Optimal Control of Certain Hyperbolic And Integral Stochastic Equations

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Using Gateaux differentiation of the quality functional we obtain necessary conditions for optimality of a control for stochastic differential equations of hyperbolic type containing two-parameter white noise and for stochastic integral equations. Bibliography: 15 titles.

Consider the optimal control problem $\{\xi_u, J(u), U\}$ with motion trajectory ξ_u, quality functional $J(u)$, and set of admissible controls U. Let $J_0'(u_0, v)$ be the Gateaux differential of the functional $J(u)$ at the point $u = u_0$, i.e.,

$$J_0'(u_0, v) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[J(u_\varepsilon) - J(u_0)], \tag{1}$$

$$u_\varepsilon = u_0 + \varepsilon v, \quad u_0 \in U, \quad v \in U. \tag{2}$$

If u_0 is an optimal control of the problem $\{\xi_u, J(u), U\}$, i.e., $J(u_0) = \inf_{u \in U} J(u)$, then the Gateaux differential (if it exists) must be nonnegative. Thus the inequality

$$J_0'(u_0, v) \geq 0, \quad v \in U, \tag{3}$$

is a necessary condition for optimality of the control u_0. If the Gateaux differential is linear in v, the representation $J_0'(u_0, v) = \langle J'(u_0), v \rangle$ holds [6], where $J'(u_0)$ is the Gateaux derivative of the functional $J(u_0)$ at the point $u = u_0$. If in addition U is a convex set and u_0 an interior point of U, then condition (3) is equivalent [2] to the condition

$$J'(u_0) = 0. \tag{4}$$

If the function $J(u)$ is convex on U, then condition (4) is sufficient for optimality of the control u_0 [2].

In some cases condition (3) (or (4)) enables us to obtain a synthesis of the optimal control in explicit form.

In the deterministic case this approach has been used in solving optimal control problems for processes described by ordinary and partial differential equations (cf., for example, [1, 2]).

In the author's papers [9–13] and the paper of Warfield [15] the limit (1) was computed for various (differential, integral, partial differential) stochastic systems with a noise-free control. In the construction of of the control u_ε in these papers the McShane variation was used instead of relation (2). For systems with a noisy control this limit does not exist.

The existence of a limit of the form (1), (2) for systems with a noisy control was proved by Varsan [14] for ordinary stochastic differential equations. However, since optimal control theory for stochastic ordinary differential equations (cf., for example, [4, 8]) is more developed than optimal control theory for stochastic integral equations and stochastic partial differential equations, the study of the latter is of particular interest.

The purpose of the present article is to calculate the limit (1), (2) for systems with a noisy control described by stochastic differential equations of hyperbolic type and stochastic integral equations.

Differential equations of hyperbolic type. Consider the optimal control problem for the stochastic differential equation

$$\frac{\partial^2 \xi(z)}{\partial x \partial y} = a(z, \xi(z), u(z)) + b(z, \xi(z), u(z))\frac{\partial^2 w(z)}{\partial x \partial y}, \tag{5}$$

$$\xi(x, 0) = \varphi(x), \quad \xi(0, y) = \psi(y), \quad \varphi(0) = \psi(0),$$

with quality functional

$$J(u) = \mathbf{M}[F(\xi(Z)) + \iint_D G(z, \xi(z), u(z))\, dz].$$

Here $a(z, p, u)$ is an n-dimensional vector, $b(z, p, u)$ is an $n \times m$ matrix, $F(p)$ and $G(z, p, u)$ are nonnegative functions, $z = (x, y) \in D = [0, X] \times [0, Y]$, $Z = (X, Y)$, $p \in R^n$, $u \in R^l$, $\dfrac{\partial^2 w(z)}{\partial x \partial y}$ is m-dimensional two-parameter white noise. Equations of the form (5) have been studied, for example, in [3, 9].

Let $\{\Omega, \sigma, \mathbf{P}\}$ be a probability space, $\{\mathcal{F}_z\}$ a flow of σ-algebras, $\mathcal{F}_z \in \sigma$, $z = (x, y) \le r = (s, t)$ if $x \le$ $y \le t$, σ_z is the minimal σ-algebra containing all the σ-algebras $\mathcal{F}_{(x,t)}$ and $\mathcal{F}_{(s,y)}$ for $r \ge z$, U the set of admissible controls, i.e., $\mathcal{F}_z$-measurable functions $u(z)$ for which $\|u\|^2 = \sup_{z \in D} \mathbf{M}|u(z)|^2 < \infty$. The letter C is used to denote arbitrary positive constants, and ∇_p (resp. ∇_u) denotes differentiation on p (resp. u).

It is assumed that the functions $\varphi(x)$ and $\psi(y)$ are σ_0-measurable and

$$\sup_{0 \le x \le X} \mathbf{M}|\varphi(x)|^2 + \sup_{0 \le y \le Y} \mathbf{M}|\psi(y)|^2 < \infty, \tag{7}$$

and that the functions $a(z, p, u)$, $b(z, p, u)$, $F(p)$, and $G(z, p, u)$ are differentiable on p and u and such that

$$|a(z, p, u)|^2 + |b(z, p, u)|^2 + F(p) + G(z, p, u) + |\nabla_p F(p)|^2$$
$$+ |\nabla_p G(z, p, u)|^2 + |\nabla_u G(z, p, u)|^2 \le C(1 + |p|^2 + |u|^2), \tag{8}$$
$$|\nabla_p a(z, p, u)| + |\nabla_p b(z, p, u)| + |\nabla_u a(z, p, u)| + |\nabla_u b(z, p, u)| \le C, \tag{9}$$
$$|\nabla_p a(z, p_1, u_1) - \nabla_p a(z, p_2, u_2)|^2 + |\nabla_p b(z, p_1, u_1) - \nabla_p b(z, p_2, u_2)|^2$$
$$+ |\nabla_u a(z, p_1, u_1) - \nabla_u a(z, p_2, u_2)|^2 + |\nabla_u b(z, p_1, u_1) - \nabla_u b(z, p_2, u_2)|^2$$
$$+ |\nabla_p G(z, p_1, u_1) - \nabla_p G(z, p_2, u_2)|^2 + |\nabla_u G(z, p_1, u_1) - \nabla_u G(z, p_2, u_2)|^2$$
$$+ |\nabla_p F(p_1) - \nabla_p F(p_2)|^2 \le C[|p_1 - p_2|^2 + |u_1 - u_2|^2]. \tag{10}$$

Let $\xi_0(z)$ be an $\mathcal{F}_z$-measurable solution of Eq. (5) with the control u_0 and $\xi_\epsilon(z)$ the solution with the control u_ϵ.

THEOREM 1. *Let $u_0 \in U$, $v \in U$. Then the limit (1), (2) for the control problem (5), (6) exists and is equal to*

$$J_0'(u_0, v) = \mathbf{M}[q_0^*(Z)\nabla_p F(\xi_0(Z)) + \iint_D (q_0^*(z)\nabla_p G(z, \xi_0(z), u_0(z)) + v^*(z)\nabla_u G(z, \xi_0(z), u_0(z)))\, dz].$$

$$\tag{11}$$

Here $q_0(z)$ is a solution of the equation

$$\frac{\partial^2 q_0}{\partial x \partial y} = \nabla_p a(z, \xi_0(z), u_0(z))q_0(z) + \nabla_u a(z, \xi_0(z), u_0(z))v(z)$$

$$+ [\nabla_p b(z, \xi_0(z), u_0(z))q_0(z) + \nabla_u b(z, \xi_0(z), u_0(z))v(z)]\frac{\partial^2 w}{\partial x \partial y} \tag{12}$$

with initial condition $q_0(x, 0) = q_0(0, y) = 0$.

To prove Theorem 1 we need to study the behavior of the process $q_\epsilon(z) = \dfrac{1}{\epsilon}(\xi_\epsilon(z) - \xi_0(z))$.

LEMMA. *Let $u_0 \in U$, $v \in U$. Then $\|q_\epsilon\| < \infty$ uniformly on ϵ and $\lim_{\epsilon \to 0} \|q_\epsilon - q_0\| = 0$.*

PROOF: It follows from (5) that

$$q_\epsilon(z) = \frac{1}{\epsilon}\iint_0^z [a(r, \xi_\epsilon(r), u_\epsilon(r)) - a(r, \xi_0(r), u_0(r))]\, dr + \frac{1}{\epsilon}\iint_0^z [b(r, \xi_\epsilon(r), u_\epsilon(r)) - b(r, \xi_0(r), u_0(r))]\, dw(r).$$

$$\tag{13}$$

Applying the Newton-Leibniz formula on the interval $0 \le \tau \le 1$ to the functions

$$\alpha(\tau) = a(r, \xi_\epsilon^\tau(r), u_\epsilon^\tau(r)), \quad \beta(\tau) = b(r, \xi_\epsilon^\tau(r), u_\epsilon^\tau(r)),$$

where

$$\xi_\epsilon^\tau(r) = \xi_0(r) + \tau(\xi_\epsilon(r) - \xi_0(r)),$$
$$u_\epsilon^\tau(r) = u_0(r) + \tau(u_\epsilon(r) - u_0(r)),$$

we represent (13) in the form

$$q_\epsilon(z) = \int\!\!\int_0^z [A_p^\epsilon(r) q_\epsilon(r) + A_u^\epsilon(r) v(r)]\, dr + \int\!\!\int_0^z [B_p^\epsilon(r) q_\epsilon(r) + B_u^\epsilon(r) v(r)]\, w(dr), \tag{14}$$

$$A_p^\epsilon(r) = \int_0^1 \nabla_p a(r, \xi_\epsilon^\tau(r), u_\epsilon^\tau(r))\, d\tau,$$

$$A_u^\epsilon(r) = \int_0^1 \nabla_u a(r, \xi_\epsilon^\tau(r), u_\epsilon^\tau(r))\, d\tau,$$

$$B_p^\epsilon(r) = \int_0^1 \nabla_p b(r, \xi_\epsilon^\tau(r), u_\epsilon^\tau(r))\, d\tau,$$

$$B_u^\epsilon(r) = \int_0^1 \nabla_u b(r, \xi_\epsilon^\tau(r), u_\epsilon^\tau(r))\, d\tau.$$

It follows from (14), (9), and the inequality $\|v\| < \infty$ that

$$\mathbf{M}|q_\epsilon(z)|^2 \le C\left(1 + \int\!\!\int_0^z \mathbf{M}|q_\epsilon(r)|^2\, dr\right).$$

Consequently ([9], Lemma 1), we have $\|q_\epsilon\| < C$.

It follows from (12) and (14) that for $l_\epsilon = q_\epsilon - q_0$

$$l_\epsilon(z) = h_\epsilon(z) + \int\!\!\int_0^z A_p^\epsilon(r) l_\epsilon(r)\, dr + \int\!\!\int_0^z B_p^\epsilon(r) l_\epsilon(r) w(dr), \tag{15}$$

$$h_\epsilon(z) = \int\!\!\int_0^z [(A_p^\epsilon(r) - A_p^0(r)) q_0(r) + (A_u^\epsilon(r) - A_u^0(r)) v(r)]\, dr$$

$$+ \int\!\!\int_0^z [(B_p^\epsilon(r) - B_p^0(r)) q_0(r) + (B_u^\epsilon(r) - B_u^0(r)) v(r)]\, w(dr).$$

From (15) and (9) we obtain

$$\mathbf{M}|l_\epsilon(z)|^2 \le C\left[\|h_\epsilon\|^2 + \int\!\!\int_0^z \mathbf{M}|l_\epsilon(r)|^2\, dr\right], \tag{16}$$

from which ([9], Lemma 1) we have $\|l_\epsilon\| \le C\|h_\epsilon\|$.

It remains to show that $\lim_{\epsilon \to 0} \|h_\epsilon\| = 0$. Since

$$\|h_\epsilon\|^2 \le C \int\!\!\int_D [\mathbf{M}|A_p^\epsilon(z) - A_p^0(z)) q_0(z)|^2 + \mathbf{M}|(B_p^\epsilon(z) - B_p^0(z)) q_0(z)|^2$$

$$+ \mathbf{M}|(A_u^\epsilon(z) - A_u^0(z)) v(z)|^2 + \mathbf{M}|(B_u^\epsilon(z) - B_u^0(z)) v(z)|^2]\, dz,$$

it suffices to prove that one of the terms on the right-hand side of (16) tends to zero. The proof is similar for the other terms.

Let $\chi_N(z)$ be the indicator function of the set $\{\omega : |q_0(z)| > N\}$. Representing $q_0(z)$ in the form $q_0(z) = q_0(z)\chi_N(z) + q_0(z)(1 - \chi_N(z))$ and using the inequalities $|A_p^\varepsilon(z) - A_p^0(z)| \leq C$ and $\|A_p^\varepsilon - A_p^0\| \leq C$ respectively according as $|q_0(z)| > N$ or $|q_0(z)| \leq N$ (these inequalities follow from (9) and (10) and the inequalities $\|q_\varepsilon\| < \infty$ and $\|v\| < \infty$), we obtain

$$\iint\limits_D \mathbf{M}|(A_p^\varepsilon(z) - A_p^0(z))q_0(z)|^2 \, dz \leq C[\|q_0\chi_N\|^2 + N^2\varepsilon^2]. \tag{17}$$

Since $\|q_0\| < \infty$, it follows that for any $\delta > 0$ there exists N such that $\|q_0\chi_N\|^2 < \delta/2C$. Fixing this N, we choose ε so that $N^2\varepsilon^2 < \varepsilon/2C$. Thus for any $\delta > 0$ there exists ε such that the right-hand side of inequality (17) becomes less than δ. The lemma is now proved.

PROOF OF THEOREM 1: By analogy with (14) it follows from (6) that

$$J_\varepsilon'(u_0, v) = \frac{1}{\varepsilon}[J(u_\varepsilon) - J(u_0)] = \mathbf{M}[q_\varepsilon^*(Z)F_p^\varepsilon(Z) + \iint\limits_D (q_\varepsilon^*(z)G_p^\varepsilon(z) + v^*(z)G_u^\varepsilon(z)) \, dz], \tag{18}$$

$$F_p^\varepsilon(z) = \int_0^1 \nabla_p F(\xi_\varepsilon^\tau(z)) \, d\tau,$$

$$G_p^\varepsilon(z) = \int_0^1 \nabla_p G(z, \xi_\varepsilon^\tau(z), u_\varepsilon^\tau(z)) \, d\tau,$$

$$G_u^\varepsilon(z) = \int_0^1 \nabla_u G(z, \xi_\varepsilon^\tau(z), u_\varepsilon^\tau(z)) \, d\tau.$$

Using (8) and (10) together with the inequalities $\|v\| < \infty$ and $\|q_\varepsilon\| < \infty$, it is easy to obtain the inequality

$$|J_\varepsilon'(u_0, v) - J_0'(u_0, v)| \leq C[\varepsilon + \|l_\varepsilon\|]$$

from (18) and (11). The theorem is now proved.

REMARK 1. It follows from (11) and (12) that $J_0'(u_0, v)$ is linear in v. Consequently the representation $J_0'(u_0, v) = \langle J_0'(u_0), v \rangle$, holds, where $J'(u_0)$ is the Gateaux derivative of the functional (6) at the point $u = u_0$. If the functional (6) is convex (for example, quadratic), then the condition $J'(u_0) = 0$ is necessary and sufficient for the control u_0 to be optimal [2].

EXAMPLE 1. For the control problem

$$\frac{\partial^2 \xi}{\partial x \partial y} = \alpha(z) + (\beta(z) + \gamma(z)u_0(z))\frac{\partial^2 w}{\partial x \partial y},$$

$$J(u) = \mathbf{M}\Big[\xi^2(Z) + \lambda \iint\limits_D u^2(z) \, dz\Big], \quad \lambda > 0,$$

the Gateaux differential has the form

$$J_0'(u_0, v) = 2\mathbf{M}\Big[\xi_0(Z)q_0(Z) + \lambda \iint\limits_D u_0(z)v(z) \, dz\Big],$$

$$q_0(z) = \iint\limits_0^z \gamma(r)v(r) \, w(dr).$$

om Itô's formula [5], taking account of the equality $Mq_0(z) = 0$, we obtain

$$M\{\xi_0 Z_v\} Z_v = M \iint_D (\beta(z) + \gamma(z)u_0(z))\gamma(z)v(z)\,dz.$$

onsequently

$$J_0'(u_0, v) = 2M \iint_D [(\beta(z) + \gamma(z)u_0(z))\gamma(z) + \lambda u_0(z)]v(z)\,dz = 0,$$

ience $u_0(z) = -\gamma(z)\beta(z)/(\lambda + \gamma^2(z))$.

Integral equations. Consider the optimal control problem for the stochastic integral equation ($t \in T$])

$$\xi(t) = \eta(t) + \int_0^t a(t, s, \xi(s), u(s))\,ds + \int_0^t b(t, s, \xi(s), u(s))\,dw(s) \tag{19}$$

th quality functional

$$J(u) = M[F(\xi(T)) + \int_0^T G(s, \xi(s), u(s))\,ds.] \tag{20}$$

The stochastic process $\eta(t)$ with values in R^n and the m-dimensional Wiener process $w(t)$ are inde-ndent and $\mathcal{F}_t$-measurable, $\{\mathcal{F}_t\}$ is a flow of σ-algebras on the probability space $\{\Omega, \sigma, P\}$ and U is the set admissible controls, i.e., $\mathcal{F}_t$-measurable functions $u(t)$ with values in R^l for which

$$\|u\|^2 = \sup_{0 \le t \le T} M|u(t)|^2 < \infty.$$

We shall assume that the functions $\eta(t)$ $a(t, s, x, u)$, $b(t, s, x, u)$, $F(x)$, and $G(t, x, u)$ have suitable nensions and satisfy conditions analogous to (7)–(10). We then have the following result.

EOREM 2. *Suppose $u_0 \in U$ and $v \in U$. Then the limit (1) and (2) for the control problem (19), (20)* sts *and equals*

$$J_0'(u_0, v) = M[q_0^*(T)\nabla_x F(\xi_0(T)) + \int_0^T (q_0^*(t)\nabla_x G(t, \xi_0(t), u_0(t)) + v^*(t)\nabla_u G(t, \xi_0(t), u_0(t)))\,dt].$$

Here $q_0(t)$ is a solution of the equation

$$q_0(t) = \int_0^t (\nabla_x a(t, s, \xi_0(s), u_0(s))q_0(s) + \nabla_u a(t, s, \xi_0(s), u_0(s))v(s))\,ds$$

$$+ \int_0^t (\nabla_x b(t, s, \xi_0(s), u_0(s))q_0(s) + \nabla_u b(t, s, \xi_0(*s), u_0(s))v(s))\,dw(s).$$

The proof is analogous to the proof of Theorem 1.

ʍARK 2. Theorem 2 can easily be generalized to equations with Poisson perturbations, and also to gro-functional equations of neutral type [11, 13].

ıMPLE 2. For the control problem

$$\xi(t) = \eta(t) + \int_0^1 (\beta(t, s) + \gamma(t, s)u(s))\,dw(s),$$

$$J(u) = M\left[\xi^2(T) + \lambda \int_0^T u^2(s)\,ds\right], \quad \lambda > 0,$$

we have

$$J_0'(u_0, v) = 2\mathbf{M}\left[\xi_0(T)q_0(T) + \lambda \int_0^T u_0(s)v(s)\,ds\right],$$

$$q_0(T) = \int_0^T \gamma(T, s)v(s)\,dw(s).$$

Computing the expression $\mathbf{M}\xi_0(T)q_0(T)$ directly, we obtain

$$J_0'(u_0, v) = 2\mathbf{M}\int_0^T [(\beta(T, s) + \gamma(T, s)u_0(s))\gamma(T, s) + \lambda u_0(s)]v(s)\,ds.$$

Hence $u_0(s) = -\beta(T, s)\gamma(T, s)/(\lambda + \gamma^2(T, s))$.

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