A NEW VIEW ON ONE PROBLEM OF ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF DELAY DIFFERENCE EQUATIONS

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One known theorem on the asymptotic behavior of solution of linear delay difference equation is considered where a stability criterion is derived via a positive root of the corresponding characteristic equation. Two new directions for further investigation are proposed. The first direction is connected with a weakening of the known stability criterion; the second one is connected with consideration of negative and complex roots of the characteristic equation. A lot of pictures with stability regions and trajectories of considered processes are presented for visual demonstration of the proposed directions.

1. Introduction: statement of the problem

There is a series of papers (see, e.g., [5–12]) where a similar method is used for investigation of asymptotic behavior of solutions of difference equations [5, 7], and differential equations [9, 10, 12], integro-differential equations [6, 8], and difference equations with continuous time [11]. The basic assumption in this method is that the positive root of the corresponding characteristic equation satisfies a special sufficient condition for asymptotic stability of some auxiliary equation. Here on the example of Volterra difference equation it is proposed to improve the results of these investigations in two directions. Firstly it is shown that the basic assumption on the positive root of the corresponding characteristic equation can be essentially weaken using different conditions for asymptotic stability. Besides of that it is shown that consideration of negative and complex roots of the characteristic equation gives some new horizons for investigation. For visual demonstration of the proposed ideas, a lot of pictures with numerical calculations of stability regions and trajectories of considered processes are presented.

Consider the Volterra difference equation

$$\Delta x_n = ax_n + \sum_{j=1}^{\infty} K_j x_{n-j}, \quad n \geq 0, \tag{1.1}$$

with the initial condition

$$x_j = \phi_j, \quad j \leq 0. \tag{1.2}$$

Here $\Delta x_n = x_{n+1} - x_n$, a and K_j, $j = 1, 2, \ldots$, are real numbers. The equation

$$\lambda - 1 = a + \sum_{j=1}^{\infty} \lambda^{-j} K_j \tag{1.3}$$

is called the characteristic equation of difference equation (1.1).

THEOREM 1.1. *Let λ_0 be a positive root of characteristic equation (1.3) with the property*

$$\frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} j |K_j| < 1. \tag{1.4}$$

Then for any initial sequence ϕ_j, $j \leq 0$, the solution of (1.1), (1.2) satisfies the condition

$$\lim_{n \to \infty} \lambda_0^{-n} x_n = Q_{\lambda_0}(\phi), \tag{1.5}$$

where

$$Q_{\lambda_0}(\phi) = \frac{L_{\lambda_0}(\phi)}{1 + \gamma_{\lambda_0}}, \qquad \gamma_{\lambda_0} = \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} j K_j,$$

$$L_{\lambda_0}(\phi) = \phi_0 + \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} K_j \left(\sum_{r=-j}^{-1} \lambda_0^{-r} \phi_r \right). \tag{1.6}$$

The proof of Theorem 1.1 follows from [5] where, in particular, it is shown that the sequence

$$z_n = \lambda_0^{-n} x_n - Q_{\lambda_0}(\phi) \tag{1.7}$$

is a solution of the linear difference equation

$$z_n = -\frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} K_j \left(\sum_{r=n-j}^{n-1} z_r \right), \quad n > 0, \tag{1.8}$$

and by condition (1.4) z_n, defined by (1.7), converges to zero that is equivalent to (1.5).

Two following questions arise here.

Firstly, it is clear that condition (1.4) is a sufficient condition for asymptotic stability of the trivial solution of (1.8). But is condition (1.4) a unique or the best sufficient condition?

Secondly, why only a positive root of (1.3) is considered here? Which is a situation in the case of negative or complex root?

Below it is shown that condition (1.4) of Theorem 1.1 can be weaken and the negative and complex roots of (1.3) also can be useful for investigation of asymptotic behavior of the solution of (1.1), (1.2).

2. Improvement of the known result

Rewrite equation (1.8) in the form

$$z_n = \sum_{l=1}^{\infty} a_l z_{n-l}, \quad n > 0, \quad a_l = -\frac{1}{\lambda_0} \sum_{j=l}^{\infty} \lambda_0^{-j} K_j. \tag{2.1}$$

Different sufficient conditions for asymptotic stability of the trivial solution of difference Volterra equation type of (2.1) were obtained in [1–4, 13] via the general method of Lyapunov functionals construction.

In particular, if

$$\sum_{l=1}^{\infty} |a_l| < 1, \tag{2.2}$$

then the trivial solution of (2.1) is asymptotically stable [1]. Condition (2.2) is weaker than (1.4). Really,

$$\sum_{l=1}^{\infty} |a_l| \le \frac{1}{\lambda_0} \sum_{l=1}^{\infty} \sum_{j=l}^{\infty} \lambda_0^{-j} |K_j| = \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \sum_{l=1}^{j} \lambda_0^{-j} |K_j| = \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} j |K_j| < 1. \tag{2.3}$$

Another sufficient condition for asymptotic stability of the trivial solution of difference Volterra equation (2.1) has [1] the following form: if $2\alpha - 1 < \beta < 1$, where

$$\beta = \sum_{l=1}^{\infty} a_l, \quad \alpha = \sum_{l=1}^{\infty} B_l, \quad B_l = \left| \sum_{j=l+1}^{\infty} a_j \right|, \tag{2.4}$$

then the trivial solution of (2.1) is asymptotically stable.

So, the following theorem holds.

THEOREM 2.1. *Let λ_0 be a positive root of characteristic equation (1.3) that satisfies the property*

$$\frac{1}{\lambda_0} \sum_{l=1}^{\infty} \left| \sum_{j=l}^{\infty} \lambda_0^{-j} K_j \right| < 1 \tag{2.5}$$

or the property

$$2\alpha - 1 < \beta < 1,$$

$$\alpha = \frac{1}{\lambda_0} \sum_{l=1}^{\infty} \left| \sum_{j=l+1}^{\infty} (j-l)\lambda_0^{-j} K_j \right|, \qquad \beta = -\frac{1}{\lambda_0} \sum_{l=1}^{\infty} \sum_{j=l}^{\infty} \lambda_0^{-j} K_j. \tag{2.6}$$

Then for any initial sequence ϕ_j, $j \le 0$, the solution of (1.1), (1.2) satisfies condition (1.5), (1.6).

From (2.3) it follows that condition (2.5) is weaker than (1.4). To compare conditions (1.4), (2.5), and (2.6) consider the following example.

Example 2.2. Consider the difference equation

$$\Delta x_n = a x_n + K_1 x_{n-1} + K_2 x_{n-2}. \tag{2.7}$$

Auxiliary difference equation (2.1) in this case has the form

$$z_n = -(K_1 \lambda_0^{-2} + K_2 \lambda_0^{-3}) z_{n-1} - K_2 \lambda_0^{-3} z_{n-2}. \tag{2.8}$$

Conditions (1.4), (2.5), and (2.6) are correspondingly

$$|K_1| \lambda_0^{-2} + 2|K_2| \lambda_0^{-3} < 1, \tag{2.9}$$

$$|K_1 \lambda_0^{-2} + K_2 \lambda_0^{-3}| + |K_2| \lambda_0^{-3} < 1, \tag{2.10}$$

$$-1 < K_1 \lambda_0^{-2} + 2 K_2 \lambda_0^{-3} < 1 - 2|K_2| \lambda_0^{-3}. \tag{2.11}$$

It is well known also [13] that the necessary and sufficient condition for asymptotic stability of the trivial solution of (2.8) is

$$|K_1 \lambda_0^{-2} + K_2 \lambda_0^{-3}| < 1 + K_2 \lambda_0^{-3}, \qquad |K_2| \lambda_0^{-3} < 1. \tag{2.12}$$

One can see that condition (2.12) follows from each of conditions (2.9), (2.10), and (2.11). From each of these conditions it follows also that $1 + \gamma_{\lambda_0} = 1 + K_1 \lambda_0^{-2} + 2 K_2 \lambda_0^{-3} > 0$, so $Q_{\lambda_0}(\phi)$ in (1.6) is defined.
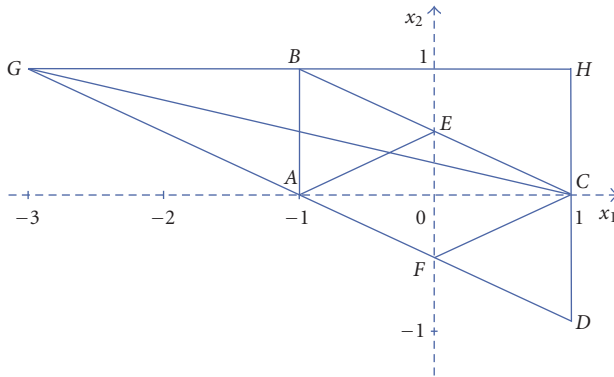
Figure 2.1. Different stability regions.

On Figure 2.1 stability regions for (2.8) are shown constructed by conditions (2.9) (region AECF), (2.10) (region ABCD), (2.11) (region GCD), and (2.12) (region GHD) in the space (x_1, x_2), where $x_1 = K_1 \lambda_0^{-2}$, $x_2 = K_2 \lambda_0^{-3}$.

3. Different situations with roots of the characteristic equation

To demonstrate the different situations of the use not only positive but also negative and complex roots of characteristic equation (1.3) consider the simple difference equation

$$\Delta x_n = a x_n + b x_{n-1}, \quad n = 0, 1, \ldots,$$

$$x_j = \phi_j, \quad j = -1, 0.$$

(3.1)

The corresponding characteristic equation is

$$\lambda - 1 = a + b \lambda^{-1}.$$

(3.2)

The following theorem deals with behavior of the sequences x_n and $y_n = \lambda_0^{-n} x_n$, where x_n is a solution of (3.1) and λ_0 is a root of characteristic equation (3.2).

THEOREM 3.1. *There are four different situations with a solution of (3.1).*
    (1) *If*

$$a + 1 \neq 0, \quad (a+1)^2 + 4b > 0,$$

(3.3)

then

$$\lim_{n \to \infty} y_n = Q_{\lambda_0}(\phi),$$

(3.4)

where

$$Q_{\lambda_0}(\phi) = \frac{L_{\lambda_0}(\phi)}{1 + \lambda_0^{-2}b}, \qquad L_{\lambda_0}(\phi) = \phi_0 + \lambda_0^{-1}b\phi_{-1}, \tag{3.5}$$

$$\lambda_0 = \frac{a+1}{2}\left(1 + \sqrt{1 + \frac{4b}{(a+1)^2}}\right). \tag{3.6}$$

(2) If

$$a + 1 = 0, \quad b > 0, \tag{3.7}$$

then $\lambda_0 = \pm\sqrt{b}$ *and*

$$y_{2k} = \phi_0, \quad y_{2k+1} = \lambda_0\phi_{-1}, \quad k = 0, 1, \dots. \tag{3.8}$$

(3) If

$$a + 1 \neq 0, \qquad (a+1)^2 + 4b = 0, \tag{3.9}$$

then

$$y_n = \phi_0 + nL_{\lambda_0}(\phi), \quad n = 0, 1, \dots, \tag{3.10}$$

where $L_{\lambda_0}(\phi)$ *is defined by (3.5) and* $\lambda_0 = (1/2)(a+1)$.
   (4) If

$$(a+1)^2 + 4b < 0, \tag{3.11}$$

then

$$\left| y_n - Q_{\lambda_0}(\phi) \right| = \left| \phi_0 - Q_{\lambda_0}(\phi) \right|, \quad n = 0, 1, \dots, \tag{3.12}$$

where

$$Q_{\lambda_0}(\phi) = \frac{\phi_0}{2} \mp i\frac{\phi_0(a+1) + 2b\phi_{-1}}{2\sqrt{|(a+1)^2 + 4b|}}, \tag{3.13}$$

and λ_0 *is one of two conjugate complex roots*

$$\lambda_0 = \frac{a + 1 \pm i\sqrt{|(a+1)^2 + 4b|}}{2}, \quad i = \sqrt{-1}, \tag{3.14}$$

of characteristic equation (3.2). It means that the values of the process y_n *are located in a complex plane on the circle with the center* $Q_{\lambda_0}(\phi)$ *and the radius* $r = |\phi_0 - Q_{\lambda_0}(\phi)|$. *This circle includes the points* 0 *and* ϕ_0.

Proof. (1) Let us suppose that condition (3.3) holds. Put

$$z_n = y_n - Q_{\lambda_0}(\phi), \quad y_n = \lambda_0^{-n} x_n, \tag{3.15}$$

where x_n is a solution of (3.1) and λ_0 is a root of characteristic equation (3.2).

By condition (3.3), (3.2) has two real roots

$$\lambda_{1,2} = \frac{a+1 \pm \sqrt{(a+1)^2 + 4b}}{2} = \frac{a+1}{2}\left(1 \pm \sqrt{1 + \frac{4b}{(a+1)^2}}\right). \tag{3.16}$$

From (2.1) it follows that sequence (3.15) satisfies the equation

$$z_n = -\lambda_0^{-2} b z_{n-1}, \quad n = 0, 1, \dots. \tag{3.17}$$

The necessary and sufficient condition for asymptotic stability of the trivial solution of (3.17) is

$$|\lambda_0^{-2} b| < 1. \tag{3.18}$$

From (3.2) it follows that condition (3.18) is equivalent to $|1 - (a+1)\lambda_0^{-1}| < 1$ or

$$\lambda_0 (a+1)^{-1} > \frac{1}{2}. \tag{3.19}$$

It is easy to see that from two roots (3.16) of (3.2) root (3.6) only satisfies condition (3.19). So (3.4) is proven.

(2) By conditions (3.7) from (3.2) it follows that $\lambda_0^{-2} b = 1$. Equation (3.17) takes the form $z_n = -z_{n-1}$. Therefore, $z_n = (-1)^n z_0$, $n = 1, 2, \dots$. Via (3.15), (3.5) from here we have

$$\begin{aligned} y_n &= Q_{\lambda_0}(\phi) + (-1)^n [\phi_0 - Q_{\lambda_0}(\phi)] \\ &= (-1)^n \phi_0 + \frac{1}{2}[1 - (-1)^n][\phi_0 + (\lambda_0^{-2} b)\lambda_0 \phi_{-1}] \\ &= \frac{1}{2}[1 + (-1)^n]\phi_0 + \frac{\lambda_0}{2}[1 - (-1)^n]\phi_{-1} \end{aligned} \tag{3.20}$$

that is equivalent to (3.8).

(3) By condition (3.9) the solution of (3.2) is $\lambda_0 = (1/2)(a+1)$. From here and (3.9) it follows that $1 + \lambda_0^{-2} b = 0$ and, therefore, $Q_{\lambda_0}(\phi)$ in (3.5) is undefined. It means that sequence (3.15) undefined too. Using $y_j = \lambda_0^{-j} x_j$, $j = 0, 1, \dots$, (3.1), (3.2), and $\lambda_0^{-1} b = -\lambda_0$, we have

$$\begin{aligned} \Delta x_j - a x_j - b x_{j-1} &= \Delta(\lambda_0^j y_j) - a\lambda_0^j y_j - b\lambda_0^{j-1} y_{j-1} \\ &= \lambda_0^j [\lambda_0 \Delta y_j + (\lambda_0 - 1 - a) y_j - b\lambda_0^{-1} y_{j-1}] \\ &= \lambda_0^{j+1} [\Delta y_j - \Delta y_{j-1}] = 0. \end{aligned} \tag{3.21}$$
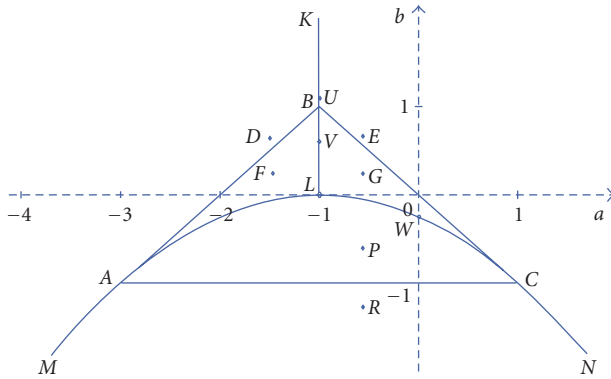
Figure 3.1. Regions with different behavior of x_n and y_n.

From here via (3.5) it follows that

$$\Delta y_j = \Delta y_{j-1} = y_0 - y_{-1} = x_0 - \lambda_0 x_{-1} = \phi_0 + \lambda_0^{-1} b \phi_{-1} = L_{\lambda_0}(\phi) \tag{3.22}$$

or $y_j = y_{j-1} + L_{\lambda_0}(\phi)$. Summing this equality with respect to $j = 1, 2, \ldots, n$, we obtain (3.10).

(4) Let us suppose now that condition (3.11) holds. Then the conjugate complex roots of (3.2) are defined by (3.14) and satisfy the condition $|\lambda_0|^2 = -b = |b|$ or $|\lambda_0^{-2} b| = 1$. From (3.17) it follows that process (3.15) satisfies the equation $|z_n| = |z_{n-1}|$ or $|z_n| = |z_0|$. It is equivalent to (3.12). Now it is enough to show that $Q_{\lambda_0}(\phi)$ defined by (3.5) equals $Q_{\lambda_0}(\phi)$ defined by (3.13). Really, putting $\delta = \sqrt{|(a+1)^2 + 4b|}$, from (3.14) we obtain $2\lambda_0 - (a+1) = \pm i\delta$. Using (3.2), (3.5), and (3.14), one can transform $Q_{\lambda_0}(\phi)$ by the following way:

$$Q_{\lambda_0}(\phi) = \frac{L_{\lambda_0}(\phi)}{2 - \lambda_0^{-1}(a+1)} = \frac{\lambda_0 L_{\lambda_0}(\phi)}{2\lambda_0 - (a+1)} = \frac{\lambda_0 L_{\lambda_0}(\phi)}{\pm i\delta} = \frac{2i\lambda_0 L_{\lambda_0}(\phi)}{\mp 2\delta}$$

$$= \frac{i(\phi_0(a+1) \pm i\phi_0\delta + 2b\phi_{-1})}{\mp 2\delta} = \frac{\phi_0}{2} \mp i\frac{\phi_0(a+1) + 2b\phi_{-1}}{2\delta}. \tag{3.23}$$

The theorem is proven.   □

Four regions described in Theorem 3.1 are shown on Figure 3.1: (1) at the left of the curve KLM and from the right of the curve KLN; (2) the line KL; (3) the curve MLN; (4) under the curve MLN. The point L with the coordinates $a = -1$, $b = 0$ is excluded from the consideration since in this point $\lambda_0 = 0$. The inside of the triangle ABC is the region of asymptotic stability of the trivial solution of (3.1).

Below on Figures 3.2, 3.4, and 3.5 the first situation from Theorem 3.1 is shown.

On Figure 3.2 the trajectories of the processes x_n and y_n are shown in the point D (it is shown on Figure 3.1) with the coordinates $a = -1.5$, $b = 0.65$. Here $\phi_{-1} = 2$, $\phi_0 = 0.5$, $\lambda_0 = -1.094$ (a negative root). The point P does not belong to the stability region (the
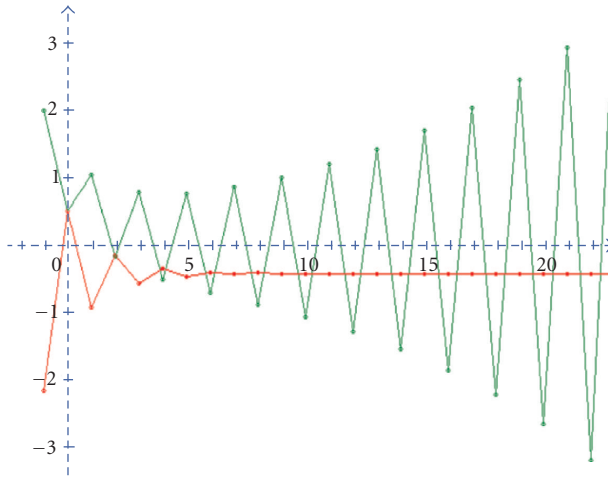
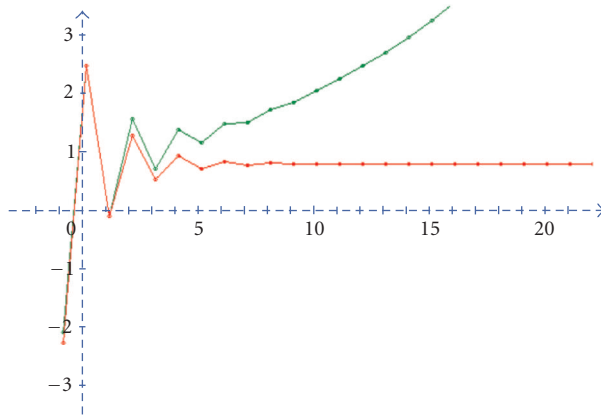Figure 3.2. Regions with different behavior of x_n and y_n in the point D.

Figure 3.3. Behavior of x_n and y_n in the point E.

triangle ABC) of the trivial solution of (3.1), so the process x_n (green) goes to $\pm\infty$. The process y_n (red) enough quickly converges to $Q_{\lambda_0}(\phi) = -0.446$.

On Figure 3.3 the similar trajectories of the processes x_n and y_n are shown in the point E (Figure 3.1). Here $a = -0.5$, $b = 0.65$, $\phi_{-1} = -2$, $\phi_0 = 2.5$, $\lambda_0 = 1.094$ (a positive root), the process x_n (green) goes to $+\infty$, the process y_n (red) quickly converges to $Q_{\lambda_0}(\phi) = 0.850$.

On Figure 3.4 the trajectories of the processes x_n and y_n are shown in the point F (Figure 3.1) with the coordinates $a = -1.5$, $b = 0.25$. Here $\phi_{-1} = 3$, $\phi_0 = -1.5$, $\lambda_0 = -0.809$ (a negative root). The point F belongs to the stability region (the triangle ABC)
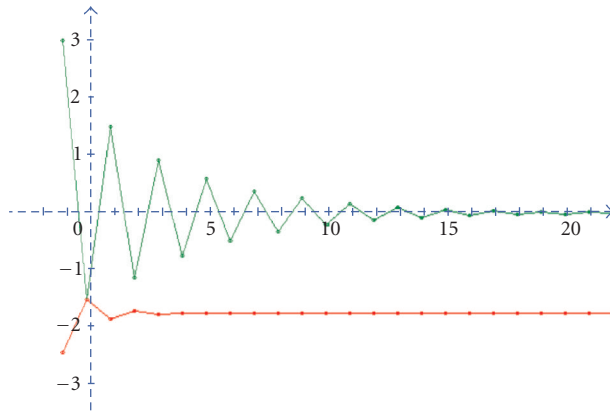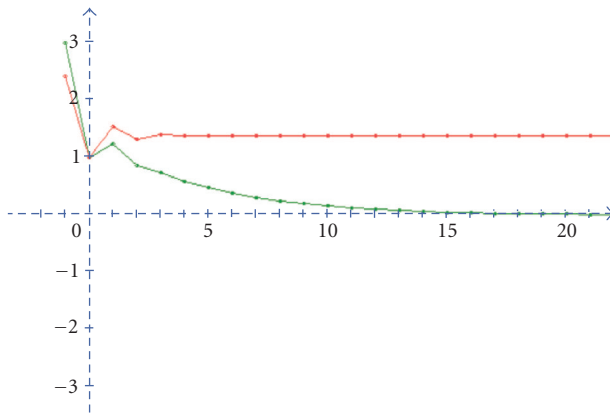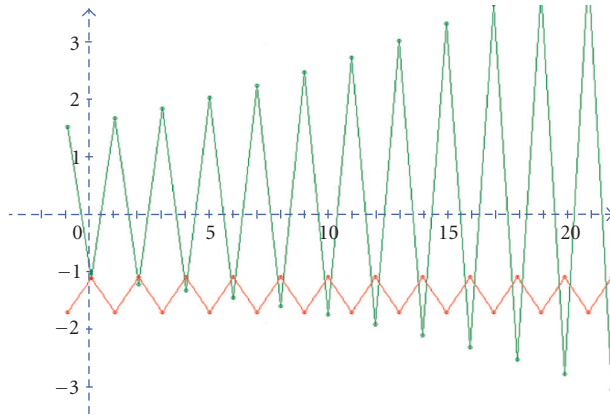
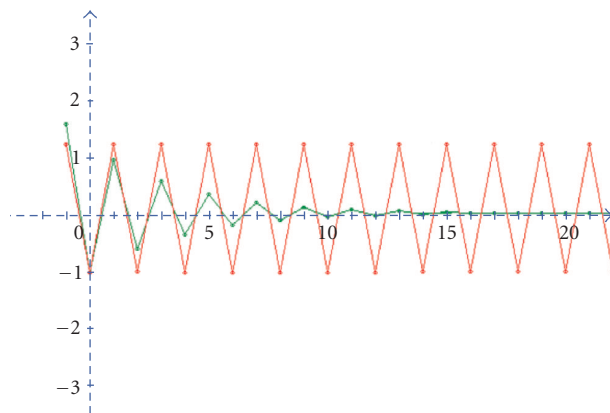Figure 3.4. Behavior of x_n and y_n in the point F.

Figure 3.5. Behavior of x_n and y_n in the point G.

of the trivial solution of (3.1), so, the process x_n (green) converges to zero. The process y_n (red) quickly converges to $Q_{\lambda_0}(\phi) = -1.756$.

On **Figure 3.5** the similar trajectories of the processes x_n and y_n are shown in the point G (**Figure 3.1**). Here $a = -0.5$, $b = 0.25$, $\phi_{-1} = 3$, $\phi_0 = 1$, $\lambda_0 = 0.809$ (a positive root), the process x_n (green) converges to zero, the process y_n (red) quickly converges to $Q_{\lambda_0}(\phi) = 1.394$.

On Figures 3.6 and 3.7 the second situation from **Theorem 3.1** is shown.

On **Figure 3.6** the trajectories of the processes x_n and y_n are shown in the point U (**Figure 3.1**) with the coordinates $a = -1$, $b = 1.1$. Here $\phi_{-1} = 1.5$, $\phi_0 = -1$, $\lambda_0 = -1.049$ (a negative root). The point U does not belong to the stability region (the triangle ABC) of the trivial solution of (3.1), so the process x_n (green) goes to $\pm\infty$. The process y_n (red) has two values: $\phi_0 = -1$ and $\lambda_0\phi_{-1} = -1.573$.

Figure 3.6. Behavior of x_n and y_n in the point U.

Figure 3.7. Behavior of x_n and y_n in the point V.

On **Figure 3.7** the similar trajectories of the processes x_n and y_n are shown in the point V (**Figure 3.1**). Here $a = -1$, $b = 0.6$, $\phi_{-1} = 1.5$, $\phi_0 = -1$, $\lambda_0 = 0.775$ (a positive root), the process x_n (green) converges to zero, the process y_n (red) has two values: $\phi_0 = -1$ and $\lambda_0\phi_{-1} = 1.162$.

On **Figures 3.8** and **3.9** the third situation from **Theorem 3.1** is shown.

On **Figure 3.8** the trajectories of the processes x_n and y_n are shown in the point W (**Figure 3.1**) with the coordinates $a = 0$, $b = -0.25$. Here $\phi_{-1} = 3.5$, $\phi_0 = 1.6$, $\lambda_0 = 0.5$ (a positive root), $L_{\lambda_0}(\phi) = -0.15$. The point W belongs to the stability region (the triangle ABC) of the trivial solution of (3.1), so the process x_n (green) converges to zero. The process y_n (red) is a straight line.

On **Figure 3.9** the trajectories of the processes x_n and y_n are shown in the point A (**Figure 3.1**) with the coordinates $a = -3$, $b = -1$. Here $\phi_{-1} = 1.2$, $\phi_0 = -1$, $\lambda_0 = -1$
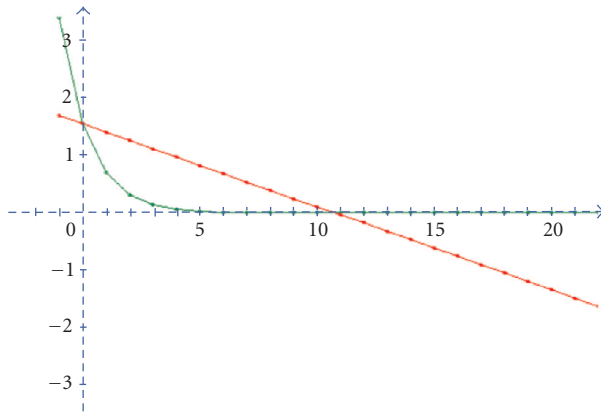
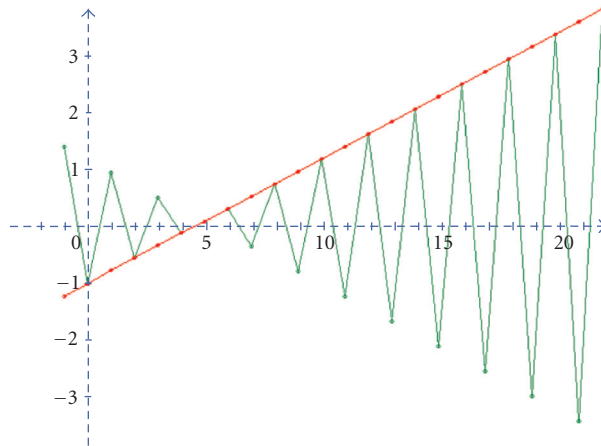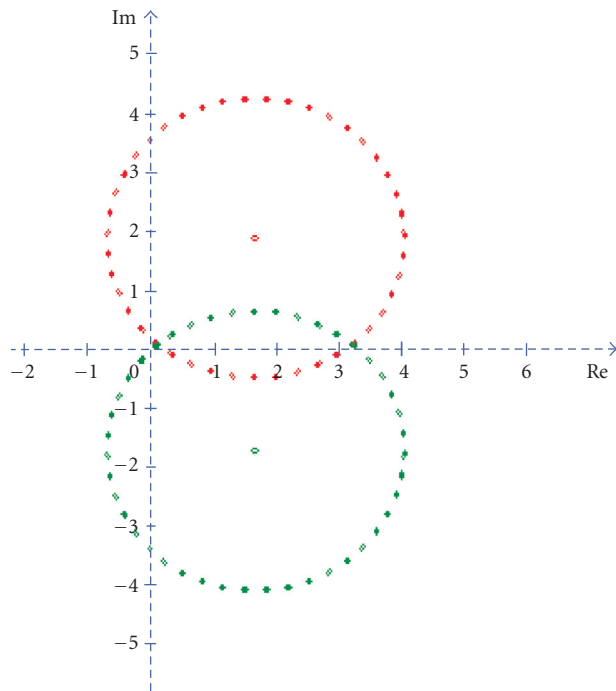Figure 3.8. Behavior of x_n and y_n in the point W.

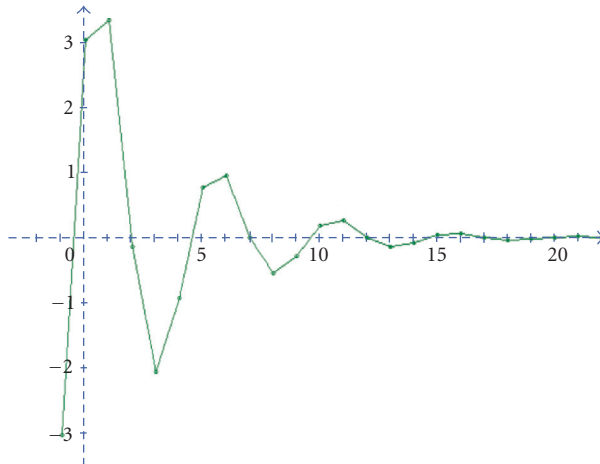Figure 3.9. Behavior of x_n and y_n in the point A.

(a negative root), $L_{\lambda_0}(\phi) = 0.2$. The point A does not belong to the stability region (the triangle ABC) of the trivial solution of (3.1), so the process x_n (green) goes to $\pm\infty$. The process y_n (red) is a straight line.

On Figures 3.10 and 3.11 the fourth situation from **Theorem 3.1** is shown.

On **Figure 3.10(a)** the trajectory of the complex process y_n is shown in the point P (**Figure 3.1**) with the coordinates $a = -0.5$, $b = -0.6$. Here $\phi_{-1} = -3$, $\phi_0 = 3$. One can see that the values of the process y_n are located in the complex plane on the circle with radius $r = 2.297$ and the center $Q_{\lambda_0}(\phi) = 1.5 - i1.739$ (green) if $\lambda_0 = 0.25 + i0.733$ and $Q_{\lambda_0}(\phi) = 1.5 + i1.739$ (red) if $\lambda_0 = 0.25 - i0.733$. On **Figure 3.10(b)** the trajectory of the process x_n is shown in the same point P (**Figure 3.1**). This point belongs to the stability
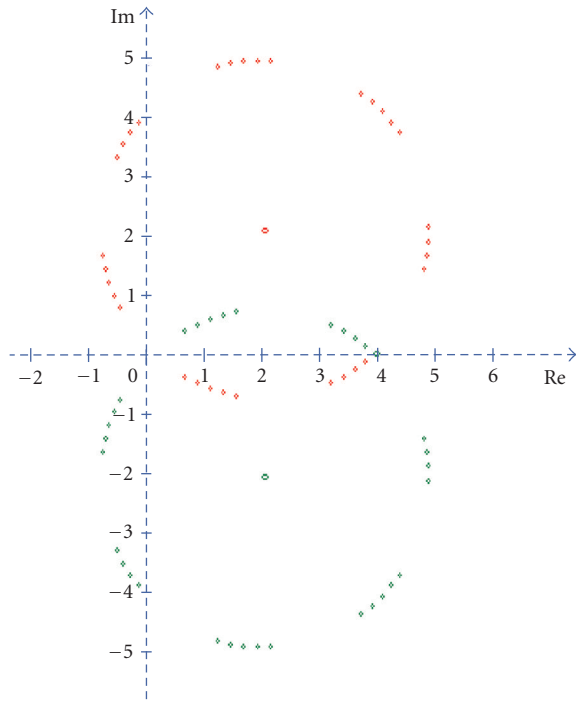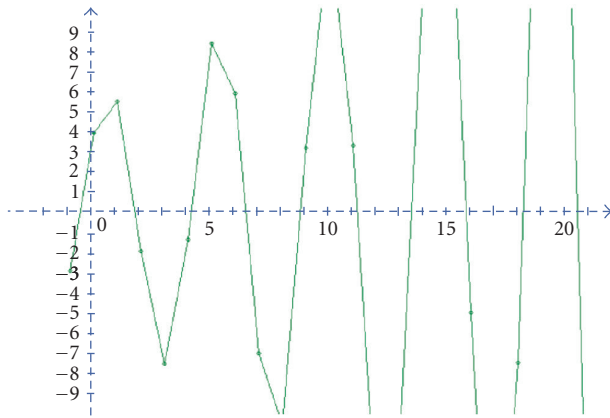
(a)

(b)

Figure 3.10. (a) Behavior of y_n in the point P; (b) behavior of x_n in the point P.

(a)

(b)

Figure 3.11. (a) Behavior of y_n in the point R; (a) behavior of x_n in the point R.

region (the triangle ABC) of the trivial solution of (3.1), so the process x_n converges to zero.

On Figure 3.11(a) the trajectory of the complex process y_n is shown in the point R (Figure 3.1) with the coordinates $a = -0.5$, $b = -1.2$. Here $\phi_{-1} = -3$, $\phi_0 = 4$. One can see that the values of the process y_n are located in the complex plane on the circle with radius $r = 2.941$ and the center $Q_{\lambda_0}(\phi) = 2 - i2.157$ (green) if $\lambda_0 = 0.25 + i1.067$ and $Q_{\lambda_0}(\phi) = 2 + i2.157$ (red) if $\lambda_0 = 0.25 - i1.067$. On Figure 3.11(b) the trajectory of the process x_n is shown in the same point R (Figure 3.1). This point does not belong to the stability region (the triangle ABC) of the trivial solution of (3.1), so the process x_n goes to $\pm\infty$.

4. Conclusion

In this paper it is shown that the known results type of Theorem 1.1 (see [5–12]) can be improved similar to Theorem 2.1 by virtue of different stability conditions obtained via general method of Lyapunov functionals construction [1–4, 13]. On the other hand, it is noted that the results of the papers [5–12] in general case can be essentially extended similar to Theorem 3.1 via consideration of not only positive but also negative and complex roots of the corresponding characteristic equation. These ideas can be applied both for difference and for functional-differential equations.

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