ASYMPTOTIC STABILITY ANALYSIS OF A STOCHASTIC VOLTERRA INTEGRO-DIFFERENTIAL EQUATION WITH FADING MEMORY

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Abstract. We investigate the long term behavior of solutions to a stochastic Volterra integro-differential equation with a fading memory; the fading memory is represented by using a decaying exponential convolution kernel. We give sufficient conditions for asymptotic mean square stability of the solution.

In a similar spirit, we investigate the long term behavior of solutions to discrete analogues of the above continuous problem; our discrete analogues are based on the Euler-Maruyama scheme for stochastic differential equations and θ-methods for approximating the integral term. We give necessary and sufficient conditions for asymptotic mean square stability of the trivial solution, obtaining our results by means of the general method of Lyapunov functionals construction. We focus also on the geometric interpretations of our findings, such as the sizes of stability regions. This enables us to make some conclusions with regards to choosing an appropriate θ-method for obtaining numerical approximations.

Keywords. Volterra, Lyapunov functionals, stochastic, integro-differential equation, difference equation, numerical analysis

1 Introduction

It is well known that Volterra functional equations are used to model systems with memory; such applications arise in fields as diverse as population dynamics, nuclear reactor dynamics, viscoelasticity and mathematical finance. See, for example, [4], [10], [12] and [15] for further discussions of applications. Randomness has a variety of roles to play in the modelling process; whether it be to simulate the random effects observed in the real-world system (such as the random occurrence of natural disasters having an effect on a population) or whether it be to simulate a parameter in the model which is only known to a particular degree of accuracy (see [19]).

Since mathematical models can be rather complex it is often necessary to obtain approximate solutions to the model equations through discretisation. When obtaining such approximate solutions one may be interested in obtaining a very precise solution over a short period of time or obtaining information about the qualitative behavior of the solution over a long period of time. Additionally, one must also be sure that the solution to the discrete problem is indeed providing a solution to the original continuous problem. The introduction of randomness to the problem introduces further considerations; multiple solutions to the problem will exist, dependent upon the random numbers generated and so we need to consider some sort of averaging process when making general statements about solutions.

Our particular equation under consideration is

$$dx(t) = \left(\alpha x(t) + \beta \int_0^t e^{-\lambda(t-s)} x(s) ds \right) dt + \sigma x(t-\tau) dw(t), \qquad (1.1)$$

with the initial condition

$$x(s) = \varphi_0(s), \quad s \in [-\tau, 0], \qquad (1.2)$$

where $\alpha, \beta, \lambda, \sigma, \tau \geq 0$ are real constants and $w(t)$ is a standard Wiener process.

This equation has been chosen for a number of reasons. Firstly, it is a natural extension of previous work (see [22]) where the equation (1.1) is considered with $\lambda = 0$. Secondly, the kernel is of fading memory type; fading memory systems have a role to play in modelling hereditary systems in which the past has greater influence the more recent it is (see for example [16]). Thirdly, previous work on similar deterministic systems have yielded a rich variety of outcomes, influenced by choices of the original representation of the system and various discretisations ([5], [6], [7], [8], [9], [11], [17] are examples of some of the work available).

Stochastic Volterra equations is a growth area that is attracting interest, particularly in the study of the qualitative behaviour of solutions. Such works include the following publications: [1], [2], [3], [18], [23].

Our approach is to investigate the asymptotic mean square stability of the zero solution to the equation (1.1), (1.2) using Kolmanovskii's and Shaikhet's general method of Lyapunov functionals construction for hereditary systems. Following this, we discretise the continuous problem, leading to a system of Volterra difference equations. We then take a similar approach (i.e. Lyapunov functionals construction) to obtain conditions for asymptotic mean square stability of this system. Our discretisation process is a hybrid scheme using the Euler-Maruyama scheme for the SDE part and a θ-method for the Volterra integral operator. We note that our delay term in the stochastic system does not complicate the analysis further and is a straightforward extension. This method is described and applied in, amongst others, the following papers: [13], [14], [20], [22].

The Existence and uniqueness of a solution for equations of type (1.1) are considered in [12]. Below we will use the following definitions of stability.

Definition 1.1 *The trivial solution of equation (1.1), (1.2) is called:*

(i) mean square stable if for each $\epsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|x(t)|^2 < \epsilon$ for all $t \geq 0$ if $\sup_{s \in [-\tau, 0]} \mathbf{E}|\varphi_0(s)|^2 < \delta$;

(ii) asymptotically mean square stable if it is mean square stable and $\lim_{t \to \infty} \mathbf{E}|x(t)|^2 = 0$ for each initial function φ_0.

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a basic probability space, $\mathbf{E}$ be an expectation, $i \in Z \bigcup Z_0$ be a discrete time, $Z = \{0, 1, ...\}$, $Z_0 = \{-m, ..., 0\}$, $\mathfrak{F}_i \in \mathfrak{F}$, $i \in Z$, be a family of σ-algebras, ξ_i, $i \in Z$, be a sequence of $\mathfrak{F}_i$-adapted random variables, process $x_i \in \mathbf{R^n}$ be a solution of the stochastic difference equation

$$x_{i+1} = F(i, x_{i-m}, ..., x_i) + G(i, x_{i-m}, ..., x_i)\xi_{i+1}, \qquad i \in Z, \qquad (1.3)$$

with the initial condition

$$x_i = \varphi_{0i}, \qquad i \in Z_0. \qquad (1.4)$$

It is assumed that $F(i, 0, ..., 0) = 0$, $G(i, 0, ..., 0) = 0$.

Definition 1.2 *The solution of equation (1.3) with the initial function (1.4) is called:*

(i) uniformly mean square bounded if $\sup_{i \in Z} \mathbf{E}|x_i|^2 < \infty$;

(ii) asymptotically mean square trivial if $\lim_{i \to \infty} \mathbf{E}|x_i|^2 = 0$.

Definition 1.3 *The trivial solution of equation (1.3) with the initial function (1.4) is called:*

(i) mean square stable if for each $\epsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|x_i|^2 < \epsilon$, $i \in Z$, if $\sup_{i \in Z_0} \mathbf{E}|\varphi_{0i}|^2 < \delta$;

(ii) asymptotically mean square stable if it is mean square stable and for each initial function φ_0 the solution of equation (1.3) is asymptotically mean square trivial.

2   Stability of the integro-differential equation

Consider the equation (1.1), (1.2) and put

$$y_1(t) = \int_0^t e^{-\lambda(t-s)} x(s)ds, \qquad y_2(t) = x(t). \qquad (2.1)$$

From (1.1), (2.1) it follows that

$$\begin{aligned}
dy_1(t) &= (-\lambda y_1(t) + y_2(t))dt, \\
dy_2(t) &= (\beta y_1(t) + \alpha y_2(t))dt + \sigma y_2(t-\tau)dw(t).
\end{aligned} \qquad (2.2)$$

If $\sigma = 0$ then from (2.2) we obtain

$$\begin{aligned}
\ddot{y}_2(t) &= \beta \dot{y}_1(t) + \alpha \dot{y}_2(t) \\
&= \beta(-\lambda y_1(t) + y_2(t)) + \alpha \dot{y}_2(t) \\
&= -\lambda\beta y_1(t) + \beta y_2(t) + \alpha \dot{y}_2(t) \\
&= -\lambda(\dot{y}_2(t) - \alpha y_2(t)) + \beta y_2(t) + \alpha \dot{y}(t) \\
&= (\alpha - \lambda)\dot{y}_2(t) + (\alpha\lambda + \beta)y_2(t).
\end{aligned}$$

As a result

$$\ddot{y}_2(t) = (\alpha - \lambda)\dot{y}_2(t) + (\alpha\lambda + \beta)y_2(t). \qquad (2.3)$$

Thus, if $\sigma = 0$ then the inequalities

$$\alpha < \lambda, \qquad \alpha\lambda + \beta < 0, \qquad (2.4)$$

are the necessary and sufficient conditions for asymptotic stability of the trivial solution of equation (2.3), or the system (2.2).

Besides, if $\beta = 0$ then the inequality

$$\alpha + \delta < 0, \qquad \delta = \frac{1}{2}\sigma^2 \qquad (2.5)$$

is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of the equation (1.1), (1.2).

To obtain a stability condition for the trivial solution of the equation (1.1), (1.2) in the general case we put

$$A = \begin{pmatrix} -\lambda & 1 \\ \beta & \alpha \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}, \quad y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}. \qquad (2.6)$$

Then system (2.1) can be represented in the form

$$\dot{y}(t) = Ay(t) + By(t-\tau)dw(t). \qquad (2.7)$$

Following the general method of Lyapunov functionals construction [13] consider a Lyapunov functional for equation (2.7) in the form $V(t) = V_1(t) + V_2(t)$, where $V_1(t) = y'(t)Py(t)$ and the matrix P is a positive definite solution of the matrix equation

$$A'P + PA + B'PB = -C. \qquad (2.8)$$

Here the matrices A and B are defined in (2.6),

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}, \quad c > 0. \qquad (2.9)$$

Let L be the infinitesimal operator [12] of equation (2.7). Via (2.7) we have

$$LV_1(t) = y'(t)PAy(t) + y'(t)A'Py(t) + y'(t-\tau)B'PBy(t-\tau). \qquad (2.10)$$

Choosing $V_2(t)$ in the form

$$V_2(t) = \int_{t-\tau}^{t} y'(s)B'PBy(s)ds, \qquad (2.11)$$

we obtain

$$LV_2(t) = y'(t)B'PBy(t) - y'(t-\tau)B'PBy(t-\tau). \qquad (2.12)$$

From (2.8), (2.9), (2.10), (2.11), (2.12) for the functional $V(t) = V_1(t) + V_2(t)$ it follows that

$$LV = -\left(y_1^2(t) + cy_2^2(t)\right). \qquad (2.13)$$

So, the following lemma is now proven:

Lemma 2.1 *If for some positive number c the matrix equation (2.8) has a positive definite solution P then the trivial solution of the equation (1.1), (1.2) is asymptotically mean square stable.*

Note that via (2.5), (2.6), (2.9) the matrix equation (2.8) is equivalent to the system of equations

$$\begin{aligned}
2\left(\beta p_{12} - \lambda p_{11}\right) &= -1, \\
p_{11} + (\alpha - \lambda)p_{12} + \beta p_{22} &= 0, \\
2\left(p_{12} + (\alpha + \delta)p_{22}\right) &= -c,
\end{aligned} \qquad (2.14)$$

with the solution

$$\begin{aligned}
p_{11} &= \frac{(\alpha - \lambda)(\alpha + \delta) - \beta(1 - c\beta)}{2\left((\alpha\lambda + \beta - \lambda^2)(\alpha + \delta) - \lambda\beta\right)}, \\
p_{12} &= -\frac{\alpha + \delta - c\lambda\beta}{2\left((\alpha\lambda + \beta - \lambda^2)(\alpha + \delta) - \lambda\beta\right)}, \\
p_{22} &= \frac{1 - c\left(\alpha\lambda + \beta - \lambda^2\right)}{2\left((\alpha\lambda + \beta - \lambda^2)(\alpha + \delta) - \lambda\beta\right)}.
\end{aligned} \qquad (2.15)$$

Note also that the matrix P is a positive definite one if and only if

$$p_{11} > 0, \quad p_{22} > 0, \quad p_{11}p_{22} > p_{12}^2. \qquad (2.16)$$

Consider the following inequalities:

$$\begin{aligned}
\left(\alpha\lambda + \beta - \lambda^2\right)(\alpha + \delta) &> \lambda\beta, & (2.17) \\
(\alpha - \lambda)(\alpha + \delta) &> \beta(1 - c\beta), & (2.18) \\
\left(\alpha\lambda + \beta - \lambda^2\right)(\alpha + \delta) &< \lambda\beta, & (2.19) \\
(\alpha - \lambda)(\alpha + \delta) &< \beta(1 - c\beta), & (2.20)
\end{aligned}$$

and

$$((\alpha - \lambda)(\alpha + \delta) - \beta(1 - c\beta))(1 - c(\alpha\lambda + \beta - \lambda^2)) > (\alpha + \delta - c\beta\lambda)^2.$$

$$(2.21)$$

From (2.15), (2.16) and Lemma 2.1 we obtain the following lemma:

Lemma 2.2 *If for some $c > 0$ the parameters α, β, λ and δ satisfy the inequalities (2.17), (2.18), (2.21) then the trivial solution of (1.1), (1.2) is asymptotically mean square stable.*

We can now prove the following corollary:

Corollary 2.3 *If*

$$\lambda > 0, \quad \beta < 0, \quad \alpha + \delta < 0 \tag{2.22}$$

then the trivial solution of (1.1), (1.2) is asymptotically mean square stable.

Proof: Substitute into (2.18), (2.21)

$$c = \frac{\alpha + \delta}{\lambda\beta}. \tag{2.23}$$

From conditions (2.22) it follows that $c > 0$ and inequalities (2.17), (2.18), (2.21) hold. Via Lemma 2.2 the proof is completed.

Remark 2.4 *If c is defined as in (2.23) then the solution (2.15) of the system (2.14) is*

$$p_{11} = \frac{1}{2\lambda}, \quad p_{22} = -\frac{1}{2\beta\lambda}, \quad p_{12} = 0, \tag{2.24}$$

and via conditions (2.22) it satisfies inequalities (2.16).

Remark 2.5 *If conditions (2.22) hold then conditions (2.4) and (2.5) hold too.*

Corollary 2.6 *If*

$$\alpha < \lambda - \delta, \quad \beta < -\frac{\lambda(\lambda - \alpha)(\alpha + \delta)}{\lambda - \alpha - \delta} \tag{2.25}$$

then the trivial solution of the equation (1.1), (1.2) is asymptotically mean square stable.

Proof: Via Lemma 2.2 it is enough to show that conditions (2.17), (2.18) and (2.21) hold. From (2.25) it follows that the inequality (2.17) holds. We represent the inequality (2.18) in the form

$$(\lambda - \alpha)(\alpha + \delta) + \beta < c\beta^2 \tag{2.26}$$

and show that the left hand side of (2.26) is negative, as follows: from (2.25) it follows that

$$
\begin{aligned}
(\lambda - \alpha)(\alpha + \delta) + \beta \quad &< \quad (\lambda - \alpha)(\alpha + \delta)\left(1 - \frac{\lambda}{\lambda - \alpha - \delta}\right) \\
&= \quad -\frac{(\lambda - \alpha)(\alpha + \delta)^2}{\lambda - \alpha - \delta} \qquad (2.27) \\
&< \quad 0.
\end{aligned}
$$

From (2.27) it follows that (2.26) holds for arbitrary $c > 0$. In order to show that for some $c > 0$ (2.21) holds too we represent (2.21) in the form

$$
q_1 c^2 + \left((q_1 + \lambda^2)\, q_2 + 2q_3\lambda\beta + \beta^2\right) c + q_2 - q_3^2 > 0, \qquad (2.28)
$$

where $q_1 = -\beta^2(\alpha\lambda + \beta)$, $q_2 = -((\lambda - \alpha)(\alpha + \delta) + \beta)$, $q_3 = \alpha + \delta$. From (2.25) it follows that

$$
\beta < -\lambda(\alpha + \delta)\frac{\lambda - \alpha}{\lambda - \alpha - \delta} \le -\lambda\alpha.
$$

Thus, $q_1 > 0$. This means that if (2.25) holds then there exists a large enough $c > 0$ such that (2.28) (and (2.21) too) holds. The proof is now completed.

Remark 2.7 *If $\delta = 0$ then condition (2.25) coincides with (2.4).*

Remark 2.8 *From the inequalities (2.17), (2.18) it follows that $p_{11} > 0$, from the inequalities (2.17), (2.18), (2.21) it follows that the matrix P with the elements (2.15) is a positive definite matrix. From the inequalities (2.19), (2.20) it follows that $p_{11} > 0$ too, but the inequalities (2.19), (2.20) are incompatible with (2.21).*

We can show that the inequalities (2.19), (2.20) are incompatible with (2.21) as follows. Assume firstly that $\beta = 0$. Then from (2.19), (2.20) and (2.21) it follows that

$$
\lambda > 0, \quad (\alpha - \lambda)(\alpha + \delta) < 0, \quad c\lambda(\alpha - \lambda) > 1.
$$

So, $\alpha > \lambda$, $\alpha + \delta < 0$ and we obtain the following contradiction:

$$
0 < \lambda < \alpha < -\delta < 0. \qquad (2.29)
$$

Assume now that $\beta \ne 0$. Then from (2.19), (2.20), (2.21) it follows that

$$
0 < \frac{1}{q_2} < c < \frac{q_1}{\beta^2}, \quad (c\beta^2 - q_1)(1 - cq_2) > (\alpha + \delta - c\beta\lambda)^2, \qquad (2.30)
$$

where

$$
q_1 = \beta - (\alpha - \lambda)(\alpha + \delta) > 0, \quad q_2 = \beta + \lambda(\alpha - \lambda) > 0. \qquad (2.31)
$$

If $\beta < 0$ then from $q_2 > 0$ it follows that $\alpha > \lambda$ and from $q_1 > 0$ it follows that $\alpha + \delta < 0$. So, we obtain again the contradiction (2.29). Let be $\beta > 0$. If $\alpha < \lambda$ then from (2.30), (2.31), (2.19), (2.20) it follows that

$$
\begin{aligned}
\beta^2 &< q_1 q_2 \\
&= (\beta + (\lambda - \alpha)(\alpha + \delta))(\beta - \lambda(\lambda - \alpha)) \\
&= \beta^2 + (\lambda - \alpha)\left[(\alpha\lambda + \beta - \lambda^2)(\alpha + \delta) - \beta\lambda\right] \\
&< \beta^2
\end{aligned}
$$

and we obtain a contradiction. If $\alpha \geq \lambda$ then from (2.19), (2.20) it follows that

$$
\beta(\alpha + \delta) \leq (\alpha\lambda + \beta - \lambda^2)(\alpha + \delta) < \lambda\beta,
$$

and we obtain the contradiction $\alpha + \delta < \lambda \leq \alpha$.

3　Stability of a difference analogue

3.1　Construction of a difference analogue

Consider a difference analogue to the equation (1.1), (1.2), constructed by combining the Euler-Maruyama scheme for stochastic differential equations with a θ-method for the integral term. So, we obtain the following system with discrete time:

$$
x_1 = (a + b)x_0 + \sigma_0 x_{-m}\xi_1,
$$

$$
x_2 = ax_1 + b\left(\theta\gamma x_0 + (1 - \theta) x_1\right) + \sigma_0 x_{1-m}\xi_2,
$$

$$
x_{i+1} = ax_i + b\left(\theta\gamma^i x_0 + \sum_{j=1}^{i-1}\gamma^{i-j}x_j + (1 - \theta) x_i\right) + \sigma_0 x_{i-m}\xi_{i+1}, \quad (3.1)
$$

$$
i \geq 2, \quad x_j = \varphi_{0j}, \quad j = -m, ..., 0,
$$

where

$$
a = 1 + \alpha h, \quad b = \beta h^2, \quad \gamma = e^{-\lambda h}, \quad \sigma_0 = \sigma h^{\frac{1}{2}}, \quad \theta \in [0, 1]. \quad (3.2)
$$

Here $m = \tau h^{-1}$ is an integer, $h = t_{i+1} - t_i$ is a step of quantization, $x_i = x(t_i)$, $\xi_i = h^{-\frac{1}{2}}(w(t_i) - w(t_{i-1}))$, $i \in Z$, are $\mathfrak{F}_i$-adapted and mutually independent random variables such that $\mathbf{E}\xi_i = 0$, $\mathbf{E}\xi_i^2 = 1$.

We transform the system (3.1) for $i \geq 2$ in the following way:

$$
\begin{aligned}
x_{i+1} &= (a + b(1 - \theta)) x_i + b\gamma x_{i-1} + b\theta\gamma^i x_0 + \gamma b \sum_{j=1}^{i-2}\gamma^{i-1-j}x_j + \sigma_0 x_{i-m}\xi_{i+1} \\
&= (a + b(1 - \theta)) x_i + b\gamma x_{i-1} + b\theta\gamma^i x_0 + \sigma_0 x_{i-m}\xi_{i+1} \\
&\quad + \gamma\left(x_i - (a + b(1 - \theta)) x_{i-1} - b\theta\gamma^{i-1}x_0 - \sigma_0 x_{i-1-m}\xi_i\right) \\
&= (\gamma + a + b(1 - \theta)) x_i + \gamma(b\theta - a) x_{i-1} + \sigma_0 x_{i-m}\xi_{i+1} - \gamma\sigma_0 x_{i-1-m}\xi_i.
\end{aligned}
$$

As a result we obtain

$$x_{i+1} = Ax_i + Bx_{i-1} + \sigma_1 x_{i-m}\xi_{i+1} + \sigma_2 x_{i-1-m}\xi_i, \tag{3.3}$$

where

$$A = \gamma + a + b(1-\theta), \quad B = \gamma(b\theta - a), \quad \sigma_1 = \sigma_0, \quad \sigma_2 = -\gamma\sigma_0. \tag{3.4}$$

3.2 Stability conditions in the space of the auxiliary parameters: analytical representation

It is known [20] that the necessary and sufficient conditions for asymptotic mean square stability of the trivial solution of equation (3.3) with $\sigma_2 = 0$ are

$$|A| < 1 - B, \quad |B| < 1, \tag{3.5}$$

$$\sigma_1^2 < \frac{1+B}{1-B}\left((1-B)^2 - A^2\right). \tag{3.6}$$

To obtain stability conditions for equation (3.3) with arbitrary σ_2 we put

$$x(i) = \begin{pmatrix} x_{i-1} \\ x_i \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 \\ B & A \end{pmatrix}, \quad B_k = \begin{pmatrix} 0 \\ \sigma_k \end{pmatrix}, \quad k = 1, 2, \tag{3.7}$$

and represent equation (3.3) in the matrix form

$$x(i+1) = A_1 x(i) + B_1 x_{i-m}\xi_{i+1} + B_2 x_{i-1-m}\xi_i. \tag{3.8}$$

Following the general method of Lyapunov functionals construction [14] consider the auxiliary difference equation without delay:

$$x(i+1) = A_1 x(i), \tag{3.9}$$

i.e. equation (3.8) with $B_1 = B_2 = 0$ and suppose that the matrix equation

$$A_1' DA_1 - D = -U, \quad D = \begin{pmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.10}$$

has a positive semi-definite solution D with $d_{22} > 0$. The function $v_i = x'(i)Dx(i)$ is a Lyapunov function for equation (3.9) since $\Delta v_i = -x_i^2$. We will construct a Lyapunov functional V_i for equation (3.8) in the form $V_i = V_{1i} + V_{2i}$, where the main part $V_{1i} = v_i$ and the additional part V_{2i} will be chosen below. Calculating $\mathbf{E}\Delta V_{1i} = \mathbf{E}(V_{1,i+1} - V_{1i})$, via (3.8), (3.10) we obtain

$$
\begin{aligned}
\mathbf{E}\Delta V_{1i} &= \mathbf{E}\left(x'(i+1)Dx(i+1) - x'(i)Dx(i)\right) \\
&= \mathbf{E}((A_1 x(i) + B_1 x_{i-m}\xi_i)'D(A_1 x(i) \\
&\quad + B_1 x_{i-m}\xi_{i+1} + B_2 x_{i-1-m}\xi_i) - x'(i)Dx(i) \\
&= \mathbf{E}(x'(i)(A_1'DA_1 - D)x(i) + B_1'DB_1 x_{i-m}^2\xi_{i+1}^2 + B_2'DB_2 x_{i-1-m}^2\xi_i^2 \\
&\quad + 2B_1'DA_1 x(i)x_{i-m}\xi_{i+1} + 2B_2'DA_1 x(i)x_{i-1-m}\xi_i \\
&\quad + 2B_1'DB_2 x_{i-m} x_{i-1-m}\xi_{i+1}\xi_i).
\end{aligned}
$$

From (3.7), (3.8), (3.10) and the properties of ξ_i it follows that

$$x'(i)\left(A_1'DA_1 - D\right)x(i) = -x_i^2,$$

$$B_k'DB_k = \sigma_k^2 d_{22}, \ \ k = 1, 2,$$

$$\mathbf{E}x_{i-m}^2 \xi_{i+1}^2 = \mathbf{E}x_{i-m}^2,$$

$$B_2'DA_1 = \sigma_2\left(Bd_{22}, d_{12} + Ad_{22}\right),$$

$$\mathbf{E}x(i)x_{i-1-m}\xi_i = \left(0, \mathbf{E}x_i x_{i-1-m}\xi_i\right)',$$

$$\begin{aligned}
\mathbf{E}x_i x_{i-1-m}\xi_i &= \mathbf{E}(Ax_{i-1} + Bx_{i-2} + \sigma_1 x_{i-1-m}\xi_i \\
&\quad + \sigma_2 x_{i-2-m}\xi_{i-1})x_{i-1-m}\xi_i \\
&= \sigma_1 \mathbf{E}x_{i-1-m}^2.
\end{aligned}$$

$$B_2'DA_1 \mathbf{E}x(i)x_{i-1-m}\xi_i = \sigma_1\sigma_2\left(d_{12} + Ad_{22}\right)\mathbf{E}x_{i-1-m}^2.$$

Thus, we have

$$\begin{aligned}
\mathbf{E}\Delta V_{1i} &= -\mathbf{E}x_i^2 + \sigma_1^2 d_{22}\mathbf{E}x_{i-m}^2 \\
&\quad + \left(\sigma_2^2 d_{22} + 2\sigma_1\sigma_2\left(d_{12} + Ad_{22}\right)\right)\mathbf{E}x_{i-1-m}^2.
\end{aligned} \qquad (3.11)$$

Note that via (3.7), (3.10) we have

$$A_1'DA_1 = \begin{pmatrix} B^2 d_{22} & B\left(d_{12} + Ad_{22}\right) \\ B\left(d_{12} + Ad_{22}\right) & d_{11} + 2Ad_{12} + A^2 d_{22} \end{pmatrix}.$$

Thus, the matrix equation (3.10) is equivalent to the system of equations

$$B^2 d_{22} - d_{11} = 0,$$

$$B\left(d_{12} + Ad_{22}\right) - d_{12} = 0,$$

$$d_{11} + 2Ad_{12} + A^2 d_{22} - d_{22} = -1,$$

with the solution

$$d_{11} = B^2 d_{22}, \ \ d_{12} = \frac{AB}{1 - B}d_{22}, \ \ d_{22} = \left(\frac{1 + B}{1 - B}\left((1 - B)^2 - A^2\right)\right)^{-1}, \qquad (3.12)$$

where $d_{22} > 0$ if and only if conditions (3.5) hold. Via (3.12)

$$d_{12} + Ad_{22} = \frac{A}{1 - B}d_{22}.$$

From here and (3.11) we obtain

$$\mathbf{E}\Delta V_{1i} = -\mathbf{E}x_i^2 + d_{22}\left(\sigma_1^2 \mathbf{E}x_{i-m}^2 + \nu \mathbf{E}x_{i-1-m}^2\right), \qquad (3.13)$$

where

$$\nu = 2\sigma_1\sigma_2 \frac{A}{1-B} + \sigma_2^2. \tag{3.14}$$

Choosing the additional functional V_{2i} in the form

$$V_{2i} = d_{22}\sigma_1^2 \sum_{j=1}^{m} x_{i-j}^2,$$

we have

$$\Delta V_{2i} = d_{22}\sigma_1^2 \left(\sum_{j=1}^{m} x_{i+1-j}^2 - \sum_{j=1}^{m} x_{i-j}^2 \right) = d_{22} \left(\sigma_1^2 x_i^2 - \sigma_1^2 x_{i-m}^2 \right). \tag{3.15}$$

From (3.13), (3.15) for the functional $V_i = V_{1i} + V_{2i}$ it follows that

$$\mathbf{E}\Delta V_i = -\left(1 - \sigma_1^2 d_{22}\right) \mathbf{E}x_i^2 + \nu d_{22}\mathbf{E}x_{i-1-m}^2. \tag{3.16}$$

Note that via (3.14), (3.5), (3.4) we get

$$\sigma_1^2 + \nu = \sigma_1^2 + 2\sigma_1\sigma_2\frac{A}{1-B} + \sigma_2^2 > \sigma_1^2 - 2|\sigma_1\sigma_2| + \sigma_2^2 = \sigma_0^2(1-\gamma)^2 \geq 0$$

and therefore

$$\nu = \sigma_0^2 \left(\gamma^2 - 2\frac{\gamma + b - b\theta + a}{\gamma^{-1} - b\theta + a} \right) > -\sigma_0^2. \tag{3.17}$$

Via [21], Lemma 2.1 and (3.12), (3.14), (3.16) we obtain the following:

Lemma 3.1 *If $\nu \geq 0$ then the conditions (3.5) and*

$$\sigma_1^2 + 2\sigma_1\sigma_2\frac{A}{1-B} + \sigma_2^2 < \frac{1+B}{1-B}\left((1-B)^2 - A^2\right) \tag{3.18}$$

are the necessary and sufficient conditions for asymptotic mean square stability of the trivial solution of equation (3.3). If $\nu < 0$ then the conditions (3.5) and

$$\sigma_1^2 < \frac{1+B}{1-B}\left((1-B)^2 - A^2\right) \tag{3.19}$$

are sufficient conditions for asymptotic mean square stability of the trivial solution of equation (3.3). If $\nu < 0$,

$$\sigma_1^2 \geq \frac{1+B}{1-B}\left((1-B)^2 - A^2\right) \tag{3.20}$$

and conditions (3.5), (3.18) hold then each mean square bounded solution of equation (3.3) is asymptotically mean square trivial.

Let us rewrite conditions (3.5), (3.18) in terms of the parameters of equation (3.1). From (3.4), (3.5) it follows that

$$
\begin{aligned}
b\theta - \gamma^{-1} &< a < b\theta + \gamma^{-1}, \\
b\left(\theta - (1+\gamma)^{-1}\right) - 1 &< a < 1 + b\left(\theta - (1-\gamma)^{-1}\right), \\
b &< \gamma^{-1} - \gamma,
\end{aligned}
$$

or

$$
b\left(\theta - (1+\gamma)^{-1}\right) - 1 < a < \begin{cases} b\theta + \gamma^{-1} & \text{if} \quad b < 2 - \gamma - \gamma^{-1}, \\ 1 - b\left((1-\gamma)^{-1} - \theta\right) & \text{if} \quad b \geq 2 - \gamma - \gamma^{-1}, \end{cases}
$$

$$
-(2 + \gamma + \gamma^{-1}) < b < \gamma^{-1} - \gamma. \tag{3.21}
$$

Using (3.4) we transform (3.18) into the form

$$
\sigma_0^2 \left(1 - 2\gamma\frac{A}{1-B} + \gamma^2\right) < \frac{1+B}{1-B}\left((1-B)^2 - A^2\right)
$$

or

$$
\sigma_0^2 < \frac{(1+B)\left((1-B)^2 - A^2\right)}{(1+\gamma^2)(1-B) - 2\gamma A}.
$$

Via (3.4) we have also that

$$
\gamma A = \gamma^2 + b\gamma - B,
$$

$$
\left(1 + \gamma^2\right)(1-B) - 2\gamma A = \left(1 - \gamma^2\right)(1+B) - 2\gamma b.
$$

Thus,

$$
\sigma_0^2 < \frac{(1+B)\left((1-B)^2 - A^2\right)}{(1-\gamma^2)(1+B) - 2\gamma b} = \frac{(1-B-A)(1-B+A)}{1 - \gamma^2 - 2\gamma b(1+B)^{-1}}.
$$

From here and (3.4) as a result for (3.18) we obtain

$$
\sigma_0^2 < \frac{((1-\gamma)(1+b\theta-a)-b)\left((1+\gamma)(1-b\theta+a)+b\right)}{1 - \gamma^2 - 2\gamma b(1 + \gamma b\theta - \gamma a)^{-1}}. \tag{3.22}
$$

Similar to (3.22) condition (3.6) takes the form

$$
\sigma_0^2 < \frac{[(1-\gamma)(1+b\theta-a)-b][(1+\gamma)(1-b\theta+a)+b]}{(1 - \gamma b\theta + \gamma a)(1 + \gamma b\theta - \gamma a)^{-1}}. \tag{3.23}
$$

From Lemma 3.1 via (3.17), (3.21)-(3.23) we obtain

Theorem 3.2 *If*

$$
\gamma^2 \geq 2\frac{\gamma + b - b\theta + a}{\gamma^{-1} - b\theta + a} \tag{3.24}
$$

then conditions (3.21), (3.22) are the necessary and sufficient conditions for asymptotic mean square stability of the trivial solution of equation (3.3).

If condition (3.24) does not hold then conditions (3.21), (3.23) are sufficient conditions for asymptotic mean square stability of the trivial solution of equation (3.3).

If conditions (3.23), (3.24) do not hold and conditions (3.21), (3.22) hold then each mean square bounded solution of equation (3.3) is asymptotically mean square trivial.

3.3 Stability conditions in the space of the auxiliary parameters: geometrical representation

Conditions (3.21) for each fixed θ give in the space of the parameters (a, b) the triangle with the vertices

$$A_1(-\gamma^{-1} + \theta(\gamma^{-1} - \gamma), \gamma^{-1} - \gamma),$$

$$A_2(\gamma^{-1} + \theta(2 - \gamma - \gamma^{-1}), 2 - \gamma - \gamma^{-1}), \qquad (3.25)$$

$$A_3(\gamma^{-1} - \theta(2 + \gamma + \gamma^{-1}), -2 - \gamma - \gamma^{-1}),$$

and the sides

$$A_1 A_2 : \quad a = 1 - b\left(\frac{1}{1 - \gamma} - \theta\right), \qquad A_2 A_3 : \quad a = b\theta + \frac{1}{\gamma},$$

$$A_1 A_3 : \quad a = b\left(\theta - \frac{1}{1 + \gamma}\right) - 1. \qquad (3.26)$$

It is easy to check that the area S of this triangle does not depend on θ and equals $S = 4\gamma^{-1}$. In Fig.3.1 one can see three copies of triangle $A_1 A_2 A_3$ for $\lambda = 22$, $h = 0.05$ and different values of θ: 1) $\theta = 0$, 2) $\theta = 0.5$, 3) $\theta = 1$. In Fig.3.2 inside of the triangle $A_1 A_2 A_3$ for $\theta = 0.25$, $\lambda = 22$, $h = 0.05$ and $\sigma = 2$ three different types of the regions are shown: 1) the region of the necessary and sufficient conditions for asymptotic mean square stability of the trivial solution of equation (3.3) (yellow); 2) the region of the sufficient conditions for asymptotic mean square stability of the trivial solution of equation (3.3) (red); 3) the region, where each mean square bounded solution of equation (3.3) is asymptotically mean square trivial (green).

If to calculate the infimum with respect to θ in the left hand side and the supremum in the right hand side of inequalities (3.21) we obtain

$$\frac{\gamma b}{1 + \gamma} - 1 < a < \begin{cases} \gamma^{-1} & \text{if} \quad b \in (-2 - \gamma - \gamma^{-1}, 2 - \gamma - \gamma^{-1}), \\ 1 - \frac{b}{1 - \gamma} & \text{if} \qquad\qquad b \in [2 - \gamma - \gamma^{-1}, 0), \end{cases}$$

$$-\frac{b}{1 + \gamma} - 1 < a < 1 - \frac{\gamma b}{1 - \gamma} \quad \text{if} \quad b \in [0, \gamma^{-1} - \gamma). \qquad (3.27)$$
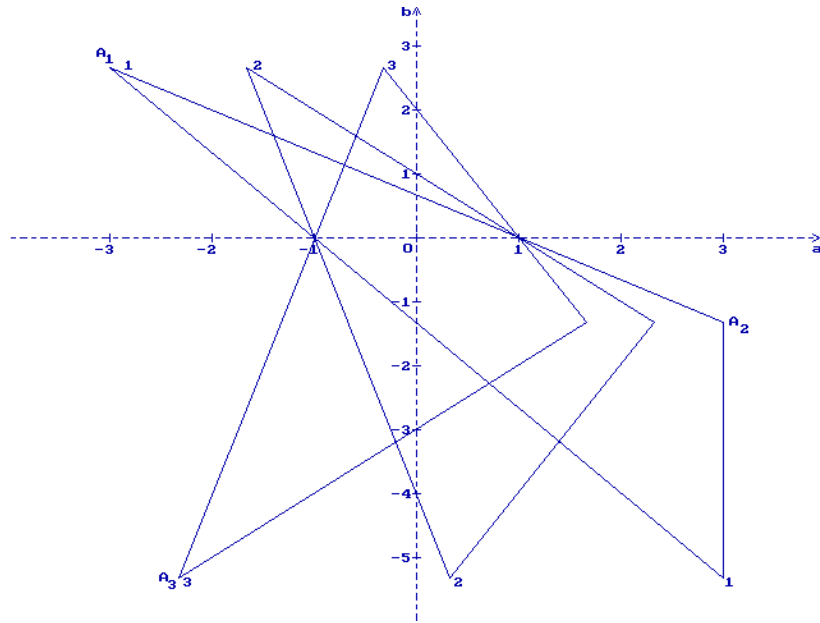
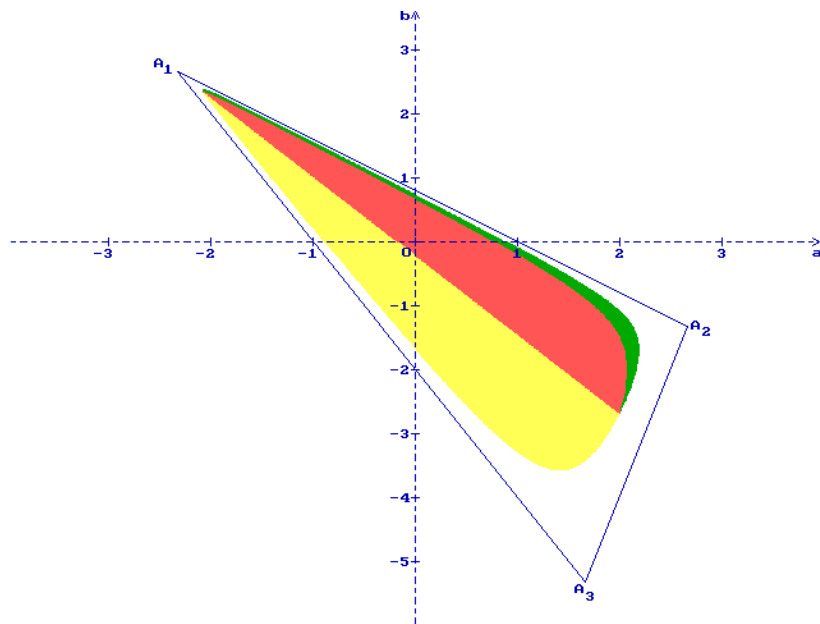Figure 3.1: $\lambda = 22$, $h = 0.05$, 1) $\theta = 0$, 2) $\theta = 0.5$, 3) $\theta = 1$.

Figure 3.2: $\theta = 0.25$, $\lambda = 22$, $h = 0.05$, $\sigma = 2$.

Conditions (3.27) gives the septangle with the vertices

$$A_1(-\gamma^{-1}, \gamma^{-1} - \gamma), \quad A_2(-\gamma, \gamma^{-1} - \gamma), \quad A_3(1, 0),$$

$$A_4(\gamma^{-1}, 2 - \gamma - \gamma^{-1}), \quad A_5(\gamma^{-1}, -2 - \gamma - \gamma^{-1}), \quad (3.28)$$

$$A_6(-2 - \gamma, -2 - \gamma - \gamma^{-1}), \qquad A_7(-1, 0),$$

and the sides

$$A_1A_2: \ \ b = \frac{1}{\gamma} - \gamma, \quad A_1A_7: \ \ a = -1 - \frac{b}{1+\gamma},$$

$$A_2A_3: \ \ a = 1 - \frac{\gamma b}{1-\gamma}, \quad A_3A_4: \ \ a = 1 - \frac{b}{1-\gamma}, \quad A_4A_5: \ \ a = \frac{1}{\gamma}, \ (3.29)$$

$$A_5A_6: \ \ b = -2 - \gamma - \frac{1}{\gamma}, \quad A_6A_7: \ \ a = -1 + \frac{\gamma b}{1+\gamma}.$$

It is easy to check that if $b \geq 0$ then the right hand side of (3.24) decreases and if $b < 0$ it increases. So, if $b \geq 0$ then condition (3.24) gives the maximal region in the space of the parameters (a, b) if $\theta = 1$ and if $b < 0$ then condition (3.24) gives the maximal region in the space of the parameters (a, b) if $\theta = 0$. Thus, via (3.24) we can find for the trivial solution of equation (3.3) with $\sigma_0 = 0$ the region of the necessary and sufficient stability conditions

$$\frac{b}{1+\gamma^{-1}} - 1 < a \begin{cases} < & \gamma^{-1} & \text{if} & b \in (-2 - \gamma - \gamma^{-1}, -\gamma^{-1}), \\ \leq & \frac{\gamma+2b}{\gamma^2-2} & \text{if} & b \in [-\gamma^{-1}, 0), \end{cases}$$

$$-\frac{b}{1+\gamma} - 1 < a < \frac{\gamma + \gamma^2 b}{\gamma^2 - 2} \quad \text{if} \quad b \in [0, \gamma^{-1} - \gamma), \qquad (3.30)$$

and the region of sufficient stability conditions

$$\frac{\gamma + 2b}{\gamma^2 - 2} < a < \begin{cases} \gamma^{-1} & \text{if} & b \in (-\gamma^{-1}, 2 - \gamma - \gamma^{-1}), \\ 1 - \frac{b}{1-\gamma} & \text{if} & b \in [2 - \gamma - \gamma^{-1}, 0), \end{cases}$$

$$\frac{\gamma + \gamma^2 b}{\gamma^2 - 2} < a < 1 - \frac{\gamma b}{1-\gamma} \quad \text{if} \quad b \in [0, \gamma^{-1} - \gamma). \qquad (3.31)$$

Conditions (3.30) gives the septangle $A_1A_2A_8A_9A_5A_6A_7$ (in Fig.3.2) with vertices (3.28) and

$$A_8(\gamma(\gamma^2 - 2)^{-1}, 0), \qquad A_9(\gamma^{-1}, -\gamma^{-1}), \qquad (3.32)$$

and sides (3.29) and

$$A_2A_8: \ \ a = \frac{\gamma + \gamma^2 b}{\gamma^2 - 2}, \qquad A_8A_9: \ \ a = \frac{\gamma + 2b}{\gamma^2 - 2}. \qquad (3.33)$$
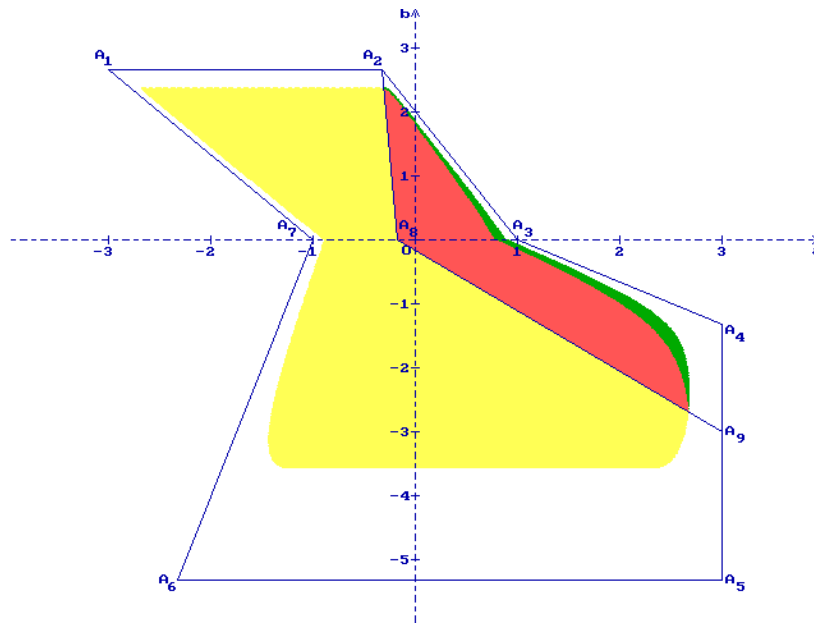
Figure 3.3: $\lambda = 22$, $h = 0.05$, $\sigma = 2$.

Conditions (3.31) gives the pentagon $A_2 A_3 A_4 A_9 A_8$ with vertices (3.28), (3.32) and sides (3.29), (3.33).

In Fig.3.3 one can see the septangle $A_1 A_2 A_3 A_4 A_5 A_6 A_7$ for $\lambda = 22$ and $h = 0.05$ with the area $S = \gamma^2 + 2\gamma + 2 + 6\gamma^{-1} + \gamma^{-2}$. Inside of this septangle for $\sigma = 2$ three different types of the regions are shown: 1) if and only if the point (a, b) belongs to this region then there exist some $\theta \in [0, 1]$ such that the trivial solution of equation (3.3) is asymptotically mean square stable (yellow); 2) if the point (a, b) belongs to this region then there exist some $\theta \in [0, 1]$ such that the trivial solution of equation (3.3) is asymptotically mean square stable (red); 3) if the point (a, b) belongs to this region then there exist some $\theta \in [0, 1]$ such that each mean square bounded solution of equation (3.3) is asymptotically mean square trivial (green). In all these cases a value of θ can be easily determined via the condition (3.21).

3.4 Stability conditions in the space of the initial parameters

Transform conditions (3.28), (3.29), (3.32), (3.33) to the space of the parameters (α, β) with $\gamma = e^{-\lambda h}$. Then

$$A_1 \left(-\frac{1 + \gamma^{-1}}{h}, \frac{\gamma^{-1} - \gamma}{h^2} \right), \quad A_2 \left(-\frac{1 + \gamma}{h}, \frac{\gamma^{-1} - \gamma}{h^2} \right), \quad A_3(0, 0),$$

$$A_4 \left(\frac{\gamma^{-1} - 1}{h}, \frac{2 - \gamma - \gamma^{-1}}{h^2} \right), \quad A_5 \left(\frac{\gamma^{-1} - 1}{h}, -\frac{2 + \gamma + \gamma^{-1}}{h^2} \right),$$

$$A_6 \left(-\frac{4 + \gamma + 3\gamma^{-1}}{(1 + \gamma^{-1})h}, -\frac{2 + \gamma + \gamma^{-1}}{h^2} \right), \quad A_7 \left(-\frac{2}{h}, 0 \right), \qquad (3.34)$$

$$A_8 \left(\frac{2 + \gamma - \gamma^2}{(\gamma^2 - 2)h}, 0 \right), \quad A_9 \left(\frac{\gamma^{-1} - 1}{h}, -\frac{\gamma^{-1}}{h^2} \right).$$

and sides

$$A_1 A_2 : \ \beta = \frac{\gamma^{-1} - \gamma}{h^2}, \quad A_2 A_3 : \ \alpha = -\frac{\beta h}{\gamma^{-1} - 1},$$

$$A_3 A_4 : \ \alpha = -\frac{\beta h}{1 - \gamma}, \quad A_4 A_5 : \ \alpha = \frac{\gamma^{-1} - 1}{h},$$

$$A_5 A_6 : \ \beta = -\frac{(1 + \gamma)^2}{\gamma h^2}, \quad A_6 A_7 : \ \alpha = \frac{\beta h}{1 + \gamma^{-1}} - \frac{2}{h}, \qquad (3.35)$$

$$A_1 A_7 : \ \alpha = -\frac{\beta h}{1 + \gamma} - \frac{2}{h}, \quad A_2 A_8 : \ \alpha = \frac{1}{h} \left(\frac{\gamma + \gamma^2 \beta h^2}{\gamma^2 - 2} - 1 \right),$$

$$A_8 A_9 : \ \alpha = \frac{1}{h} \left(\frac{\gamma + 2\beta h^2}{\gamma^2 - 2} - 1 \right),$$

Necessary and sufficient asymptotic mean square stability conditions (3.30) and sufficient asymptotic mean square stability conditions (3.31) respectively take the forms

$$\frac{\beta h}{1 + \gamma^{-1}} - \frac{2}{h} < \alpha \begin{cases} < & \frac{\gamma^{-1} - 1}{h} \quad \text{if} \quad \beta \in \left(-\frac{(1+\gamma)^2}{\gamma h^2}, -\frac{1}{\gamma h^2} \right), \\[2mm] \leq & \frac{1}{h} \left(\frac{\gamma + 2\beta h^2}{\gamma^2 - 2} - 1 \right) \quad \text{if} \quad \beta \in \left[-\frac{1}{\gamma h^2}, 0 \right), \end{cases}$$

$$-\frac{\beta h}{1 + \gamma} - \frac{2}{h} < \alpha < \frac{1}{h} \left(\frac{\gamma + \gamma^2 \beta h^2}{\gamma^2 - 2} - 1 \right) \quad \text{if} \quad \beta \in \left[0, \frac{\gamma^{-1} - \gamma}{h^2} \right), \quad (3.36)$$

and

$$\frac{1}{h} \left(\frac{\gamma + 2\beta h^2}{\gamma^2 - 2} - 1 \right) < \alpha < \begin{cases} \frac{\gamma^{-1} - 1}{h} \quad \text{if} \quad \beta \in \left(-\frac{1}{\gamma h^2}, -\frac{(1-\gamma)^2}{\gamma h^2} \right), \\[2mm] -\frac{\beta h}{1 - \gamma} \quad \text{if} \quad \beta \in \left[-\frac{(1-\gamma)^2}{\gamma h^2}, 0 \right), \end{cases}$$

$$\frac{1}{h} \left(\frac{\gamma + \gamma^2 \beta h^2}{\gamma^2 - 2} - 1 \right) < \alpha < -\frac{\gamma \beta h}{1 - \gamma} \quad \text{if} \quad \beta \in \left[0, \frac{\gamma^{-1} - \gamma}{h^2} \right). \qquad (3.37)$$

Conditions (3.22), (3.23) in the space of the parameters (α, β) are respectively:

$$\sigma^2 < \frac{\left[(1 - e^{-\lambda h})(\beta \theta h - \alpha) - \beta h \right] \left[(1 + e^{-\lambda h})(2 + \alpha h - \beta \theta h^2) + \beta h^2 \right]}{1 - e^{-2\lambda h} - 2\beta h^2 e^{-\lambda h} \left[1 - e^{-\lambda h} + e^{-\lambda h} h(\beta \theta h - \alpha) \right]^{-1}} =$$

$$= \frac{\left[\frac{1-e^{-\lambda h}}{h}(\beta\theta h - \alpha) - \beta\right]\left[(1 + e^{-\lambda h})(2 + \alpha h - \beta\theta h^2) + \beta h^2\right]}{\frac{1-e^{-2\lambda h}}{h} - 2\beta e^{-\lambda h}\left[\frac{1-e^{-\lambda h}}{h} + e^{-\lambda h}(\beta\theta h - \alpha)\right]^{-1}} \quad (3.38)$$

and

$$\sigma^2 < \frac{\left[(1 - e^{-\lambda h})(\beta\theta h - \alpha) - \beta h\right]\left[(1 + e^{-\lambda h})(2 + \alpha h - \beta\theta h^2) + \beta h^2\right]}{\left[1 + e^{-\lambda h} - e^{-\lambda h}h(\beta\theta h + \alpha)\right]\left[1 - e^{-\lambda h} + e^{-\lambda h}h(\beta\theta h - \alpha)\right]^{-1}} =$$

$$= \frac{\left[\frac{1-e^{-\lambda h}}{h}(\beta\theta h - \alpha) - \beta\right]\left[(1 + e^{-\lambda h})(2 + \alpha h - \beta\theta h^2) + \beta h^2\right]}{\left[1 + e^{-\lambda h} - e^{-\lambda h}h(\beta\theta h + \alpha)\right]\left[\frac{1-e^{-\lambda h}}{h} + e^{-\lambda h}(\beta\theta h - \alpha)\right]^{-1}}. \quad (3.39)$$

So, we obtain the following

Theorem 3.3 *If condition (3.36) holds then for each $\theta \in [0,1]$ the inequality (3.38) is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of equation (3.3).*

If condition (3.37) holds and for some $\theta \in [0,1]$ the inequality (3.39) holds too then the trivial solution of equation (3.3) is asymptotically mean square stable.

If condition (3.37) holds and for some $\theta \in [0,1]$ the condition (3.38) holds too but condition (3.39) does not hold then each mean square bounded solution of equation (3.3) is asymptotically mean square trivial.

In Figure 3.4 one can see the septangle $A_1 A_2 A_3 A_4 A_5 A_6 A_7$ in the space of the parameters (α, β) for $\lambda = 12$ and $h = 0.1$. Inside of this septangle for $\sigma = 2$ three different types of the regions are shown:

1) if and only if the point (α, β) belongs to this region then there exist some $\theta \in [0,1]$ such that the trivial solution of equation (3.3) is asymptotically mean square stable (yellow);

2) if the point (α, β) belongs to this region then there exist some $\theta \in [0,1]$ such that the trivial solution of equation (3.3) is asymptotically mean square stable (red);

3) if the point (α, β) belongs to this region then there exist some $\theta \in [0,1]$ such that each mean square bounded solution of equation (3.3) is asymptotically mean square trivial (green).

In addition, in the Figure 3.4 one can see two copies of the bound of asymptotic mean square stability region for the trivial solution of the initial equation (1.1) that is defined by conditions (2.25): 1) for $\sigma = 0$; 2) for $\sigma = 2$.

Remark 3.4 *If $h \to 0$ then via (2.5) the inequality (3.38) is changed into the form*

$$\delta < \frac{\lambda - \alpha}{1 - \lambda^2(\alpha\lambda + \beta)^{-1}}$$

that is equivalent to (2.25), condition (3.37) is turned into (2.4), the bound $A_2 A_8 A_9$ goes to infinity and the pentagon $A_2 A_3 A_4 A_9 A_8$ (Fig.3.4) with the area $S = (2 - \gamma)h^{-3}$ fills all space that is defined by condition (2.4).
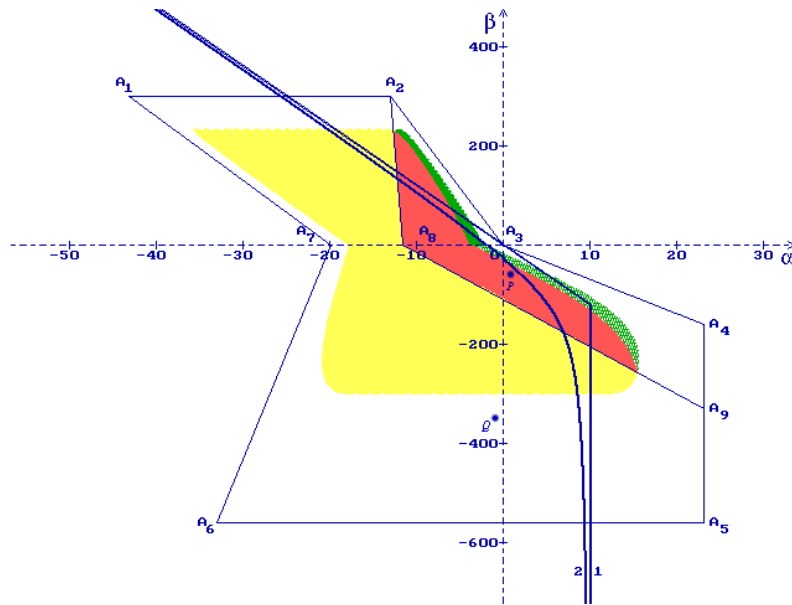
Figure 3.4: $\lambda = 12$, $h = 0.1$, $\sigma = 2$.

Two points, $P = (1, -50)$ and $Q = (-1, -350)$, are also marked on Figure 3.4. At each of these points in the (α, β)-plane we have computed 50 trajectories (plotted in Figures 3.5 and 3.6, respectively along with their means) of the difference analogue (3.1) with the following parameters: $\theta = 0.5$, $\lambda = 12$, $\sigma = 2$, $\tau = 0.5$, $h = 0.1$. For Figure 3.5 we used an initial function of $x(s) = 1$, $s \in [-0.5, 0]$ and for Figure 3.6 we used an initial function of $x(s) = 0.1$, $s \in [-0.5, 0]$. From Figure 3.4 we can see that the point P is in a region where we expect solutions which are mean-square asymptotically stable. In Figure 3.5 we see that is the case. In contrast, the point Q in Figure 3.4 is in a region where we do not expect asymptotic mean-square stability for our solutions; Figure 3.6 illustrates this case.

4   Conclusions

In this paper we have provided, via the general method of Lyapunov functionals construction [13], sufficient conditions for the asymptotic mean square stability of the trivial solution of a stochastic Volterra integro-differential equation (1.1), (1.2) which is of convolution type (with fading memory kernel). Using an analogous approach, and utilising results from [20] and [21], we are able to derive necessary and sufficient conditions for the trivial solution of a discrete analogue of the original Volterra equation.

The discrete system provides a means for obtaining numerical approximations to solutions of the original equation. The analysis contained herein makes it possible to analyze whether or not these approximations will exhibit the same stability properties of the solution to the original continuous-time problem. Indeed, our analysis shows that care needs to be taken when choosing an appropriate θ-method for obtaining the numerical approxi-
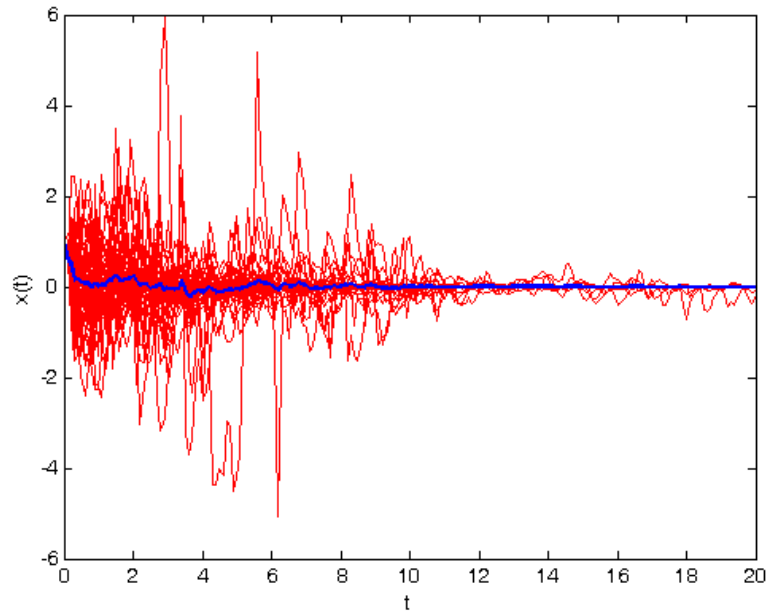
Figure 3.5: $\lambda = 12$, $h = 0.1$, $\sigma = 2$, $\alpha = 1$, $\beta = -50$, $\theta = 0.5$, $\tau = 0.5$, $x(s) = 1$, $s \in [-0.5, 0]$.
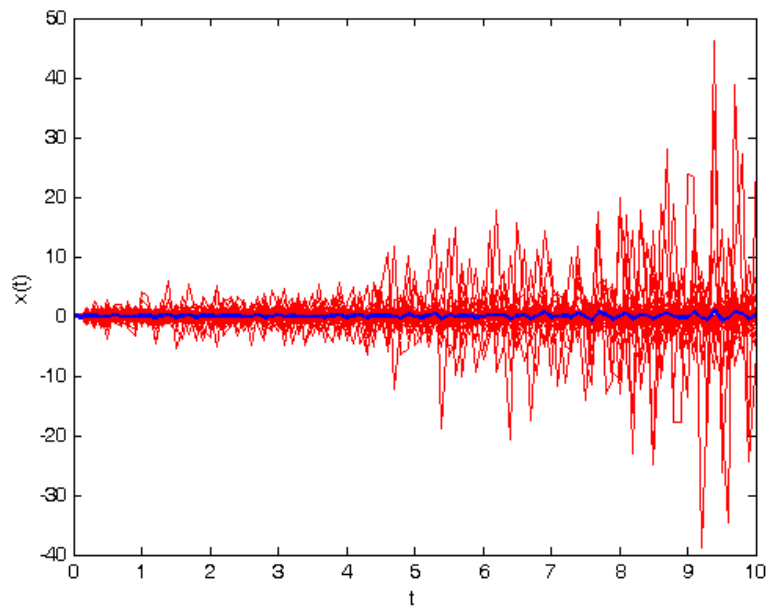

Figure 3.6: $\lambda = 12$, $h = 0.1$, $\sigma = 2$, $\alpha = -1$, $\beta = -350$, $\theta = 0.5$, $\tau = 0.5$, $x(s) = 0.1$, $s \in [-0.5, 0]$.

mation. Our stability conditions also provide a means by which an appropriate θ-method can be identified.

5 References

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