. \\ & \left. + h_j(0)x'(t - h_i(t))B'_iPB_jS_{ij}^{-1}B'_jPB_i x(t - h_i(t)) \right] + \\ & + \sum_{l=1}^{m_0} \alpha_l [x'(t - h_l(t))U_l x(t - h_l(t)) + \sum_{j=1}^{m_0} \int_{t-h_j(t)}^t x'(s)Z_{lj} x(s)ds] + \\ & + \sum_{l=1}^{m_0} \sum_{j=1}^{m_0} \alpha_l h_j(0)x'(t - h_l(t))B'_lPB_jZ_{lj}^{-1}B'_jPB_lx(t - h_l(t)). \end{aligned}$$

Choosing the functional V_2 in the form

$$\begin{aligned}
 V_2 = & \int_{t-\tau(t)}^t x'(s)C'PCx(s)ds + \\
 & + \sum_{i=m_0+1}^m \int_{t-h_i(t)}^t x'(s)R_i x(s)ds + \sum_{j=1}^{m_0} \int_{t-h_j(t)}^t (s-t+h_j(t))x'(s)G_j x(s)ds + \\
 & + \sum_{i=m_0+1}^m \sum_{j=1}^{m_0} [h_j(0) \int_{t-h_i(t)}^t x'(s)B'_iPB_jS_{ij}^{-1}B'_jPB_i x(s)ds + \\
 & + \int_{t-h_j(t)}^t (s-t+h_j(t))x'(s)S_{ij}x(s)ds] + \\
 & + \sum_{l=1}^{m_0} \alpha_l [\int_{t-h_l(t)}^t x'(s)U_l x(s)ds + \sum_{j=1}^{m_0} \int_{t-h_j(t)}^t (s-t+h_j(t))x'(s)Z_{lj}x(s)ds] + \\
 & + \sum_{l=1}^{m_0} \sum_{j=1}^{m_0} \alpha_l h_j(0) \int_{t-h_j(t)}^t x'(s)B'_lPB_jZ_{lj}^{-1}B'_jPB_l x(s)ds.
 \end{aligned}$$

for $V = V_1 + V_2$ we obtain $LV \leq -x'(t)Qx(t)$, where

$$\begin{aligned}
 Q = & -[(A+B)'P + P(A+B) + C'PC + \\
 & + \sum_{l=1}^{m_0} \alpha_l (U_l + PB_l U_l^{-1} B'_l P) + \sum_{i=m_0+1}^m (R_i + PB_i R_i^{-1} B'_i P) + \\
 (25) \quad & + \sum_{j=1}^{m_0} h_j(0) (G_j + (A+B)'PB_j G_j^{-1} B'_j P (A+B)) + \\
 & + \sum_{j=1}^{m_0} h_j(0) \left(\sum_{l=1}^{m_0} \alpha_l (Z_{lj} + B'_l PB_j Z_{lj}^{-1} B'_j PB_l) + \sum_{i=m_0+1}^m (S_{ij} + B'_i PB_j S_{ij}^{-1} B'_j PB_i) \right)],
 \end{aligned}$$

$$B = \sum_{j=1}^{m_0} B_j, \quad 0 \leq m_0 \leq m.$$

Thus, it is proved

THEOREM 4. *Let the condition (19) and the inequality*

$$\sum_{j=1}^{m_0} h_j(0) \|B_j\| < 1$$

for some $m_0 \in [0, m]$ hold and for some symmetric matrices $U_l > 0$, $R_i > 0$, $G_j > 0$, $S_{ij} > 0$, $Z_{lj} > 0$ and $Q > 0$ there exists a positive definite solution P of the matrix Riccati equation (25). Then the zero solution of the equation (18) is asymptotic mean square stable.

REMARK 4. *Using other variants of the inequalities (20)-(24) and choosing a corresponding form of the functional V_2 in place of the equation (25) in Theorem 4 we can use one from the sixteen different matrix Riccati equations. For example,*

$$\begin{aligned} Q = & -[(A + B)'P + P(A + B) + C'PC + \\ & + \sum_{l=1}^{m_0} \alpha_l (U_l + B_l' P U_l^{-1} P B_l) + \sum_{i=m_0+1}^m (R_i + B_i' P R_i^{-1} P B_i) + \\ & + \sum_{j=1}^{m_0} h_j(0) (G_j + B_j' P (A + B) G_j^{-1} (A + B)' P B_j) + \\ & + \sum_{j=1}^{m_0} h_j(0) (\sum_{l=1}^{m_0} \alpha_l (Z_{lj} + B_j' P B_l Z_{lj}^{-1} B_l' P B_j) + \sum_{i=m_0+1}^m (S_{ij} + B_j' P B_i S_{ij}^{-1} B_i' P B_j))]. \end{aligned}$$

REMARK 5. *It is easily to see that representing the equation (18) in the form*

$$\begin{aligned} \frac{d}{dt}(x(t) + \sum_{j=m_0+1}^m \int_{t-h_j(t)}^t x(s) ds) = & (A + B)x(t) + \\ + \sum_{j=m_0+1}^m \dot{h}_j(t) B_j x(t - h_j(t)) + \sum_{l=1}^{m_0} & B_l x(t - h_l(t)) + Cx(t - \tau(t)) \dot{\xi}(t), \end{aligned}$$

where

$$B = \sum_{j=m_0+1}^m B_j, \quad 0 \leq m_0 \leq m,$$

we obtain the new modification of Theorem 4 after replacement in the equation (25) the all sums type of $\sum_{j=1}^{m_0}$ on $\sum_{j=m_0+1}^m$ and the all sums type of $\sum_{i=m_0+1}^m$ on $\sum_{i=1}^{m_0}$. Using other combinations of the summands for representations of the equation (18), we will obtain other modifications of Theorem 4.

EXAMPLE 5. Consider the scalar case of the equation (18) for $m = 2$. Using Theorem 4 and Remark 5 for different values of $m_0 = 0, 1, 2$, we obtain four different sufficient conditions of asymptotic mean square stability of the scalar equation (18) zero solution:

$$A + |B_1| + |B_2| + \frac{1}{2}C^2 < 0,$$

$$(A+B_1)(1-h_1(0)|B_1|)+(\alpha_1|B_1|+|B_2|)(1+h_1(0)|B_1|)+\frac{1}{2}C^2 < 0, \quad h_1(0)|B_1| < 1,$$

$$(A+B_2)(1-h_2(0)|B_2|)+(|B_1|+\alpha_2|B_2|)(1+h_2(0)|B_2|)+\frac{1}{2}C^2 < 0, \quad h_2(0)|B_2| < 1,$$

$$(A + B_1 + B_2)(1 - h_1(0)|B_1| - h_2(0)|B_2|) + (\alpha_1|B_1| + \alpha_2|B_2|)(1 + h_1(0)|B_1| + h_2(0)|B_2|) + \frac{1}{2}C^2 < 0, \quad h_1(0)|B_1| + h_2(0)|B_2| < 1.$$

REMARK 6. Note that sufficient asymptotic mean square stability conditions for stochastic systems with several delays in stochastic part of the equation under consideration or for systems with distributed delay can be obtained by analogously way.

REFERENCES

- [1] V.B. Kolmanovskii, V.R. Nosov, *Stability of Functional Differential Equations*, N.-Y., Academic Press, 1986.
- [2] V.B. Kolmanovskii, A.D. Myshkis, *Applied Theory of Functional Differential Equations*, Boston, Kluwer Academic Publishers, 1992.
- [3] V.B. Kolmanovskii, L.E. Shaikhet, *Control of Systems with Aftereffect*, Translations of mathematical monographs, vol.157, American Mathematical Society, Providence, RI, 1996.
- [4] V.B. Kolmanovskii, L.E. Shaikhet, Stability of stochastic hereditary systems, *Avtomatika i telemekhanika*, **7** (1993), 66-85 (in Russian).
- [5] V.B. Kolmanovskii, About stability of some systems with aftereffect, *Avtomatika i telemekhanika*, **11** (1993), 45-59 (in Russian).

- [6] V.B. Kolmanovskii, L.E. Shaikhet, On one method of Lyapunov functionals construction for stochastic hereditary systems, *Differentsialniye uravneniya*, **11** (1993), 1909-1920 (in Russian).
- [7] V.B. Kolmanovskii, L.E. Shaikhet, New results in stability theory for stochastic functional differential equations (SFDEs) and their applications, *Dynamic Publishers Inc. Proceedings of Dynamic Systems and Applications*, **1** (1994), 167-171.
- [8] V.B. Kolmanovskii, L.E. Shaikhet, General method of Lyapunov functionals construction for stability investigation of stochastic difference equations, *Dynamic Systems and Applications*, *WSSIAA* **4** (1995), 397-439.
- [9] L.E. Shaikhet, Stability in probability of nonlinear stochastic systems with delay, *Matematicheskkiye zametki*, **57**, N.1 (1995), 142-146 (in Russian).
- [10] L.E. Shaikhet, Stability in probability of nonlinear stochastic hereditary systems, *Dynamic Systems and Applications*, **4**, N.2 (1995), 199-204.
- [11] V.B. Kolmanovskii, L.E. Shaikhet, A method of Lyapunov functionals construction for stochastic differential equations of neutral type, *Differentsialniye uravneniya*, **11** (1995) (in Russian).
- [12] V.B. Kolmanovskii, L.E. Shaikhet, Asymptotic behaviour of some systems with discrete time, *Avtomatika i telemekhanika*, **12** (1996) (in Russian).
- [13] L.E. Shaikhet, Modern state and development perspectives of Lyapunov functionals method in the stability theory of stochastic hereditary systems, *Theory of Stochastic Processes*, **2(18)**, N.1-2 (1996), 248-259.
- [14] L.E. Shaikhet, Necessary and sufficient conditions of asymptotic mean square stability for stochastic linear difference equations, *Applied Mathematics Letters*, **10**, N.3 (1997), 111-115.
- [15] R.Z. Khasminskii, *Stability of differential equations systems by stochastic perturbations of its parameters*, Moscow, "Nauka", 1969.