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STABILITY OF A POSITIVE EQUILIBRIUM STATE FOR A STOCHASTICALLY PERTURBED MATHEMATICAL MODEL OF GLASSY-WINGED SHARPSHOOTER POPULATION

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Abstract. The known nonlinear mathematical model of the Glassy-winged Sharpshooter is considered. It is assumed that this model is influenced by stochastic perturbations of the white noise type and these perturbations are directly proportional to the deviation of the system state from the positive equilibrium point. A necessary and sufficient condition for asymptotic mean square stability of the equilibrium point of the linear part of the considered stochastic differential equation is obtained. This condition is at the same time a sufficient one for stability in probability of the equilibrium point of the initial nonlinear equation. Numerical calculations and figures illustrate the obtained results.

1. **Introduction.** The incentive of this work was the paper [11] where the nonlinear mathematical model of the Glassy-winged Sharpshooter is investigated in the form

$$\dot{N}(t) = -rN(t)\ln\frac{N(t-\tau)}{K} - cN(t) + I, \tag{1}$$

with the initial condition

$$N(s) = N_0(s), \qquad s \in [-\tau, 0]. \tag{2}$$

Here $N(t)$ is the population size (number of insects) and the parameters and units are:

- τ - time delay (month),
- K - carrying capacity (number of insects),
- r - intrinsic rate of growth (1/month),
- c - harvesting effort (1/month),
- I - immigration rate (number of insects/month).

In the paper [11] the conditions for asymptotic stability of the positive equilibrium point of the equation (1), (2) are obtained. Here these stability conditions are generalized on the case when the equation (1), (2) is influenced by stochastic perturbations of the white noise type that are directly proportional to a deviation of the system state $N(t)$ from the positive equilibrium point. Similar investigations

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have been made previously for well known Nicholson's blowflies equation and some other nonlinear models (see, for example, [3, 9, 10] and references therein).

2. Equilibrium point and stochastic perturbations. The positive equilibrium point N^* of the system (1), (2) is defined by the equation

$$N^* \left(r \ln \frac{N^*}{K} + c \right) = I \tag{3}$$

that follows from (1), (2) by the condition $N(t) \equiv N^*$. The equation (3) cannot be solved analytically but can be easily solved numerically.

Let us assume that the equation (1) is influenced by stochastic perturbations of white noise type that are directly proportional to the deviation of the system state $N(t)$ from the equilibrium point N^* and influence $\dot{N}(t)$ immediately. Note that stochastic perturbations of such type were first proposed in [2] and successfully used later by other researches for different mathematical models (see, for example, [1, 4, 6, 7, 8]).

So, the equation (1) takes the form of Ito's stochastic differential equation [5]

$$\dot{N}(t) = -rN(t) \ln \frac{N(t-\tau)}{K} - cN(t) + I + \sigma(N(t) - N^*)\dot{w}(t), \tag{4}$$

where σ is a constant and $w(t)$ is the standard Wiener process [5]. Note that via the chosen type of stochastic perturbations the equilibrium point N^* of the equation (1) is also the equilibrium point of the equation (4).

3. Centering and linearization. Putting in (4) $N(t) = N^* + x(t)$ we will center the equation (4) in the neighborhood of the equilibrium point N^*. From (4) we have

$$\begin{aligned}
\dot{x}(t) =& -r(N^* + x(t)) \ln \frac{N^* + x(t-\tau)}{K} - c(N^* + x(t)) + I + \sigma x(t)\dot{w}(t) \\
=& -r(N^* + x(t)) \left(\ln \frac{N^* + x(t-\tau)}{K} - \ln \frac{N^*}{K} \right) \\
& -r(N^* + x(t)) \ln \frac{N^*}{K} - c(N^* + x(t)) + I + \sigma x(t)\dot{w}(t) \\
=& -r(N^* + x(t)) \left(\ln \frac{N^* + x(t-\tau)}{N^*} \right) \\
& -\left(r \ln \frac{N^*}{K} + c \right) x(t) - N^* \left(r \ln \frac{N^*}{K} + c \right) + I + \sigma x(t)\dot{w}(t).
\end{aligned}$$

As a result using (3) we obtain the nonlinear equation

$$\begin{gathered}
\dot{x}(t) = -r(N^* + x(t)) \ln \left(1 + \frac{x(t-\tau)}{N^*} \right) - \frac{I}{N^*} x(t) + \sigma x(t)\dot{w}(t), \\
x(s) = \phi(s), \qquad s \in [-\tau, 0],
\end{gathered} \tag{5}$$

with the trivial solution $x(t) \equiv 0$.

It is clear that stability of the zero solution of the equation (5) is equivalent to stability of the positive equilibrium N^* of the equation (4).

Using the representation $\ln(1+y) = y+o(y)$ (where $o(y)$ means that $\lim\limits_{y\to 0}\frac{o(y)}{y} = 0$) and neglecting $o(y)$, we obtain the linear part of the equation (5) in the form

$$\begin{aligned} \dot{x}(t) &= -\frac{I}{N^*}x(t) - rx(t-\tau) + \sigma x(t)\dot{w}(t), \\ x(s) &= \phi(s), \qquad s \in [-\tau, 0]. \end{aligned} \tag{6}$$

4. **Stability analysis.** Consider some definitions and auxiliary statements where $\mathbf{E}$ denotes the expectation and $\mathbf{P}\{A\}$ denotes the probability of an event A.

Definition 4.1. The zero solution of the equation (6) is called:

- mean square stable if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that the solution $x(t) = x(t,\phi)$ of the equation (6) satisfies the condition $\mathbf{E}|x(t,\phi)|^2 < \varepsilon$, $t \geq 0$, for arbitrary initial function $\phi(s)$ such that $\sup\limits_{s\leq 0}\mathbf{E}|\phi(s)|^2 < \delta$;

- asymptotically mean square stable if it is mean square stable and for each initial function $\phi(s)$ such that $\sup\limits_{s\leq 0}\mathbf{E}|\phi(s)|^2 < \infty$ the solution $x(t) = x(t,\phi)$ of the equation (6) satisfies the condition $\lim\limits_{t\to\infty}\mathbf{E}|x(t)|^2 = 0$.

Definition 4.2. The zero solution of the equation (5) is called stable in probability if for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there exists $\delta > 0$ such that the solution $x(t) = x(t,\phi)$ of the equation (5) satisfies the condition $\mathbf{P}\{\sup\limits_{t\geq 0}|x(t,\phi)| > \varepsilon_1\} < \varepsilon_2$ for any initial function $\phi(s)$ such that $\mathbf{P}\{\sup\limits_{s\leq 0}|\phi(s)| < \delta\} = 1$.

Consider now the linear Ito stochastic differential equation with delays

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + \sigma x(t-h)\dot{w}(t), \tag{7}$$

where A, B, σ, $\tau \geq 0$, $h \geq 0$ are known constants. A necessary and sufficient condition for asymptotic mean square stability of the trivial solution of the equation (7) is defined by the following lemma [9, 10].

Lemma 4.3. *A necessary and sufficient condition for asymptotic mean square stability of the trivial solution of (7) is*

$$A + B < 0, \qquad G^{-1} > \frac{1}{2}\sigma^2, \tag{8}$$

where

$$G = \begin{cases} \frac{Bq^{-1}\sin(q\tau)-1}{A+B\cos(q\tau)}, & B+|A| < 0, \quad q = \sqrt{B^2-A^2}, \\ \frac{1+|A|\tau}{2|A|}, & B = A < 0, \\ \frac{Bq^{-1}\sinh(q\tau)-1}{A+B\cosh(q\tau)}, & A+|B| < 0, \quad q = \sqrt{A^2-B^2}. \end{cases} \tag{9}$$

Corollary 1. *The zero solution of the equation (6) is asymptotically mean square stable if and only if*

$$\frac{1}{2}\sigma^2 < \begin{cases} \frac{I(N^*)^{-1}+r\cos(q\tau)}{1+rq^{-1}\sin(q\tau)}, & rN^* > I, \quad q = \sqrt{r^2-I^2(N^*)^{-2}}, \\ \frac{2r}{1+r\tau}, & rN^* = I, \\ \frac{I(N^*)^{-1}+r\cosh(q\tau)}{1+rq^{-1}\sinh(q\tau)}, & rN^* < I, \quad q = \sqrt{I^2(N^*)^{-2}-r^2}. \end{cases} \tag{10}$$

The proof follows from (6), (7), (8), (9).

Corollary 2. *By the condition (10) the zero solution of the equation (5) (and the equilibrium point N^* of the equation (4)) is stable in probability.*

The proof follows from [10] where it is shown that a sufficient condition for asymptotic mean square stability of the zero solution of the linear part (6) of the nonlinear equation (5) is at the same time a sufficient condition for stability in probability of the zero solution of the equation (5).

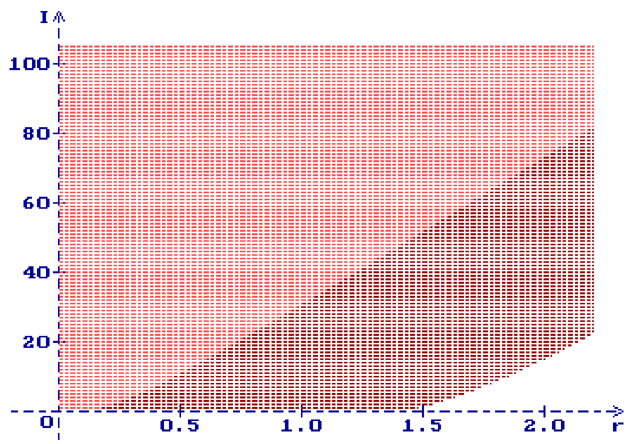

FIGURE 1. Stability region by the parameter values $K = 16$, $c = 0.35$, $\tau = 1$, $\varepsilon = 0.1$

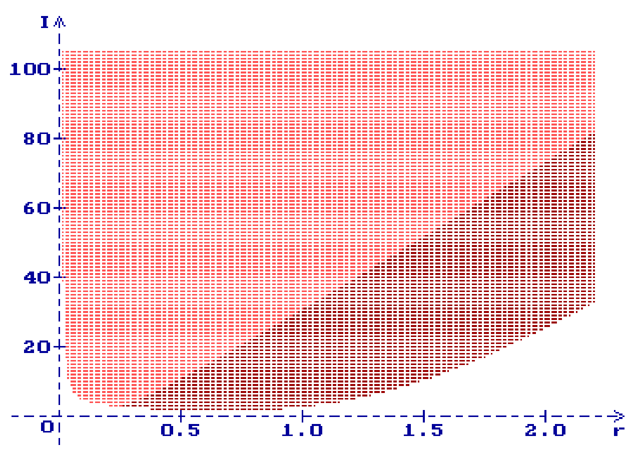

FIGURE 2. Stability region by the parameter values $K = 16$, $c = 0.35$, $\tau = 1$, $\varepsilon = 0.4$

Remark 1. Note that from the conditions (3), (10) it follows that the equilibrium point N^* satisfies to one of the two inequalities:

$$\frac{I}{r} < N^* < Ke^{1-\frac{c}{r}} \tag{11}$$

or

$$Ke^{1-\frac{c}{r}} < N^* < \frac{I}{r}. \tag{12}$$

Put, for example, $r = 0.52$, $K = 16$, $c = 0.35$, $I = 0.25$. Solving the equation (3) with these values of the parameters we obtain that $N^* = 8.63$. So, the inequality (11) holds and takes the form:

$$\frac{0.25}{0.52} = 0.48 < 8.63 < 22.19 = 16e^{1-\frac{0.35}{0.52}}.$$

Put now $r = 9$, $K = 16$, $c = 0.5$, $I = 400$. Solving the equation (3) in this case we obtain that $N^* = 42.778$. So, the inequality (12) holds and takes the form:

$$16e^{1-\frac{0.5}{9}} = 41.142 < 42.778 < 44.444 = \frac{400}{9}.$$

5. **Numerical illustrations.** The stability condition (10) together with the equation (3) can be used for numerical investigation of the considered model (4) in different directions. In particular, it allows to construct regions of stability in probability for the equilibrium point N^* of the equation (4) for different values of the parameters K, I, r, c, τ and ε, where

$$\varepsilon = \frac{1}{2}\sigma^2,$$

and to investigate dependence of stability regions on the level of noise ε, on the delay τ and on other parameters.

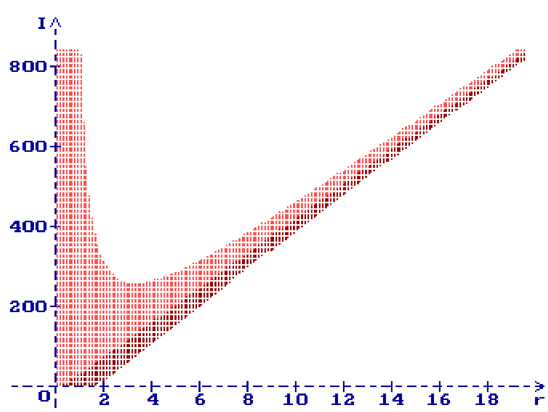

FIGURE 3. Stability region from Figure 1 in another scale

In Figure 1 the corresponding stability region is shown in the space of the parameters (r, I) for the following values of the other parameters: $K = 16$, $c = 0.35$, $\tau = 1$, $\varepsilon = 0.1$. Lighter part of the region is obtained by the condition $N^*r < I$, darker part of the region is obtained by the condition $N^*r > I$. In Figure 2 the similar picture for $\varepsilon = 0.4$ and the same values of the other parameters is shown. In Figure 3 and Figure 4 respectively the same pictures are shown in another scale.

In Figure 5 in the space of the parameters (r, N) the functions $N = \frac{I}{r}$ (the curve 1), $N = Ke^{1-\frac{c}{r}}$ (the curve 2) and the solution N^* of the equation (3) (the curve 3)

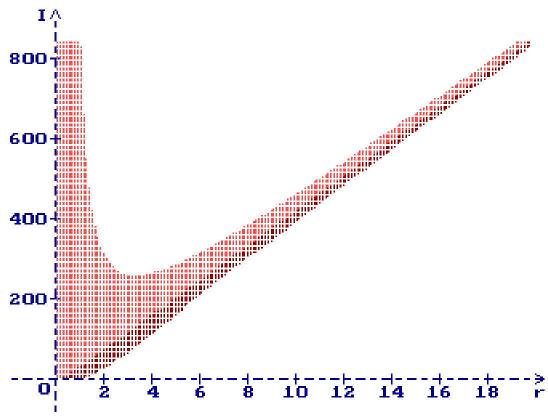

FIGURE 4. Stability region from Figure 2 in another scale

are shown by the values of the parameters: $K = 16$, $I = 13$, $c = 0.5$. In compliance with the inequalities (11), (12) the curve 3 locates between the curves 1 and 2.

The part AC of the curve 3 is a part of the stable equilibrium points N^* by $\tau = 1$ and $\varepsilon = 0.6$ (AB by the condition $rN^* < I$ and BC by the condition $rN^* > I$).

In Figure 6 a similar picture for $\tau = 1.5$ and the same values of the other parameters is shown.

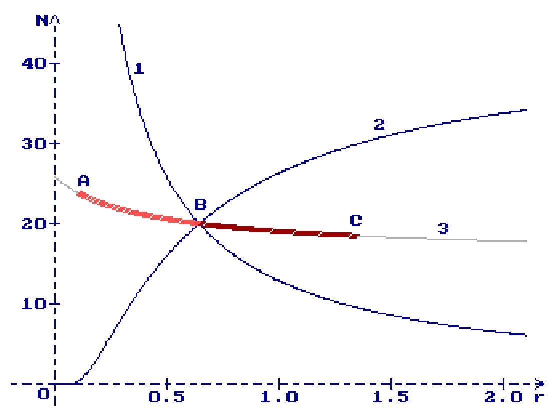

FIGURE 5. The part AC of the curve 3 is a part of the stable equilibrium points N^* by the values of the parameters $K = 16$, $I = 13$, $c = 0.5$, $\tau = 1$, $\varepsilon = 0.6$

6. **Conclusions.** It is supposed that the known nonlinear mathematical model of the Glassy-winged Sharpshooter is influenced by stochastic perturbations that are directly proportional to the deviation of the system state from the positive equilibrium point. By this the equilibrium point of the system under stochastic perturbations coincides with the equilibrium point of the deterministic system. A

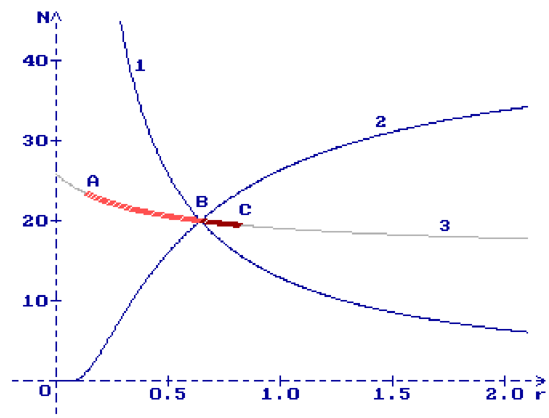

FIGURE 6. The part AC of the curve 3 is a part of the stable equilibrium points N^* by the values of the parameters $K = 16$, $I = 13$, $c = 0.5$, $\tau = 1.5$, $\varepsilon = 0.6$

necessary and sufficient condition for asymptotic mean square stability of the positive equilibrium point of the linearization system is obtained. This condition is also a sufficient one for stability in probability of the positive equilibrium point of the initial nonlinear system. The obtained results are illustrated by numerical calculations. In particular, stability regions in the space of the system parameters are constructed.

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