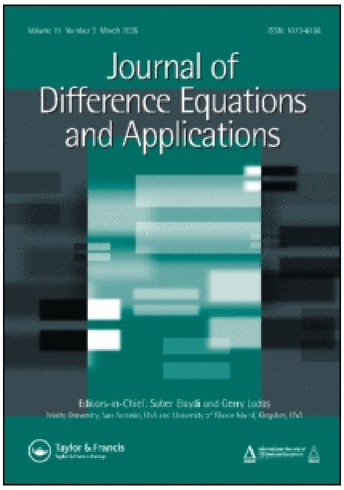

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About an unsolved stability problem for a stochastic difference equation with continuous time

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About an unsolved stability problem for a stochastic difference equation with continuous time

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An unsolved problem of stability for stochastic difference equation with continuous time is proposed for consideration.

Keywords: unsolved problem; asymptotic mean square stability; stochastic difference equation with continuous time

Contributions to the theory and application of difference equations and stochastic difference equations with continuous time are advancing (see, for instance [1–6,8–12]). At the same time, there are a number of simple problems that are easy to formulate but whose solutions are unknown. In order to attract attention to such problems, a stability problem for a stochastic difference equation is proposed. This problem is close enough to a known result, but, nevertheless, is not solved until now. To solve this problem maybe it is necessary to use some new ideas.

Let $\{\Omega, \mathfrak{F}, P\}$ be a probability space, $\{\mathfrak{F}_t, t \geq 0\}$ be a non-decreasing family of sub-σ-algebras of $\mathfrak{F}$ and E be the expectation with respect to the measure P.

Consider the scalar stochastic difference equation with continuous time

$$\begin{aligned} x(t+1) &= ax(t) + bx(t-1) + \sigma x(t)\xi(t+1), \quad t > -1, \\ x(\theta) &= \phi(\theta), \quad \theta \in \Theta = [-2, 0], \end{aligned} \tag{1}$$

where a, b and σ are known constants, the perturbation $\xi(t)$ is a $\mathfrak{F}_t$-measurable stationary stochastic process such that

$$\mathrm{E}\xi(t) = 0, \quad \mathrm{E}\xi^2(t) = 1.$$

Definition 1. *The trivial solution of equation* (1) *is called mean square stable if for any* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that* $E|x(t;\phi)|^2 < \epsilon$ *for all* $t \geq 0$ *if* $\|\phi\|^2 = \sup_{\theta \in \Theta} E|\phi(\theta)|^2 < \delta$.

Definition 2. *The trivial solution of equation* (1) *is called asymptotically mean square stable if it is mean square stable, and for each initial function* ϕ

$$\lim_{t \to \infty} E|x(t;\phi)|^2 = 0.$$

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Definition 3. *The trivial solution of equation* (1) *is called asymptotically mean square quasistable, if it is mean square stable and for each* $t \in [0, 1)$*, each initial function* ϕ *and a positive integer* j

$$\lim_{j\to\infty} E|x(t+j;\phi)|^2 = 0.$$

Remark 1. It is evident that asymptotic mean square quasistability follows from asymptotic mean square stability but the converse statement is not true [10].

Similar to [7], it can be shown that the necessary and sufficient condition for asymptotic mean square quasistability of the trivial solution of equation (1) is

$$|b| < 1, \quad |a| < 1 - b, \quad \sigma^2 < \frac{1+b}{1-b}[(1-b)^2 - a^2]. \tag{2}$$

Stability regions defined by conditions (2) are shown in Figure 1 for different values of σ^2: (1) $\sigma^2 = 0$, (2) $\sigma^2 = 0.4$ and (3) $\sigma^2 = 0.8$.

Consider now the difference equation

$$\begin{aligned} x(t+h) &= ax(t) + b\int_{t-h}^{t} x(s)\mathrm{d}s + \sigma x(t)\xi(t+h), \quad t > -h, \\ x(\theta) &= \phi(\theta), \quad \theta \in \Theta = [-2h, 0], \end{aligned} \tag{3}$$

where $h > 0$ and all other parameters are the same as in equation (1).

In the case $\sigma = 0$, the characteristic equation of equation (3) is

$$\mathrm{e}^{\lambda h} = a + \frac{b}{\lambda}(1 - \mathrm{e}^{-\lambda h}). \tag{4}$$

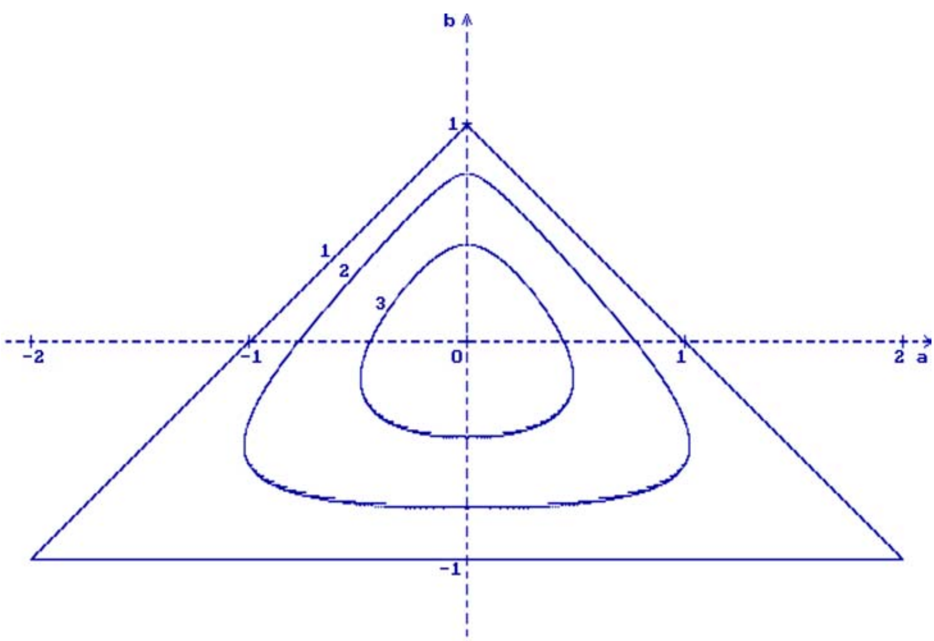

Figure 1. Regions of stability for equation (1): (1) $\sigma^2 = 0$, (2) $\sigma^2 = 0.4$ and (3) $\sigma^2 = 0.8$.

Putting $\lambda = i\omega$, $i^2 = -1$, transform equation (4) to the system of two equations

$$\cos \omega h = a + \frac{b}{\omega} \sin \omega h, \quad \sin \omega h = -\frac{b}{\omega}(1 - \cos \omega h). \tag{5}$$

It is easy to show that system (5) has three solutions:

$$\begin{gathered} a = 1, \qquad b = -\omega \tan \frac{\omega h}{2}, \\ a + bh = 1, \\ a = \cos \omega h + 2\cos^2 \frac{\omega h}{2}, \quad b = -\omega \cot \frac{\omega h}{2}. \end{gathered} \tag{6}$$

Solutions (6) define the region of asymptotic stability for the trivial solution of equation (3) if $\sigma = 0$. In Figure 2, the corresponding stability region (the bound 1) is shown for $h = 1$.

Immediately from (3), it follows $Ex^2(t+h) \leq [(|a| + |b|h)^2 + \sigma^2]\sup_{s\leq[t-h,t]} Ex^2(s)$. Thus, the inequality

$$|a| + |b|h < \sqrt{1 - \sigma^2}, \tag{7}$$

is a sufficient condition for asymptotic mean square stability of the trivial solution of equation (3). Corresponding stability region is shown in Figure 2 (the bound 2) for $h = 1$ and $\sigma^2 = 0.4$.

The problem is: to get the necessary and sufficient conditions for asymptotic mean square stability of the trivial solution of equation (3) for $\sigma \neq 0$.

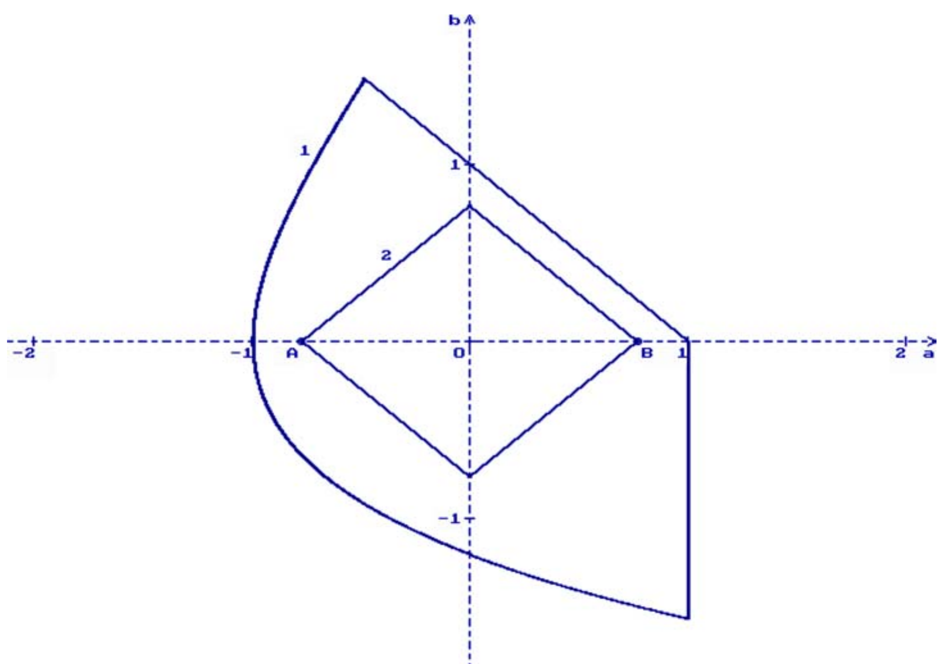

Figure 2. Regions of stability for equation (3): (1) $h = 1$, $\sigma^2 = 0$ and (2) $h = 1$, $\sigma^2 = 0.4$.

Remark 2. If $b = 0$, then condition (7) takes the form $a^2 + \sigma^2 < 1$ and coincides with (2). It is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of equation (3) in the case $b = 0$. Thus, the points A and B (in Figure 2) with the coordinates $-\sqrt{1 - \sigma^2}$ and $\sqrt{1 - \sigma^2}$, respectively, belong to the bound of the exact stability region.

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