LYAPUNOV FUNCTIONALS CONSTRUCTION FOR STOCHASTIC DIFFERENCE SECOND-KIND VOLTERRA EQUATIONS WITH CONTINUOUS TIME

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The general method of Lyapunov functionals construction which was developed during the last decade for stability investigation of stochastic differential equations with aftereffect and stochastic difference equations is considered. It is shown that after some modification of the basic Lyapunov-type theorem, this method can be successfully used also for stochastic difference Volterra equations with continuous time usable in mathematical models. The theoretical results are illustrated by numerical calculations.

1. Stability theorem

Construction of Lyapunov functionals is usually used for investigation of stability of hereditary systems which are described by functional differential equations or Volterra equations and have numerous applications [3, 4, 8, 21]. The general method of Lyapunov functionals construction for stability investigation of hereditary systems was proposed and developed (see [2, 5, 6, 7, 9, 10, 11, 12, 13, 17, 18, 19]) for both stochastic differential equations with aftereffect and stochastic difference equations. Here it is shown that after some modification of the basic Lyapunov-type stability theorem, this method can also be used for stochastic difference Volterra equations with continuous time, which are popular enough in researches [1, 14, 15, 16, 20].

Let $\{\Omega,\mathfrak{F},\mathbf{P}\}$ be a probability space, $\{\mathfrak{F}_t,\ t\geq t_0\}$ a nondecreasing family of sub-σ-algebras of $\mathfrak{F}$, that is, $\mathfrak{F}_{t_1}\subset\mathfrak{F}_{t_2}$ for $t_1<t_2$, and H a space of $\mathfrak{F}_t$-measurable functions $x(t)\in\mathbb{R}^n$, $t\geq t_0$, with norms

$$\|x\|^2=\sup_{t\geq t_0}\mathbf{E}\left|x(t)\right|^2,\qquad \|x\|_1^2=\sup_{t\in[t_0,t_0+h_0]}\mathbf{E}\left|x(t)\right|^2. \tag{1.1}$$

Consider the stochastic difference equation

$$x(t+h_0)=\eta(t+h_0)+F\big(t,x(t),x(t-h_1),x(t-h_2),\dots\big),\quad t>t_0-h_0, \tag{1.2}$$

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with the initial condition

$$x(\theta) = \phi(\theta), \quad \theta \in \Theta = \left[t_0 - h_0 - \max_{j \geq 1} h_j, t_0\right]. \tag{1.3}$$

Here, $\eta \in H$, $h_0, h_1, \dots$ are positive constants, and $\phi(\theta)$, $\theta \in \Theta$, is an $\mathfrak{F}_{t_0}$-measurable function such that

$$\|\phi\|_0^2 = \sup_{\theta \in \Theta} \mathbf{E} \left| \phi(\theta) \right|^2 < \infty, \tag{1.4}$$

the functional $F \in \mathbb{R}^n$ satisfies the condition

$$\left| F(t, x_0, x_1, x_2, \dots) \right|^2 \leq \sum_{j=0}^{\infty} a_j \left| x_j \right|^2, \qquad A = \sum_{j=0}^{\infty} a_j < \infty. \tag{1.5}$$

A solution of problem (1.2), (1.3) is an $\mathfrak{F}_t$-measurable process $x(t) = x(t; t_0, \phi)$, which is equal to the initial function $\phi(t)$ from (1.3) for $t \leq t_0$ and with probability 1 defined by (1.2) for $t > t_0$.

Definition 1.1. A function $x(t)$ from H is called

(i) uniformly mean square bounded if $\|x\|^2 < \infty$;

(ii) asymptotically mean square trivial if

$$\lim_{t \to \infty} \mathbf{E} \left| x(t) \right|^2 = 0; \tag{1.6}$$

(iii) asymptotically mean square quasitrivial if, for each $t \geq t_0$,

$$\lim_{j \to \infty} \mathbf{E} \left| x(t + jh_0) \right|^2 = 0; \tag{1.7}$$

(iv) uniformly mean square summable if

$$\sup_{t \geq t_0} \sum_{j=0}^{\infty} \mathbf{E} \left| x(t + jh_0) \right|^2 < \infty; \tag{1.8}$$

(v) mean square integrable if

$$\int_{t_0}^{\infty} \mathbf{E} \left| x(t) \right|^2 dt < \infty. \tag{1.9}$$

Remark 1.2. It is easy to see that if the function $x(t)$ is uniformly mean square summable, then it is uniformly mean square bounded and asymptotically mean square quasitrivial.

Remark 1.3. It is evident that condition (1.7) follows from (1.6), but the inverse statement is not true. The corresponding function is considered in Example 5.1.

Together with (1.2) we will consider the auxiliary difference equation

$$x(t + h_0) = F(t, x(t), x(t - h_1), x(t - h_2), \dots), \quad t > t_0 - h_0, \tag{1.10}$$

with initial condition (1.3) and the functional F, satisfying condition (1.5).

Definition 1.4. The trivial solution of (1.10) is called

(i) mean square stable if, for any $\epsilon > 0$ and $t_0 \geq 0$, there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that $\|x(t)\|^2 < \epsilon$ if $\|\phi\|_0^2 < \delta$;
(ii) asymptotically mean square stable if it is mean square stable and for each initial function ϕ, condition (1.6) holds;
(iii) asymptotically mean square quasistable if it is mean square stable and for each initial function ϕ and each $t \in [t_0, t_0 + h_0)$, condition (1.7) holds.

Theorem 1.5. *Let the process $\eta(t)$ in (1.2) satisfy the condition $\|\eta\|_1^2 < \infty$, and there exist a nonnegative functional*

$$V(t) = V(t, x(t), x(t-h_1), x(t-h_2), \dots), \tag{1.11}$$

positive numbers c_1, c_2, and nonnegative function $\gamma(t)$, such that

$$\hat{\gamma} = \sup_{s \in [t_0, t_0 + h_0)} \sum_{j=0}^{\infty} \gamma(s + jh_0) < \infty, \tag{1.12}$$

$$\mathbf{E}V(t) \leq c_1 \sup_{s \leq t} \mathbf{E}|x(s)|^2, \quad t \in [t_0, t_0 + h_0), \tag{1.13}$$

$$\mathbf{E}\Delta V(t) \leq -c_2 \mathbf{E}|x(t)|^2 + \gamma(t), \quad t \geq t_0, \tag{1.14}$$

where

$$\Delta V(t) = V(t + h_0) - V(t). \tag{1.15}$$

Then the solution of (1.2), (1.3) is uniformly mean square summable.

Proof. Rewrite condition (1.14) in the form

$$\mathbf{E}\Delta V(t + jh_0) \leq -c_2 \mathbf{E}\left|x(t + jh_0)\right|^2 + \gamma(t + jh_0), \quad t \geq t_0,\ j = 0, 1, \dots. \tag{1.16}$$

Summing this inequality from $j = 0$ to $j = i$, by virtue of (1.15), we obtain

$$\mathbf{E}V(t + (i+1)h_0) - \mathbf{E}V(t) \leq -c_2 \sum_{j=0}^{i} \mathbf{E}\left|x(t + jh_0)\right|^2 + \sum_{j=0}^{i} \gamma(t + jh_0). \tag{1.17}$$

Therefore,

$$c_2 \sum_{j=0}^{\infty} \mathbf{E}\left|x(t + jh_0)\right|^2 \leq \mathbf{E}V(t) + \sum_{j=0}^{\infty} \gamma(t + jh_0), \quad t \geq t_0. \tag{1.18}$$

We show that the right-hand side of inequality (1.18) is bounded. Really, using (1.14), (1.15), for $t \geq t_0$, we have

$$\begin{aligned} \mathbf{E}V(t) &\leq \mathbf{E}V(t-h_0) + \gamma(t-h_0) \\ &\leq \mathbf{E}V(t-2h_0) + \gamma(t-2h_0) + \gamma(t-h_0) \\ &\leq \cdots \leq \mathbf{E}V(t-ih_0) + \sum_{j=1}^{i} \gamma(t-jh_0) \\ &\leq \cdots \leq \mathbf{E}V(s) + \sum_{j=1}^{\tau} \gamma(t-jh_0), \end{aligned} \tag{1.19}$$

where

$$s = t - \tau h_0 \in [t_0, t_0 + h_0), \quad \tau = \left[\frac{t-t_0}{h_0}\right], \tag{1.20}$$

$[t]$ is the integer part of a number t.

Since $t = s + \tau h_0$, then

$$\begin{aligned} \sum_{j=0}^{\infty} \gamma(t+jh_0) &= \sum_{j=0}^{\infty} \gamma(s+(\tau+j)h_0) = \sum_{j=\tau}^{\infty} \gamma(s+jh_0), \\ \sum_{j=1}^{\tau} \gamma(t-jh_0) &= \sum_{j=1}^{\tau} \gamma(s+(\tau-j)h_0) = \sum_{j=0}^{\tau-1} \gamma(s+jh_0). \end{aligned} \tag{1.21}$$

Therefore, using (1.12), we obtain

$$\sum_{j=0}^{\infty} \gamma(t+jh_0) + \sum_{j=1}^{\tau} \gamma(t-jh_0) = \sum_{j=0}^{\infty} \gamma(s+jh_0) \leq \hat{\gamma}. \tag{1.22}$$

So, from (1.18), (1.19), and (1.22), it follows that

$$c_2 \sum_{j=0}^{\infty} \mathbf{E}\left|x(t+jh_0)\right|^2 \leq \hat{\gamma} + \mathbf{E}V(s), \quad t \geq t_0, \ s = t - \left[\frac{t-t_0}{h_0}\right], h_0 \in [t_0, t_0+h_0). \tag{1.23}$$

Using (1.13), we get

$$\sup_{s \in [t_0, t_0+h_0)} \mathbf{E}V(s) \leq c_1 \sup_{t \leq t_0+h_0} \mathbf{E}\left|x(t)\right|^2 \leq c_1\left[\|\phi\|_0^2 + \|x\|_1^2\right]. \tag{1.24}$$

From (1.2), (1.3), and (1.5), for $t \in [t_0, t_0 + h_0]$, we obtain

$$\begin{aligned} \mathbf{E}|x(t)|^2 &= \mathbf{E}|\eta(t) + F(t-h_0, x(t-h_0), x(t-h_0-h_1), x(t-h_0-h_2), \dots)|^2 \\ &\le 2\Big[\mathbf{E}|\eta(t)|^2 + \mathbf{E}|F(t-h_0, x(t-h_0), x(t-h_0-h_1), x(t-h_0-h_2), \dots)|^2\Big] \\ &\le 2\left[\mathbf{E}|\eta(t)|^2 + a_0\mathbf{E}|\phi(t-h_0)|^2 + \sum_{j=1}^{\infty} a_j \mathbf{E}|\phi(t-h_0-h_j)|^2\right] \\ &\le 2[\|\eta\|_1^2 + A\|\phi\|_0^2]. \end{aligned} \tag{1.25}$$

Using (1.23), (1.24), and (1.25), we have

$$c_2 \sum_{j=0}^{\infty} \mathbf{E}|x(t+jh_0)|^2 \le \hat{\gamma} + c_1[(1+2A)\|\phi\|_0^2 + 2\|\eta\|_1^2]. \tag{1.26}$$

From here and (1.8), it follows that the solution of (1.2), (1.3) is uniformly mean square summable. The theorem is proven. □

Remark 1.6. Replace condition (1.12) in Theorem 1.5 by the condition

$$\int_{t_0}^{\infty} \gamma(t)dt < \infty. \tag{1.27}$$

Then the solution of (1.2) for each initial function (1.3) is mean square integrable. Really, integrating (1.14) from $t = t_0$ to $t = T$, by virtue of (1.15), we have

$$\int_{t_0}^{T} \mathbf{E}(V(t+h_0) - V(t))dt \le -c_2 \int_{t_0}^{T} \mathbf{E}|x(t)|^2 dt + \int_{t_0}^{T} \gamma(t)dt \tag{1.28}$$

or

$$\int_{T}^{T+h_0} \mathbf{E}V(t)dt - \int_{t_0}^{t_0+h_0} \mathbf{E}V(t)dt \le -c_2 \int_{t_0}^{T} \mathbf{E}|x(t)|^2 dt + \int_{t_0}^{T} \gamma(t)dt. \tag{1.29}$$

From here and (1.24), (1.25), it follows that

$$\begin{aligned} c_2 \int_{t_0}^{T} \mathbf{E}|x(t)|^2 dt &\le \int_{t_0}^{t_0+h_0} \mathbf{E}V(t)dt + \int_{t_0}^{T} \gamma(t)dt \\ &\le c_1 h_0[(1+2A)\|\phi\|_0^2 + 2\|\eta\|_1^2] + \int_{t_0}^{\infty} \gamma(t)dt < \infty, \end{aligned} \tag{1.30}$$

and by $T \to \infty$, we obtain (1.9).

Remark 1.7. Suppose that for (1.10) the conditions of Theorem 1.5 hold with $\gamma(t) \equiv 0$. Then the trivial solution of (1.10) is asymptotically mean square quasistable. Really, in the case $\gamma(t) \equiv 0$ from inequality (1.26) for (1.10) ($\eta(t) \equiv 0$), it follows that $c_2\mathbf{E}|x(t)|^2 \le c_1(1+2A)\|\phi\|_0^2$ and condition (1.7) follows. It means that the trivial solution of (1.10) is asymptotically mean square quasistable.

From Theorem 1.5 and Remark 1.6, it follows that an investigation of the solution of (1.2) can be reduced to the construction of appropriate Lyapunov functionals. Below, some formal procedure of Lyapunov functionals construction for (1.2) is described.

Remark 1.8. Suppose that in (1.2) $h_0 = h > 0$, $h_j = jh$, $j = 1,2,\ldots$. Putting $t = t_0 + sh$, $y(s) = x(t_0 + sh)$, and $\xi(s) = \eta(t_0 + sh)$, one can reduce (1.2) to the form

$$\begin{gathered} y(s+1) = \xi(s+1) + G(s, y(s), y(s-1), y(s-2), \ldots), \quad s > -1, \\ y(\theta) = \phi(\theta), \quad \theta \le 0. \end{gathered} \tag{1.31}$$

Below, the equation of type (1.31) is considered.

2. Formal procedure of Lyapunov functionals construction

The proposed procedure of Lyapunov functionals construction consists of the following four steps.

Step 1. Represent the functional F at the right-hand side of (1.2) in the form

$$F(t,x(t),x(t-h_1),x(t-h_2),\ldots) = F_1(t) + F_2(t) + \Delta F_3(t), \tag{2.1}$$

where

$$\begin{gathered} F_1(t) = F_1(t,x(t),x(t-h_1),\ldots,x(t-h_k)), \\ F_j(t) = F_j(t,x(t),x(t-h_1),x(t-h_2),\ldots), \quad j = 2,3, \\ F_1(t,0,\ldots,0) \equiv F_2(t,0,0,\ldots) \equiv F_3(t,0,0,\ldots) \equiv 0, \end{gathered} \tag{2.2}$$

$k \ge 0$ is a given integer, $\Delta F_3(t) = F_3(t+h_0) - F_3(t)$.

Step 2. Suppose that for the auxiliary equation

$$y(t+h_0) = F_1(t, y(t), y(t-h_1),\ldots, y(t-h_k)), \quad t > t_0 - h_0, \tag{2.3}$$

there exists a Lyapunov functional

$$v(t) = v(t, y(t), y(t-h_1),\ldots, y(t-h_k)), \tag{2.4}$$

which satisfies the conditions of Theorem 1.5.

Step 3. Consider Lyapunov functional $V(t)$ for (1.2) in the form $V(t) = V_1(t) + V_2(t)$, where the main component is

$$V_1(t) = v(t,x(t) - F_3(t),x(t-h_1),\ldots,x(t-h_k)). \tag{2.5}$$

Calculate $\mathbf{E}\Delta V_1(t)$ and, in a reasonable way, estimate it.

Step 4. In order to satisfy the conditions of Theorem 1.5, the additional component $V_2(t)$ is chosen by some standard way.

3. Linear Volterra equations with constant coefficients

We demonstrate the formal procedure of Lyapunov functionals construction described above for stability investigation of the scalar equation

$$x(t+1) = \eta(t+1) + \sum_{j=0}^{[t]+r} a_j x(t-j), \quad t > -1, \qquad x(s) = \phi(s), \quad s \in [-(r+1), 0], \tag{3.1}$$

where $r \geq 0$ is a given integer, a_j are known constants, and the process $\eta(t)$ is uniformly mean square summable.

3.1. The first way of Lyapunov functionals construction. Following the procedure, represent (Step 1) equation (3.1) in the form (2.1) with $F_3(t) = 0$,

$$F_1(t) = \sum_{j=0}^{k} a_j x(t-j), \quad F_2(t) = \sum_{j=k+1}^{[t]+r} a_j x(t-j), \quad k \geq 0, \tag{3.2}$$

and consider (Step 2) the auxiliary equation

$$y(t+1) = \sum_{j=0}^{k} a_j y(t-j), \quad t > -1, \ k \geq 0, \qquad y(s) = \begin{cases} \phi(s), & s \in [-(r+1), 0], \\ 0, & s < -(r+1). \end{cases} \tag{3.3}$$

Take into consideration the vector $Y(t) = (y(t-k), \dots, y(t-1), y(t))'$ and represent the auxiliary equation (3.3) in the form

$$Y(t+1) = AY(t), \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ a_k & a_{k-1} & a_{k-2} & \cdots & a_1 & a_0 \end{pmatrix}. \tag{3.4}$$

Consider the matrix equation

$$A'DA - D = -U, \quad U = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{3.5}$$

and suppose that the solution D of this equation is a positive semidefinite symmetric matrix of dimension $k+1$ with the elements d_{ij} such that the condition $d_{k+1,k+1} > 0$ holds. In this case the function $v(t) = Y'(t)DY(t)$ is a Lyapunov function for (3.4), that is, it satisfies the conditions of Theorem 1.5, in particular, condition (1.14) with $\gamma(t) = 0$. Really, using (3.4), we have

$$\begin{aligned}\Delta v(t) &= Y'(t+1)DY(t+1) - Y'(t)Dy(t)\\ &= Y'(t)[A'DA - D]Y(t) = -Y'(t)UY(t) = -y^2(t).\end{aligned} \tag{3.6}$$

Following Step 3 of the procedure, we will construct a Lyapunov functional $V(t)$ for (3.1) in the form $V(t) = V_1(t) + V_2(t)$, where

$$V_1(t) = X'(t)DX(t), \qquad X(t) = \big(x(t-k),\dots,x(t-1),x(t)\big)'. \tag{3.7}$$

Rewrite now (3.1) using representation (3.2) as

$$\begin{gathered}X(t+1) = AX(t) + B(t),\\ B(t) = \big(0,\dots,0,b(t)\big)', \quad b(t) = \eta(t+1) + F_2(t),\end{gathered} \tag{3.8}$$

where the matrix A is defined by (3.4). Calculating $\Delta V_1(t)$, by virtue of (3.8), we have

$$\begin{aligned}\Delta V_1(t) &= X'(t+1)DX(t+1) - X'(t)DX(t)\\ &= \big(AX(t) + B(t)\big)'D\big(AX(t) + B(t)\big) - X'(t)DX(t)\\ &= -x^2(t) + B'(t)DB(t) + 2B'(t)DAX(t).\end{aligned} \tag{3.9}$$

Put

$$\alpha_l = \sum_{j=l}^{\infty} |a_j|, \quad l = 0, 1, \dots. \tag{3.10}$$

Using (3.8), (3.2), (3.10), and $\mu > 0$, we obtain

$$\begin{aligned}&\mathbf{E}B'(t)DB(t)\\ &\quad = d_{k+1,k+1}\mathbf{E}b^2(t) = d_{k+1,k+1}\mathbf{E}\big[\eta(t+1) + F_2(t)\big]^2\\ &\quad \le d_{k+1,k+1}\Big[(1+\mu)\mathbf{E}|\eta(t+1)|^2 + (1+\mu^{-1})\mathbf{E}F_2^2(t)\Big]\\ &\quad = d_{k+1,k+1}\left[(1+\mu)\mathbf{E}|\eta(t+1)|^2 + (1+\mu^{-1})\mathbf{E}\left(\sum_{j=k+1}^{[t]+r} a_j x(t-j)\right)^2\right]\\ &\quad \le d_{k+1,k+1}\left[(1+\mu)\mathbf{E}|\eta(t+1)|^2 + (1+\mu^{-1})\alpha_{k+1}\sum_{j=k+1}^{[t]+r} |a_j|\mathbf{E}x^2(t-j)\right].\end{aligned} \tag{3.11}$$

Since

$$DB(t) = b(t)\begin{pmatrix} d_{1,k+1} \\ d_{2,k+1} \\ \vdots \\ d_{k,k+1} \\ d_{k+1,k+1} \end{pmatrix}, \qquad AX(t) = \begin{pmatrix} x(t-k+1) \\ x(t-k+2) \\ \vdots \\ x(t) \\ \sum_{m=0}^{k} a_m x(t-m) \end{pmatrix}, \tag{3.12}$$

then

$$\begin{aligned} &\mathbf{E}B'(t)DAX(t) \\ &\quad = \mathbf{E}b(t)\left[\sum_{l=1}^{k} d_{l,k+1}x(t-k+l) + d_{k+1,k+1}\sum_{m=0}^{k} a_m x(t-m)\right] \\ &\quad = \mathbf{E}b(t)\left[\sum_{m=0}^{k-1}(a_m d_{k+1,k+1} + d_{k-m,k+1})x(t-m) + a_k d_{k+1,k+1}x(t-k)\right] \\ &\quad = d_{k+1,k+1}\sum_{m=0}^{k} Q_{km}\mathbf{E}b(t)x(t-m), \end{aligned} \tag{3.13}$$

where

$$Q_{km} = a_m + \frac{d_{k-m,k+1}}{d_{k+1,k+1}}, \quad m = 0,\dots,k-1, \qquad Q_{kk} = a_k. \tag{3.14}$$

Note that

$$\sum_{m=0}^{k} Q_{km}\mathbf{E}b(t)x(t-m) = \sum_{m=0}^{k} Q_{km}\mathbf{E}\eta(t+1)x(t-m) + \mathbf{E}F_2(t)\sum_{m=0}^{k} Q_{km}x(t-m), \tag{3.15}$$

and for $\mu > 0$,

$$2\left|\mathbf{E}\eta(t+1)x(t-m)\right| \le \mu\mathbf{E}\eta^2(t+1) + \mu^{-1}\mathbf{E}x^2(t-m). \tag{3.16}$$

Putting

$$\beta_k = \sum_{m=0}^{k}\left|Q_{km}\right| = \left|a_k\right| + \sum_{m=0}^{k-1}\left|a_m + \frac{d_{k-m,k+1}}{d_{k+1,k+1}}\right| \tag{3.17}$$

and using (3.2), (3.10), and (3.17), we obtain

$$\begin{aligned}
2\mathbf{E}F_2(t)\sum_{m=0}^{k}Q_{km}x(t-m)
&=2\sum_{m=0}^{k}\sum_{j=k+1}^{[t]+r}Q_{km}a_j\mathbf{E}x(t-m)x(t-j)\\
&\le\sum_{m=0}^{k}\sum_{j=k+1}^{[t]+r}|Q_{km}||a_j|\left(\mathbf{E}x^2(t-m)+\mathbf{E}x^2(t-j)\right)\\
&\le\sum_{m=0}^{k}|Q_{km}|\left(\alpha_{k+1}\mathbf{E}x^2(t-m)+\sum_{j=k+1}^{[t]+r}|a_j|\mathbf{E}x^2(t-j)\right)\\
&=\alpha_{k+1}\sum_{m=0}^{k}|Q_{km}|\mathbf{E}x^2(t-m)+\beta_k\sum_{j=k+1}^{[t]+r}|a_j|\mathbf{E}x^2(t-j).
\end{aligned}\tag{3.18}$$

So,

$$\begin{aligned}
2\mathbf{E}B'(t)DAX(t)\le d_{k+1,k+1}\Bigg[&\beta_k\sum_{j=k+1}^{[t]+r}|a_j|\mathbf{E}x^2(t-j)+\beta_k\mu\mathbf{E}\eta^2(t+1)\\
&+\left(\alpha_{k+1}+\mu^{-1}\right)\sum_{m=0}^{k}|Q_{km}|\mathbf{E}x^2(t-m)\Bigg].
\end{aligned}\tag{3.19}$$

From (3.9), (3.11), and (3.19), we have

$$\begin{aligned}
\mathbf{E}\Delta V_1(t)\le{}&-\mathbf{E}x^2(t)+d_{k+1,k+1}\\
&\times\Bigg[\left(\left(1+\mu^{-1}\right)\alpha_{k+1}+\beta_k\right)\sum_{j=k+1}^{[t]+r}|a_j|\mathbf{E}x^2(t-j)\\
&+\left(1+\mu\left(1+\beta_k\right)\right)\mathbf{E}\eta^2(t+1)+\left(\alpha_{k+1}+\mu^{-1}\right)\sum_{m=0}^{k}|Q_{km}|\mathbf{E}x^2(t-m)\Bigg].
\end{aligned}\tag{3.20}$$

Put now

$$R_{km}=\begin{cases}\left(\alpha_{k+1}+\mu^{-1}\right)|Q_{km}|, & 0\le m\le k,\\ \left(\left(1+\mu^{-1}\right)\alpha_{k+1}+\beta_k\right)|a_m|, & m>k.\end{cases}\tag{3.21}$$

Then (3.20) takes the form

$$\mathbf{E}\Delta V_1(t)\le-\mathbf{E}x^2(t)+d_{k+1,k+1}\left[\left(1+\mu\left(1+\beta_k\right)\right)\mathbf{E}\eta^2(t+1)+\sum_{m=0}^{[t]+r}R_{km}\mathbf{E}x^2(t-m)\right].\tag{3.22}$$

Choose now (Step 4) the functional $V_2(t)$ in the form

$$V_2(t) = d_{k+1,k+1} \sum_{m=1}^{[t]+r} q_m x^2(t-m), \quad q_m = \sum_{j=m}^{\infty} R_{kj}. \tag{3.23}$$

Then

$$\begin{aligned}
\Delta V_2(t) &= d_{k+1,k+1} \left(\sum_{m=1}^{[t]+1+r} q_m x^2(t+1-m) - \sum_{m=1}^{[t]+r} q_m x^2(t-m) \right) \\
&= d_{k+1,k+1} \left(\sum_{m=0}^{[t]+r} q_{m+1} x^2(t-m) - \sum_{m=1}^{[t]+r} q_m x^2(t-m) \right) \\
&= d_{k+1,k+1} \left(q_1 x^2(t) - \sum_{m=1}^{[t]+r} R_{km} x^2(t-m) \right).
\end{aligned} \tag{3.24}$$

From (3.22), (3.24), for the functional $V(t) = V_1(t) + V_2(t)$, we have

$$\mathbf{E}\Delta V(t) \le -(1 - q_0 d_{k+1,k+1}) \mathbf{E}x^2(t) + \gamma(t), \tag{3.25}$$

where

$$\gamma(t) = d_{k+1,k+1}(1 + \mu(1+\beta_k)) \mathbf{E}\eta^2(t+1). \tag{3.26}$$

Since the process $\eta(t)$ is uniformly mean square summable, then the function $\gamma(t)$ satisfies condition (1.12). So, if

$$q_0 d_{k+1,k+1} < 1, \tag{3.27}$$

then the functional $V(t)$ satisfies condition (1.14). It is easy to check that condition (1.13) holds too. Really, using (3.7), (3.23) for the functional $V(t) = V_1(t) + V_2(t)$ and $t \in [0,1)$, we have

$$\begin{aligned}
\mathbf{E}V(t) &\le \|D\| \sum_{j=0}^{k} \mathbf{E}x^2(t-j) + d_{k+1,k+1} \sum_{m=1}^{r} q_m \mathbf{E}x^2(t-m) \\
&\le \left((k+1)\|D\| + d_{k+1,k+1} \sum_{m=1}^{r} q_m \right) \sup_{s \le t} \mathbf{E}x^2(s).
\end{aligned} \tag{3.28}$$

So, if condition (3.27) holds, then the solution of (3.1) is uniformly mean square summable.

Using (3.23), (3.21), (3.17), and (3.10), transform q_0 in the following way:

$$\begin{aligned} q_0 &= \sum_{j=0}^{\infty} R_{kj} = \sum_{j=0}^{k} R_{kj} + \sum_{j=k+1}^{\infty} R_{kj} \\ &= (\alpha_{k+1} + \mu^{-1}) \sum_{j=0}^{k} |Q_{kj}| + ((1+\mu^{-1})\alpha_{k+1} + \beta_k) \sum_{j=k+1}^{\infty} |a_j| \\ &= (\alpha_{k+1} + \mu^{-1})\beta_k + ((1+\mu^{-1})\alpha_{k+1} + \beta_k)\alpha_{k+1} \\ &= \alpha_{k+1}^2 + 2\alpha_{k+1}\beta_k + \mu^{-1}(\alpha_{k+1}^2 + \beta_k). \end{aligned} \tag{3.29}$$

Thus, if

$$\alpha_{k+1}^2 + 2\alpha_{k+1}\beta_k < d_{k+1,k+1}^{-1}, \tag{3.30}$$

then there exists a so big $\mu > 0$ that condition (3.27) holds and, therefore, the solution of (3.1) is uniformly mean square summable.

Note that condition (3.30) can also be represented in the form

$$\alpha_{k+1} < \sqrt{\beta_k^2 + d_{k+1,k+1}^{-1}} - \beta_k. \tag{3.31}$$

Remark 3.1. Suppose that in (3.1)

$$a_j = 0, \quad j > k. \tag{3.32}$$

Then $\alpha_{k+1} = 0$. So, if condition (3.32) holds and the matrix equation (3.5) has a positive semidefinite solution D with $d_{k+1,k+1} > 0$, then the solution of (3.1) is uniformly mean square summable.

3.2. The second way of Lyapunov functionals construction. We get another stability condition. Equation (3.1) can be represented (Step 1) in the form (2.1) with $F_2(t) = 0$, $k = 0$,

$$F_1(t) = \beta x(t), \quad \beta = \sum_{j=0}^{\infty} a_j, \qquad F_3(t) = -\sum_{m=1}^{[t]+r} x(t-m) \sum_{j=m}^{\infty} a_j. \tag{3.33}$$

Really, calculating $\Delta F_3(t)$, we have

$$\begin{aligned} \Delta F_3(t) &= -\sum_{m=1}^{[t]+1+r} x(t+1-m) \sum_{j=m}^{\infty} a_j + \sum_{m=1}^{[t]+r} x(t-m) \sum_{j=m}^{\infty} a_j \\ &= -\sum_{m=0}^{[t]+r} x(t-m) \sum_{j=m+1}^{\infty} a_j + \sum_{m=1}^{[t]+r} x(t-m) \sum_{j=m}^{\infty} a_j \\ &= -x(t) \sum_{j=1}^{\infty} a_j + \sum_{m=1}^{[t]+r} x(t-m) a_m. \end{aligned} \tag{3.34}$$

Thus,

$$x(t+1) = \eta(t+1) + \beta x(t) + \Delta F_3(t). \tag{3.35}$$

In this case the auxiliary equation (2.3) (Step 2) is $y(t+1) = \beta y(t)$. The function $v(t) = y^2(t)$ is a Lyapunov function for this equation if $|\beta| < 1$. We will construct the Lyapunov functional $V(t)$ (Step 3) for (3.1) in the form $V(t) = V_1(t) + V_2(t)$, where $V_1(t) = (x(t) - F_3(t))^2$. Calculating $\mathbf{E}\Delta V_1(t)$, by virtue of representation (3.33), we have

$$\begin{aligned}
\mathbf{E}\Delta V_1(t) &= \mathbf{E}\Big[\big(x(t+1) - F_3(t+1)\big)^2 - \big(x(t) - F_3(t)\big)^2\Big] \\
&= \mathbf{E}\Big[\big(\eta(t+1) + \beta x(t) - F_3(t)\big)^2 - \big(x(t) - F_3(t)\big)^2\Big] \\
&= \mathbf{E}\big(\eta(t+1) + (\beta-1)x(t)\big)\big(\eta(t+1) + (\beta+1)x(t) - 2F_3(t)\big) \\
&= (\beta^2 - 1)\mathbf{E}x^2(t) + \mathbf{E}\eta^2(t+1) + 2\beta\mathbf{E}\eta(t+1)x(t) \\
&\quad - 2\mathbf{E}\eta(t+1)F_3(t) - 2(\beta-1)\mathbf{E}x(t)F_3(t).
\end{aligned} \tag{3.36}$$

Using $\mu > 0$, we obtain

$$2\mathbf{E}\left|\eta(t+1)x(t)\right| \le \mu\mathbf{E}\eta^2(t+1) + \mu^{-1}\mathbf{E}x^2(t). \tag{3.37}$$

Putting

$$B_m = \left|\sum_{j=m}^{\infty} a_j\right|, \qquad \alpha = \sum_{m=1}^{\infty} B_m \tag{3.38}$$

and using (3.33), (3.10), we have

$$\begin{aligned}
2\mathbf{E}\left|\eta(t+1)F_3(t)\right| &\le 2\sum_{m=1}^{[t]+r} B_m\mathbf{E}\left|\eta(t+1)x(t-m)\right| \\
&\le \sum_{m=1}^{[t]+r} B_m\big(\mu\mathbf{E}\eta^2(t+1) + \mu^{-1}\mathbf{E}x^2(t-m)\big) \\
&\le \alpha\mu\mathbf{E}\eta^2(t+1) + \mu^{-1}\sum_{m=1}^{[t]+r} B_m\mathbf{E}x^2(t-m), \\
2\mathbf{E}\left|x(t)F_3(t)\right| &\le 2\sum_{m=1}^{[t]+r} B_m\mathbf{E}\left|x(t)x(t-m)\right| \\
&\le \sum_{m=1}^{[t]+r} B_m\big(\mathbf{E}x^2(t) + \mathbf{E}x^2(t-m)\big) \\
&\le \alpha\mathbf{E}x^2(t) + \sum_{m=1}^{[t]+r} B_m\mathbf{E}x^2(t-m).
\end{aligned} \tag{3.39}$$

As a result,

$$\begin{aligned}\mathbf{E}\Delta V_1(t) \le{}& (\beta^2 - 1 + \alpha|\beta - 1| + \mu^{-1}|\beta|)\mathbf{E}x^2(t) + (1 + \mu(\alpha + |\beta|))\mathbf{E}\eta^2(t+1)\\&+ (|\beta - 1| + \mu^{-1}) \sum_{m=1}^{[t]+r} B_m \mathbf{E}x^2(t-m).\end{aligned} \tag{3.40}$$

Put now (Step 4)

$$V_2(t) = \sum_{m=1}^{[t]+r} \gamma_m x^2(t-m), \qquad \gamma_m = (|\beta - 1| + \mu^{-1}) \sum_{j=m}^{\infty} B_j. \tag{3.41}$$

Then, using (3.38), similar to (3.24), we have

$$\Delta V_2(t) = (|\beta - 1| + \mu^{-1}) \left(\alpha x^2(t) - \sum_{m=1}^{[t]+r} B_m x^2(t-m) \right). \tag{3.42}$$

So, for the functional $V(t) = V_1(t) + V_2(t)$, we obtain

$$\begin{aligned}\mathbf{E}\Delta V(t) \le{}& (\beta^2 - 1 + 2\alpha|\beta - 1| + \mu^{-1}(\alpha + |\beta|))\mathbf{E}x^2(t)\\&+ (1 + \mu(\alpha + |\beta|))\mathbf{E}\eta^2(t+1).\end{aligned} \tag{3.43}$$

Thus, if

$$\beta^2 + 2\alpha|\beta - 1| < 1, \tag{3.44}$$

then there exists a so big $\mu > 0$ that condition $\beta^2 + 2\alpha|\beta - 1| + \mu^{-1}(\alpha + |\beta|) < 1$ holds also, and, therefore, the solution of (3.1) is uniformly mean square summable.

It is easy to see that condition (3.44) can be written also in the form

$$1 + \beta > 2\alpha, \quad |\beta| < 1. \tag{3.45}$$

4. Particular cases

Here, particular cases of condition (3.31) for different $k \ge 0$ are considered.

4.1. Case $k = 0$**.** Equation (3.5) gives the solution $d_{11} = (1 - a_0^2)^{-1}$, which is a positive one if $|a_0| < 1$. From (3.17), it follows that $\beta_0 = |a_0|$. Condition (3.31) takes the form

$$\alpha_0 < 1. \tag{4.1}$$

So, under condition (4.1), the solution of (3.1) is uniformly mean square summable.

4.2. Case $k = 1$**.** The matrix equation (3.5) is equivalent to the system of equations

$$\begin{aligned}a_1^2 d_{22} - d_{11} &= 0,\\(a_1 - 1)d_{12} + a_0 a_1 d_{22} &= 0,\\d_{11} + 2a_0 d_{12} + (a_0^2 - 1)d_{22} &= -1,\end{aligned} \tag{4.2}$$

with the solution

$$d_{11} = a_1^2 d_{22}, \qquad d_{12} = \frac{a_0 a_1}{1-a_1} d_{22},$$
$$d_{22} = \frac{1-a_1}{(1+a_1)\left[(1-a_1)^2 - a_0^2\right]}. \tag{4.3}$$

The matrix D is a positive semidefinite one with $d_{22} > 0$ by conditions $|a_1| < 1$, $|a_0| < 1 - a_1$. Using (3.17), (4.3), we have

$$\beta_1 = |a_1| + \left| a_0 + \frac{d_{12}}{d_{22}} \right| = |a_1| + \left| a_0 + \frac{a_0 a_1}{1-a_1} \right| = |a_1| + \frac{|a_0|}{1-a_1},$$
$$d_{22}^{-1} = 1 - a_1^2 - a_0^2 \frac{1+a_1}{1-a_1}. \tag{4.4}$$

Condition (3.31) takes the form

$$\alpha_2 < (1 - |a_1|)\left(1 - \frac{|a_0|}{1-a_1}\right). \tag{4.5}$$

Under condition (4.5), the solution of (3.1) is uniformly mean square summable.

Note that condition (4.5) can also be written in the form

$$\alpha_0 < 1 + |a_0| \frac{|a_1| - a_1}{1-a_1}. \tag{4.6}$$

It is easy to see that condition (4.6) is not worse than (4.1). In particular, for $a_1 \geq 0$, condition (4.6) coincides with (4.1).

4.3. Case $k = 2$. The matrix equation (3.5) is equivalent to the system of equations

$$\begin{aligned}
&a_2^2 d_{33} - d_{11} = 0,\\
&a_2 d_{13} + a_1 a_2 d_{33} - d_{12} = 0,\\
&a_2 d_{23} + a_0 a_2 d_{33} - d_{13} = 0,\\
&d_{11} + 2a_1 d_{13} + a_1^2 d_{33} - d_{22} = 0,\\
&d_{12} + a_0 d_{13} + a_0 a_1 d_{33} + (a_1 - 1) d_{23} = 0,\\
&d_{22} + 2a_0 d_{23} + (a_0^2 - 1) d_{33} = -1,
\end{aligned} \tag{4.7}$$

with the solution

$$\begin{aligned} d_{11} &= a_2^2 d_{33}, \\ d_{12} &= \frac{a_2(1-a_1)(a_1+a_0a_2)}{1-a_1-a_2(a_0+a_2)} d_{33}, \\ d_{13} &= \frac{a_2(a_0+a_1a_2)}{1-a_1-a_2(a_0+a_2)} d_{33}, \\ d_{22} &= \left[a_1^2+a_2^2+\frac{2a_1a_2(a_0+a_1a_2)}{1-a_1-a_2(a_0+a_2)}\right] d_{33}, \\ d_{23} &= \frac{(a_0+a_2)(a_1+a_0a_2)}{1-a_1-a_2(a_0+a_2)} d_{33}, \end{aligned} \tag{4.8}$$

$$d_{33} = \left[1-a_0^2-a_1^2-a_2^2-2\frac{a_1a_2(a_0+a_1a_2)+a_0(a_0+a_2)(a_1+a_0a_2)}{1-a_1-a_2(a_0+a_2)}\right]^{-1}. \tag{4.9}$$

Using (3.17), (4.7), and (4.8), we have

$$\begin{aligned} \beta_2 &= |a_2| + \left|a_0+\frac{d_{23}}{d_{33}}\right| + \left|a_1+\frac{d_{13}}{d_{33}}\right| = |a_2| + \frac{|d_{13}|+|d_{12}|}{|a_2|d_{33}} \\ &= |a_2| + \frac{|a_0+a_1a_2| + |(1-a_1)(a_1+a_0a_2)|}{|1-a_1-a_2(a_0+a_2)|}. \end{aligned} \tag{4.10}$$

If the matrix D with the elements defined by (4.8) is a positive semidefinite one with $d_{33} > 0$, then under the condition

$$\alpha_3 < \sqrt{\beta_2^2 + d_{33}^{-1}} - \beta_2, \tag{4.11}$$

the solution of (3.1) is uniformly mean square summable.

5. Examples

Example 5.1. Consider the difference equation

$$\begin{aligned} x(t+1) &= \eta(t+1) + ax(t) + bx(t-1), \quad t > -1, \\ x(\theta) &= \phi(\theta), \quad \theta \in [-2,0]. \end{aligned} \tag{5.1}$$

From conditions (4.1) and (4.5) follow two sufficient conditions for uniformly mean square summability of the solution of (5.1):

$$|a| + |b| < 1, \tag{5.2}$$

$$|a| + b < 1, \quad |b| < 1. \tag{5.3}$$

Condition (4.11) for (5.1) coincides with (5.3). Condition (3.45) takes the form

$$1 + a + b > 2|b|, \quad |a+b| < 1. \tag{5.4}$$

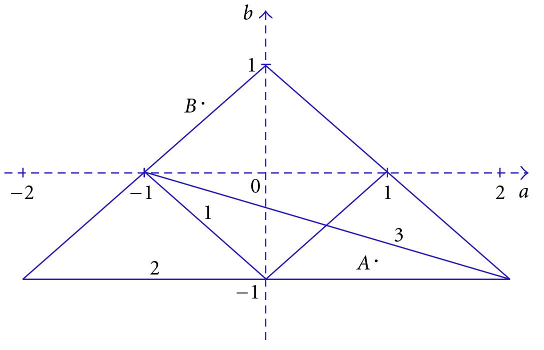

Figure 5.1

On Figure 5.1 are shown the regions of uniformly mean square summability, obtained for (5.1) by conditions (5.2) (the square number 1), (5.3) (the triangle number 2), and (5.4) (the triangle number 3).

For numerical investigation of the solution of (5.1), we determine one of the possible trajectories of the process $\eta(t)$, $t \geq t_0$, in the following way. On the interval $[t_0 + nh_0, t_0 + (n+1)h_0)$, $n = 0, 1, \ldots$, put

$$\eta(t) = 0 \tag{5.5}$$

if

$$t \in \left[t_0 + nh_0, t_0 + \left(n + 1 - \frac{1}{2^n}\right)h_0\right) \tag{5.6}$$

or

$$t \in \left[t_0 + \left(n + 1 - \frac{1}{2^{n+1}}\right)h_0, t_0 + (n+1)h_0\right), \tag{5.7}$$

put

$$\eta(t) = 2^{n+2}\left(\frac{t - t_0}{h_0} - n - 1 + \frac{1}{2^n}\right) \tag{5.8}$$

if

$$t \in \left[t_0 + \left(n + 1 - \frac{1}{2^n}\right)h_0, t_0 + \left(n + 1 - \frac{3}{2^{n+2}}\right)h_0\right), \tag{5.9}$$

and put

$$\eta(t) = 1 - 2^{n+2}\left(\frac{t - t_0}{h_0} - n - 1 + \frac{3}{2^{n+2}}\right) \tag{5.10}$$

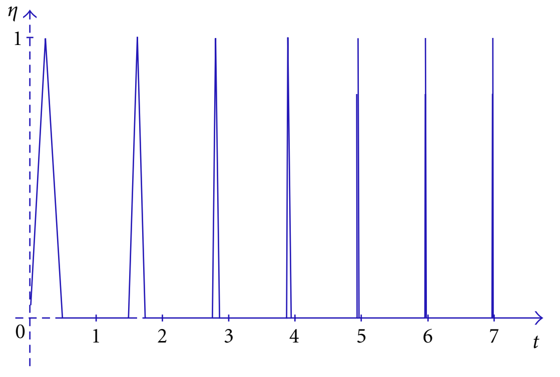

Figure 5.2

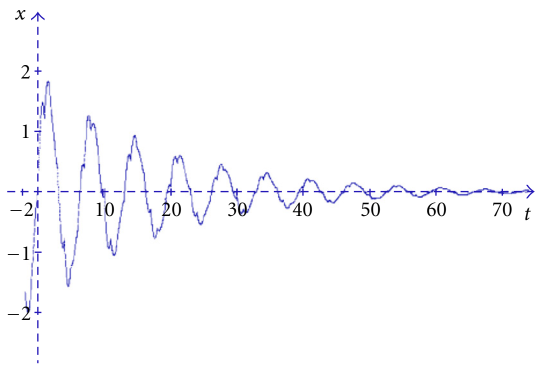

Figure 5.3

if

$$t \in \left[t_0 + \left(n+1-\frac{3}{2^{n+2}}\right)h_0, t_0 + \left(n+1-\frac{1}{2^{n+1}}\right)h_0\right). \tag{5.11}$$

The graph of the function $\eta(t)$ for $t_0 = 0$, $h_0 = 1$ is shown on Figure 5.2.

The function $\eta(t)$ constructed above satisfies the following conditions:

$$0 \le \eta(t) \le 1, \qquad \sum_{j=0}^{\infty} \eta(t+jh_0) \le 1, \qquad \int_0^{\infty} \eta(t)dt = \frac{1}{2}. \tag{5.12}$$

It is easy to see also that for each fixed $t \in [t_0, t_0 + h_0)$, the sequence $\eta_j = \eta(t + jh_0)$ has only one nonzero member, and therefore $\lim_{j\to\infty} \eta(t + jh_0) = 0$. On the other hand, for every $T > 0$, there exists a so large number n that

$$t_1 = t_0 + \left(n+1-\frac{3}{2^{n+2}}\right)h_0 > T, \quad \eta(t_1) = 1. \tag{5.13}$$

Therefore, $\lim_{t\to\infty}\eta(t)$ does not exist. So, the function $\eta(t)$ is an asymptotically quasitrivial function (satisfies condition (1.7)) but not an asymptotically trivial one (does not satisfy condition (1.6)).

The trajectory of (5.1) with the initial function $\phi(\theta)=\cos 2\theta-1$ is shown in the point $A(1.1,-0.9)$, which belongs to the summability region, on Figure 5.3 with $\eta(t)\equiv 0$ and on Figure 5.4 with $\eta(t)$ described above. The trajectory of (5.1) with the initial function $\phi(\theta)=0.05\cos 2\theta$ is shown in the point $B(-0.5,0.6)$, which does not belong to the summability region, on Figure 5.5 with $\eta(t)\equiv 0$ and on Figure 5.6 with $\eta(t)$ described above. The points A and B are shown on Figure 5.1.

Example 5.2. Consider the difference equation

$$\begin{gathered} x(t+1)=\eta(t+1)+ax(t)+\sum_{j=1}^{[t]+r} b^j x(t-j), \quad t>-1, \\ x(\theta)=\phi(\theta), \quad \theta\in[-(r+1),0], \quad r\geq 0. \end{gathered} \tag{5.14}$$

From condition (4.1), it follows that the inequality

$$|a|<\frac{1-2|b|}{1-|b|}, \quad |b|<\frac{1}{2}, \tag{5.15}$$

is a sufficient condition for uniformly mean square summability of the solution of (5.14). Condition (4.5) gives us a sufficient condition for uniformly mean square summability of the solution of (5.14) in the form

$$|a|<\left(\frac{1-2|b|}{1-|b|}\right)\left(\frac{1-b}{1-|b|}\right), \quad |b|<\frac{1}{2}. \tag{5.16}$$

From (4.8), (4.10), and (4.11), we obtain another sufficient condition for uniformly mean square summability of the solution of (5.14):

$$\begin{gathered} \frac{|b|^3}{1-|b|}<\sqrt{\beta_2^2+d_{33}^{-1}}-\beta_2, \quad |b|<1, \\ \beta_2=b^2+\frac{|a+b^3|+(1-b)|b(1+ab)|}{|1-b-b^2(a+b^2)|}, \\ d_{33}^{-1}=1-a^2-b^2-b^4-2b\frac{b^2(a+b^3)+a(a+b^2)(1+ab)}{1-b-b^2(a+b^2)}. \end{gathered} \tag{5.17}$$

Using Mathematica program for solution of the matrix equation (3.5), sufficient condition (3.31) for uniformly mean square summability of the solution of (5.14) was obtained also for $k=3$ and $k=4$. In particular, for $k=3$, this condition takes the form

$$\begin{gathered} \frac{b^4}{1-|b|}<\sqrt{\beta_3^2+d_{44}^{-1}}-\beta_3, \quad |b|<1, \\ \beta_3=|b^3|+\left|a+\frac{d_{34}}{d_{44}}\right|+\left|b+\frac{d_{24}}{d_{44}}\right|+\left|b^2+\frac{d_{14}}{d_{44}}\right|, \end{gathered} \tag{5.18}$$

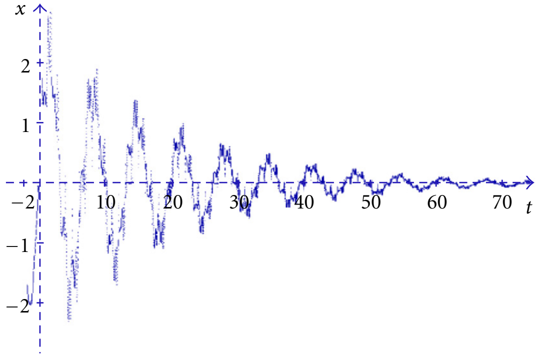

Figure 5.4

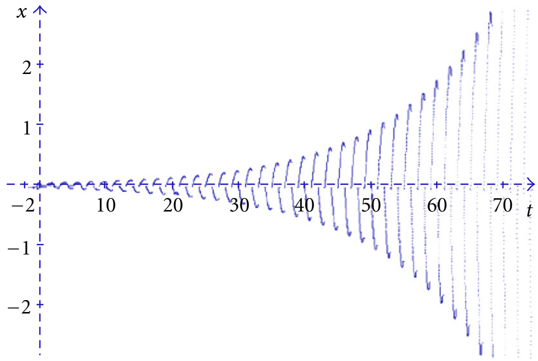

Figure 5.5

where

$$
\begin{aligned}
\frac{d_{14}}{d_{44}} &= b^3\left[b^3+b^5-b^8+a\left(1-b^3+b^4\right)\right]G^{-1},\\
\frac{d_{24}}{d_{44}} &= b^2\left[a^2b+b^2+b^5-b^6-b^8+a\left(1+b^4+b^6\right)\right]G^{-1},\\
\frac{d_{34}}{d_{44}} &= b\left[b^2+a^3b^2+b^4-b^7+a^2\left(b+b^4\right)+a\left(1+2b^3+b^5-b^6-b^8\right)\right]G^{-1},\\
d_{44} &= G\left[1-b-b^2-a^4b^3-2b^4+2b^7-2b^8+2b^9-b^{10}-b^{12}+b^{13}\right.\\
&\qquad -b^{14}+b^{17}-a^3\left(b^2+b^5\right)-a^2\left(1+b+5b^4-b^5+b^6-2b^7-b^9\right)\\
&\qquad \left.-ab^2\left(1+4b-b^2+5b^3-b^4+b^5-4b^6+4b^7-b^{10}+b^{11}\right)\right]^{-1},\\
G &= 1-b-ab^2-\left(1+a^2\right)b^3-b^4-ab^5-b^6+b^7+b^9.
\end{aligned}
\tag{5.19}
$$

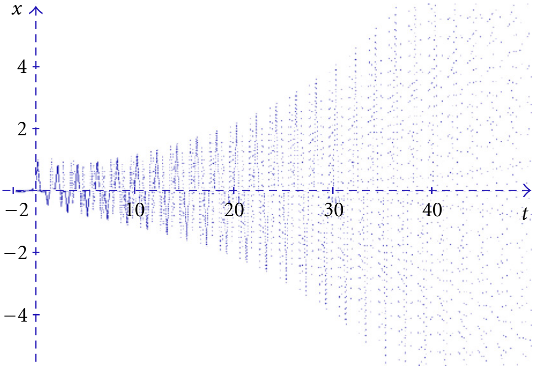

Figure 5.6

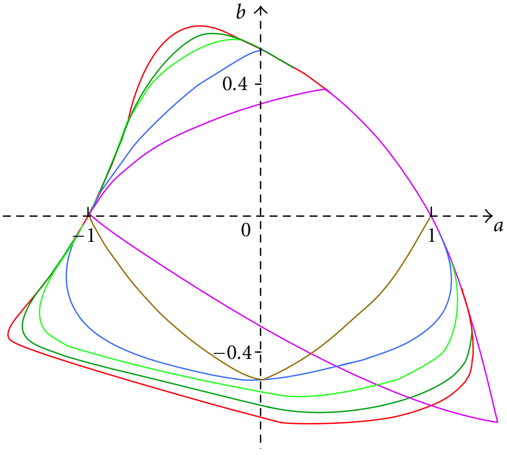

Figure 5.7

Condition (3.45) for (5.14) takes the form

$$-\frac{1-3|b|}{(1-b)(1-|b|)} < a < \frac{1-2b}{1-b}, \quad |b| < 1. \tag{5.20}$$

On Figure 5.7, the regions of uniformly mean square summability of the solution of (5.14) are shown, obtained by condition (3.31). For $k = 0$ (condition (5.15), the brown curve), for $k = 1$ (condition (5.16), the blue curve), for $k = 2$ (condition (5.17), the green curve), for $k = 3$ (condition (5.18), the cyan curve), for $k = 4$ (the red curve) and also obtained by condition (5.20) (the magenta curve).

As it is shown on Figure 5.7 (and naturally it can be shown analytically), for $b \geq 0$ condition (5.15) coincides with condition (5.16) and, for $a \geq 0$, $b \geq 0$, conditions (5.15), (5.16), (5.17), and (5.18) give the same region of uniformly mean square summability,

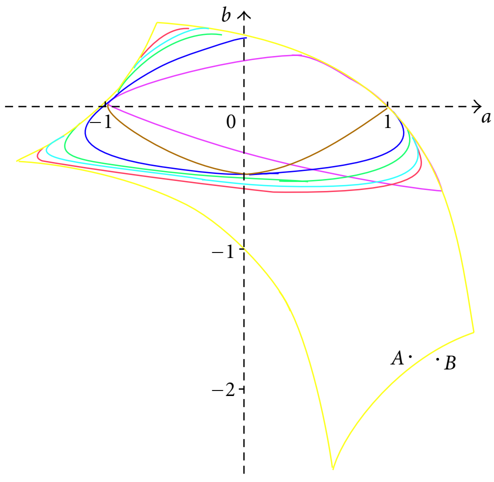

Figure 5.8

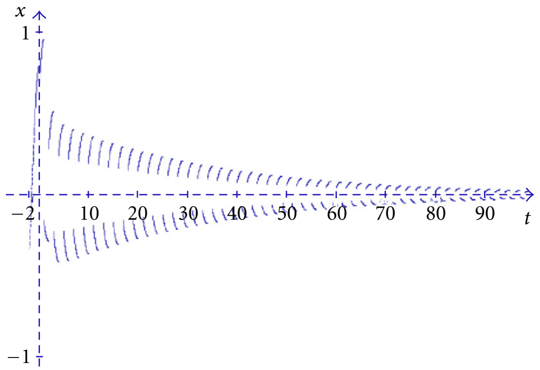

Figure 5.9

which is defined by the inequality

$$a + \frac{b}{1-b} < 1, \quad b < 1. \tag{5.21}$$

Note also that the region of uniformly mean square summability Q_k of the solution of (5.14), obtained by condition (3.31), expands if k increases, that is, $Q_0 \subset Q_1 \subset Q_2 \subset Q_3 \subset Q_4$. So, to get a greater region of uniformly mean square summability, one can use condition (3.31) for $k = 5$, $k = 6$, and so forth. But it is clear that each region Q_k can be obtained by the condition $|b| < 1$ only.

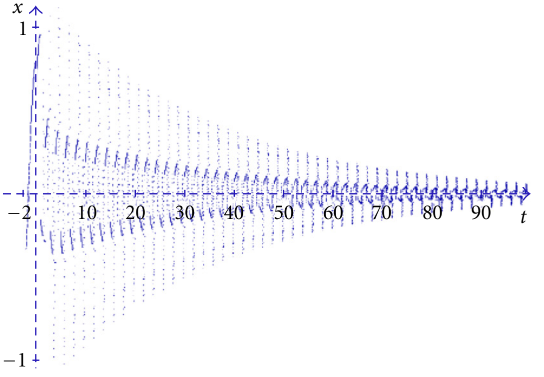

Figure 5.10

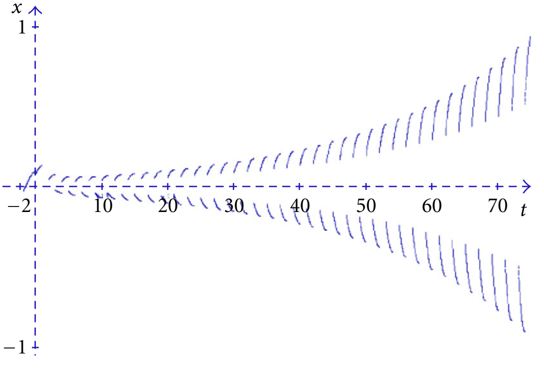

Figure 5.11

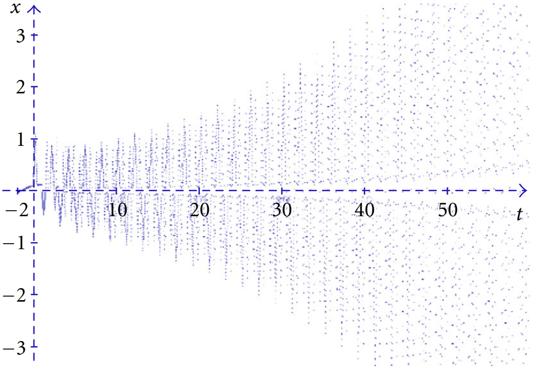

Figure 5.12

To obtain a condition of another type for uniformly mean square summability of the solution of (5.14), transform the sum from (5.14) for $t > 0$ in the following way:

$$\begin{aligned}\sum_{j=1}^{[t]+r} b^j x(t-j) &= b\sum_{j=1}^{[t]+r} b^{j-1}x(t-j)\\ &= b\sum_{j=0}^{[t]-1+r} b^j x(t-1-j)\\ &= b\left(x(t-1)+\sum_{j=1}^{[t]-1+r} b^j x(t-1-j)\right)\\ &= b\big[(1-a)x(t-1)+x(t)-\eta(t)\big].\end{aligned} \tag{5.22}$$

Substituting (5.22) into (5.14), we obtain (5.14) in the form

$$\begin{gathered} x(t+1)=\eta(t+1)+ax(t)+\sum_{j=1}^{r-1} b^j x(t-j), \quad t\in(-1,0],\\ x(t+1)=\eta_1(t+1)+(a+b)x(t)+b(1-a)x(t-1),\\ \eta_1(t+1)=\eta(t+1)-b\eta(t), \quad t>0.\end{gathered} \tag{5.23}$$

The corresponding matrix D is defined by (4.3) with $a_0 = a+b$, $a_1 = b(1-a)$, and it is a positive semidefinite one if and only if

$$\big|b(1-a)\big| < 1, \qquad |a+b| < 1-b(1-a). \tag{5.24}$$

On Figure 5.8 the graph on Figure 5.7 is shown together with the region of uniformly mean square summability obtained by condition (5.24) (the yellow curve).

The trajectory of (5.14) with $r = 1$ and the initial functional $\phi(\theta)=0.8\cos\theta$ is shown in the point $A(1.2,-1.8)$, which belongs to the summability region, on Figure 5.9 with $\eta(t)\equiv 0$ and on Figure 5.10 with $\eta(t)$ described above. The trajectory of (5.14) with $r = 1$ and the initial functional $\phi(\theta) = 0.1\cos\theta$ is shown in the point $B(1.33,-1.8)$, which does not belong to the summability region, on Figure 5.11 with $\eta(t)\equiv 0$ and on Figure 5.12 with $\eta(t)$ described above. The points A and B are shown on Figure 5.8.

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