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STABILITY OF STOCHASTIC LINEAR DIFFERENCE EQUATIONS WITH VARYING DELAY

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Abstract

Many processes in automatic regulation, physics, mechanics, biology, economy, ecology etc. can be modelled by hereditary systems (see, e.g. [1-3]). One of the main problems for the theory of such systems and their applications is connected with stability [1,2]. Many stability results were obtained by the construction of appropriate Lyapunov functionals. In [4-8] the method is proposed allowing, in some sense, to formalize the procedure of the corresponding Lyapunov functionals construction for investigation of stochastic difference equations stability. Here by virtue of proposed procedure the sufficient conditions of asymptotic mean square stability for stochastic linear difference equations with varying delays are obtained.

Keywords: Asymptotic mean square stability, stochastic linear difference equations, varying delay.

1. Consider the stochastic difference equation

$$x_{i+1} = F(i, x_{-h}, ..., x_i) + G(i, x_{-h}, ..., x_i)\xi_i, \qquad i \in Z, \tag{1}$$

$$x_i = \varphi_i, \qquad i \in Z_0.$$

Here i is the discrete time, $Z = \{0, 1, ...\}$, $Z_0 = \{-h, ..., 0\}$, $h \geq 0$, $x_i \in R^n$, the functions F and G are defined on $Z * S$, where S is a space of sequences with elements from R^n. It is assumed that $F(i, ...)$ and $G(i, ...)$ does not depend on x_j for $j > i$, $F(i, 0, ..., 0) = 0$, $G(i, 0, ..., 0) = 0$.

Let $\{\Omega, \sigma, \mathbf{P}\}$ be a probability space, $\{f_i \in \sigma\}$, $i \in Z$, be a sequence of σ-algebras, $\xi_0, \xi_1, ...$ be mutually independent random variables, ξ_i be f_{i+1}-adapted and independent on f_i, $\mathbf{E}$ be mathematical expectation, $\mathbf{E}\xi_i = 0$, $\mathbf{E}\xi_i^2 = 1$.

Definition. Zero solution of the equation (1) is called mean square stable if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}x_i^2 < \epsilon$, $i \in Z$, if $\|\varphi\|^2 = \sup_{i \in Z_0} \mathbf{E}\varphi_i^2 < \delta$. If, besides, $\lim_{i\to\infty} \mathbf{E}x_i^2 = 0$, then the equation (1) zero solution is called asymptotically mean square stable.

Theorem 1. [4]. Let there exists the nonnegative functional $V_i = V(i, x_{-h}, ..., x_i)$, $i \in Z$, for which the conditions

$$\mathbf{E}V(0, x_{-h}, ..., x_0) \leq c_1 \|\varphi\|^2,$$

$$\mathbf{E}\Delta V_i \leq -c_2 \mathbf{E}x_i^2, \qquad i \in Z,$$

where $\Delta V_i = V_{i+1} - V_i$, $c_1 > 0$, $c_2 > 0$, hold. Then the equation (1) zero solution is asymptotically mean square stable.

Consider the scalar stochastic linear difference equation

$$x_{i+1} = ax_i + bx_{i-k(i)} + \sigma x_{i-m(i)}\xi_i. \tag{2}$$

The delays $k(i)$ and $m(i)$ satisfy the conditions

$$k(i) \geq k(i+1) \geq 0, \qquad m(i) \geq m(i+1) \geq 0. \tag{3}$$

Following the general method of Lyapunov functionals construction we will construct the Lyapunov functional for the equation (2) in the form $V = V_1 + V_2$, where $V_1 = x_i^2$. Calculating $\mathbf{E}\Delta V_1$ we get

$$\mathbf{E}\Delta V_1 = \mathbf{E}(x_{i+1}^2 - x_i^2) = \mathbf{E}[(ax_i + bx_{i-k(i)} + \sigma x_{i-m(i)}\xi_i)^2 - x_i^2] =$$
$$= \mathbf{E}[(a^2-1)x_i^2 + b^2 x_{i-k(i)}^2 + \sigma^2 x_{i-m(i)}^2 + 2abx_i x_{i-k(i)}] \leq$$
$$\leq (a^2 - 1 + |ab|)\mathbf{E}x_i^2 + (b^2 + |ab|)\mathbf{E}x_{i-k(i)}^2 + \sigma^2 \mathbf{E}x_{i-m(i)}^2.$$

Choosing the functional V_2 in the form

$$V_2 = (b^2 + |ab|) \sum_{j=i-k(i)}^{i-1} x_j^2 + \sigma^2 \sum_{j=i-m(i)}^{i-1} x_j^2,$$

and using (3) we get

$$\Delta V_2 = (b^2 + |ab|)(\sum_{j=i+1-k(i+1)}^{i} x_j^2 - \sum_{j=i-k(i)}^{i-1} x_j^2) + \sigma^2(\sum_{j=i+1-m(i+1)}^{i} x_j^2 - \sum_{j=i-m(i)}^{i-1} x_j^2) =$$
$$= (b^2 + |ab| + \sigma^2)x_i^2 - (b^2 + |ab|)x_{i-k(i)}^2 - \sigma^2 x_{i-m(i)}^2 +$$
$$+(b^2 + |ab|)(\sum_{j=i+1-k(i+1)}^{i-1} x_j^2 - \sum_{j=i+1-k(i)}^{i-1} x_j^2) + \sigma^2(\sum_{j=i+1-m(i+1)}^{i-1} x_j^2 - \sum_{j=i+1-m(i)}^{i-1} x_j^2) \leq$$
$$\leq (b^2 + |ab| + \sigma^2)x_i^2 - (b^2 + |ab|)x_{i-k(i)}^2 - \sigma^2 x_{i-m(i)}^2.$$

As a result for $V = V_1 + V_2$ we have

$$\mathbf{E}\Delta V \leq ((|a| + |b|)^2 + \sigma^2 - 1)\mathbf{E}x_i^2.$$

From here and Theorem 1 it follows

Theorem 2. Let the conditions (3) and

$$(|a| + |b|)^2 + \sigma^2 < 1$$

hold. Then the zero solution of the equation (2) is asymptotically mean square stable.

2. Consider now another way of Lyapunov functional construction. Following the general method of Lyapunov functionals construction let us represent the equation (2) in the form

$$x_{i+1} = F_{1i} + F_{2i} + \Delta F_{3i} + \sigma x_{i-m(i)}\xi_i, \tag{4}$$

where

$$F_{1i} = (a+b)x_i, \quad F_{2i} = -b \sum_{j=i+1-k(i)}^{i-k(i+1)} x_j,$$

$$F_{3i} = -b \sum_{j=i-k(i)}^{i-1} x_j, \quad \Delta F_{3i} = -b \left(x_i - \sum_{j=i-k(i)}^{i-k(i+1)} x_j \right).$$

We will construct the Lyapunov functional V in the form $V = V_1 + V_2$ again, but now let

$$V_1 = (x_i - F_{3i})^2 = \left(x_i + b \sum_{j=i-k(i)}^{i-1} x_j \right)^2.$$

Calculating $\mathbf{E}\Delta V_1$ and using the representations for x_{i+1} and F_{1i} we get

$$\mathbf{E}\Delta V_1 = \mathbf{E}[(x_{i+1} - F_{3(i+1)})^2 - (x_i - F_{3i})^2] =$$

$$= \mathbf{E}(x_{i+1} - x_i - \Delta F_{3i})(x_{i+1} + x_i - F_{3(i+1)} - F_{3i}) =$$

$$= \mathbf{E}((a+b-1)x_i + F_{2i} + \sigma x_{i-m(i)}\xi_i)((a+b+1)x_i + F_{2i} - 2F_{3i} + \sigma x_{i-m(i)}\xi_i) =$$

$$= ((a+b)^2 - 1)\mathbf{E}x_i^2 + 2(a+b)\mathbf{E}x_iF_{2i} + \mathbf{E}F_{2i}^2 -$$

$$-2(a+b-1)\mathbf{E}x_iF_{3i} - 2\mathbf{E}F_{2i}F_{3i} + \sigma^2\mathbf{E}x_{i-m(i)}^2.$$

Let

$$k_0 = \sup_{i\in Z}(k(i) - k(i+1)), \qquad k_m = \inf_{i\in Z} k(i).$$

Using that $k(0) \geq k(i)$ we get

$$2|x_iF_{2i}| \leq |b| \sum_{j=i+1-k(i)}^{i-k(i+1)} (x_i^2 + x_j^2) \leq |b| \left(k_0 x_i^2 + \sum_{j=i+1-k(0)}^{i-k_m} x_j^2 \right),$$

$$F_{2i}^2 \leq b^2 k_0 \sum_{j=i+1-k(i)}^{i-k(i+1)} x_j^2 \leq b^2 k_0 \sum_{j=i+1-k(0)}^{i-k_m} x_j^2,$$

$$2|x_iF_{3i}| \leq |b| \sum_{j=i-k(i)}^{i-1} (x_i^2 + x_j^2) \leq |b| \left(k(0)x_i^2 + \sum_{j=i-k(0)}^{i-1} x_j^2 \right),$$

$$2|F_{2i}F_{3i}| \leq b^2 \sum_{j=i+1-k(i)}^{i-k(i+1)} \sum_{l=i-k(i)}^{i-1} (x_l^2 + x_j^2) \leq b^2 \left(k_0 \sum_{l=i-k(0)}^{i-1} x_l^2 + k(0) \sum_{j=i+1-k(0)}^{i-k_m} x_j^2 \right).$$

As a result we have

$$\mathbf{E}\Delta V_1 \leq (A-1)\mathbf{E}x_i^2 + B \sum_{j=i+1-k(0)}^{i-k_m} \mathbf{E}x_j^2 + C \sum_{j=i-k(0)}^{i-1} \mathbf{E}x_j^2 + \sigma^2 \mathbf{E}x_{i-m(i)}^2,$$

where

$$A = (a+b)^2 + |b(a+b)|k_0 + |b(a+b-1)|k(0),$$

$$B = |b(a+b)| + b^2(k_0 + k(0)), \qquad C = |b(a+b-1)| + b^2 k_0.$$

Let us choose the functional V_2 in the form

$$V_2 = B \left(\sum_{l=i}^{i+k_m-2} \sum_{j=l+1-k(0)}^{l-k_m} x_j^2 + \sum_{j=i-k(0)+k_m}^{i-1} (j - i + 1 + k(0) - k_m) x_j^2 \right) +$$

$$+C \sum_{j=i-k(0)}^{i-1} (j - i + 1 + k(0)) x_j^2 + \sigma^2 \sum_{j=i-m(i)}^{i-1} x_j^2.$$

It is easy to see that

$$\sum_{l=i+1}^{i+k_m-1} \sum_{j=l+1-k(0)}^{l-k_m} x_j^2 - \sum_{l=i}^{i+k_m-2} \sum_{j=l+1-k(0)}^{l-k_m} x_j^2 = \sum_{j=i-k(0)+k_m}^{i-1} x_j^2 - \sum_{j=i+1-k(0)}^{i-k_m} x_j^2,$$

$$\sum_{j=i+1-k(0)+k_m}^{i}(j-i+k(0)-k_m)x_j^2-\sum_{j=i-k(0)+k_m}^{i-1}(j-i+1+k(0)-k_m)x_j^2=(k(0)-k_m)x_i^2-\sum_{j=i-k(0)+k_m}^{i-1}x_j^2,$$

$$\sum_{j=i+1-k(0)}^{i}(j-i+k(0))x_j^2-\sum_{j=i-k(0)}^{i-1}(j-i+1+k(0))x_j^2=k(0)x_i^2-\sum_{j=i-k(0)}^{i-1}x_j^2,$$

$$\sum_{j=i+1-m(i+1)}^{i}x_j^2-\sum_{j=i-m(i)}^{i-1}x_j^2=x_i^2-x_{i-m(i)}^2-\sum_{j=i+1-m(i)}^{i-m(i+1)}x_j^2\le x_i^2-x_{i-m(i)}^2.$$

Therefore as a result for $V=V_1+V_2$ we have $\mathbf{E}\Delta V\le(A+B(k(0)-k_m)+Ck(0)+\sigma^2-1)\mathbf{E}x_i^2$. From here and Theorem 1 by virtue of the representations for A,B,C it follows

Theorem 3. Let the conditions (3) and

$$(a+b)^2+2k(0)|b(a+b-1)|+|b(a+b)|(k_0+k(0)-k_m)+$$
$$+b^2(k_0k(0)+(k_0+k(0))(k(0)-k_m))+\sigma^2<1$$

hold. Then the zero solution of the equation (2) is asymptotically mean square stable.

Remark 1. If $k(i)=k=const$ then asymptotic mean square stability condition has the form $\sigma^2<(1-a-b)(1+a+b-2k|b|)$, $|a+b|<1$. [4].

3. Consider the scalar stochastic linear difference equation

$$x_{i+1}=\sum_{j=0}^{k(i)}a_jx_{i-j}+\sum_{j=0}^{m(i)}\sigma_jx_{i-j}\xi_i, \tag{5}$$

It is supposed that

$$k(i+1)-k(i)\le 1,\quad m(i+1)-m(i)\le 1. \tag{6}$$

Let

$$\hat{k}=\sup_{i\in Z}k(i)\le\infty,\quad \hat{m}=\sup_{i\in Z}m(i)\le\infty,$$

$$a=\sum_{l=0}^{\hat{k}}|a_l|,\quad \sigma=\sum_{l=0}^{\hat{m}}|\sigma_l|,\quad A_j=a|a_j|,\quad B_j=\sigma|\sigma_j|. \tag{7}$$

We will construct the Lyapunov functional for the equation (5) in the form $V=V_1+V_2$, where $V_1=x_i^2$. Calculating $\mathbf{E}\Delta V_1$ we get

$$\mathbf{E}\Delta V_1=\mathbf{E}(x_{i+1}^2-x_i^2)=\mathbf{E}\left[\left(\sum_{j=0}^{k(i)}a_jx_{i-j}+\sum_{j=0}^{m(i)}\sigma_jx_{i-j}\xi_i\right)^2-x_i^2\right]=$$

$$=\mathbf{E}\left(\sum_{j=0}^{k(i)}a_jx_{i-j}\right)^2+\mathbf{E}\left(\sum_{j=0}^{m(i)}\sigma_jx_{i-j}\right)^2-\mathbf{E}x_i^2\le$$

$$\le\sum_{l=0}^{k(i)}|a_l|\sum_{j=0}^{k(i)}|a_j|\mathbf{E}x_{i-j}^2+\sum_{l=0}^{m(i)}|\sigma_l|\sum_{j=0}^{m(i)}|\sigma_j|\mathbf{E}x_{i-j}^2-\mathbf{E}x_i^2\le$$

$$\le\sum_{j=0}^{k(i)}A_j\mathbf{E}x_{i-j}^2+\sum_{j=0}^{m(i)}B_j\mathbf{E}x_{i-j}^2-\mathbf{E}x_i^2=(A_0+B_0-1)\mathbf{E}x_i^2+\sum_{j=1}^{k(i)}A_j\mathbf{E}x_{i-j}^2+\sum_{j=1}^{m(i)}B_j\mathbf{E}x_{i-j}^2.$$

Choose the functional V_2 in the form

$$V_2 = \sum_{j=1}^{k(i)} x_{i-j}^2 \sum_{l=j}^{\hat{k}} A_l + \sum_{j=1}^{m(i)} x_{i-j}^2 \sum_{l=j}^{\hat{m}} B_l.$$

Note that using (6) we get

$$\sum_{j=1}^{k(i+1)} x_{i+1-j}^2 \sum_{l=j}^{\hat{k}} A_l - \sum_{j=1}^{k(i)} x_{i-j}^2 \sum_{l=j}^{\hat{k}} A_l =$$

$$= x_i^2 \sum_{l=1}^{\hat{k}} A_l + \sum_{j=1}^{k(i+1)-1} x_{i-j}^2 \sum_{l=j+1}^{\hat{k}} A_l - \sum_{j=1}^{k(i)} x_{i-j}^2 \sum_{l=j+1}^{\hat{k}} A_l - \sum_{j=1}^{k(i)} A_j x_{i-j}^2 \leq$$

$$\leq x_i^2 \sum_{l=1}^{\hat{k}} A_l - \sum_{j=1}^{k(i)} A_j x_{i-j}^2.$$

Analogously

$$\sum_{j=1}^{m(i+1)} x_{i+1-j}^2 \sum_{l=j}^{\hat{m}} B_l - \sum_{j=1}^{m(i)} x_{i-j}^2 \sum_{l=j}^{\hat{m}} B_l \leq x_i^2 \sum_{l=1}^{\hat{m}} B_l - \sum_{j=1}^{m(i)} B_j x_{i-j}^2.$$

Therefore using (7) for functional $V = V_1 + V_2$ we have

$$\mathbf{E}\Delta V \leq (a^2 + \sigma^2 - 1)\mathbf{E}x_i^2.$$

From here and Theorem 1 it follows

Theorem 4. Let the conditions (6) and

$$a^2 + \sigma^2 < 1$$

hold. Then the zero solution of the equation (5) is asymptotically mean square stable.

Remark 2. From (6) it follows that $k(i) \leq k(0) + i$, $m(i) \leq m(0) + i$.

Remark 3. Using for the equation (5) the representation type of (4) we can get the stability condition depending on delays.

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