



About Stability of Nonlinear Stochastic Difference Equations

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Abstract—Using the method of Lyapunov functionals construction, it is shown that investigation of stability in probability of nonlinear stochastic difference equation with order of nonlinearity more than one can be reduced to the investigation of asymptotic mean square stability of the linear part of this equation. © 2000 Elsevier Science Ltd. All rights reserved.

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1. GENERAL THEOREMS

Difference equations usually appear by investigation of systems with discrete time or by numerical solution of systems with continue time [1]. Lyapunov functionals are used for investigation of hereditary systems in problems of stability and optimal control [2,3]. One method of Lyapunov functionals construction has been proposed and developed for differential and difference equations in [4–12]. This method is used here to construct stability in probability conditions for nonlinear stochastic difference equations. It is shown that investigation of stability in probability of nonlinear stochastic difference equation with order of nonlinearity more than one can be reduced to the investigation of asymptotic mean square stability of the linear part of this equation. For stochastic nonlinear differential equations, a similar result was obtained in [6].

Let $\{\Omega, \mathbf{P}, \sigma\}$ be a basic probability space, i be a discrete time, $i \in Z_0 \cup Z$, $Z_0 = \{-h, \dots, 0\}$, $Z = \{0, 1, \dots\}$, h be a given nonnegative number. $f_i \in \sigma$, $i \in Z$, be a sequence of σ -algebras, \mathbf{E} be the mathematical expectation, $\mathbf{E}_i = \mathbf{E}\{./f_i\}$, ξ_0, ξ_1, \dots , be a sequence of mutually independent random variables, ξ_i be f_{i+1} -adapted, $\mathbf{E}_i \xi_i = 0$, $\mathbf{E}_i \xi_i^2 = 1$, $i \in Z$, $\|\varphi\|_0 = \max_{-h \leq j \leq 0} |\varphi_j|$, $\|\varphi\|_1^2 = \max_{-h \leq j \leq 0} \mathbf{E}|\varphi_j|^2$, $U_\epsilon = \{x : |x| \leq \epsilon\}$. For arbitrary functional $V_i = V(i, x_{-h}, \dots, x_i)$,

we will denote $\Delta V_i = V(i+1, x_{-h}, \dots, x_{i+1}) - V(i, x_{-h}, \dots, x_i)$. Consider the difference equation

$$x_{i+1} = \sum_{j=0}^{i+h} a_j x_{i-j} + \sum_{j=0}^{i+h} \sigma_j x_{i-j} \xi_i + g_i(x_{-h}, \dots, x_i) \quad (1)$$

with initial function

$$x_j = \varphi_j, \quad j \in Z_0. \quad (2)$$

It is supposed that

$$|g_i(x_{-h}, \dots, x_i)| \leq \sum_{j=0}^{i+h} \gamma_j |x_{i-j}|^{\nu_j}, \quad \nu_j > 1, \quad j \in Z. \quad (3)$$

DEFINITION 1. *The zero solution of equation (1) is called stable in probability if for any $\epsilon > 0$ and $\epsilon_1 > 0$, there exists a $\delta > 0$ such that the solution $x_i = x_i(\varphi)$ of equation (1) satisfies the inequality*

$$\mathbf{P} \left\{ \sup_{i \in Z} |x_i| > \epsilon / f_0 \right\} < \epsilon_1 \quad (4)$$

for any initial function φ which are less than δ with probability 1, i.e.,

$$\mathbf{P} \{ \|\varphi\|_0 < \delta \} = 1. \quad (5)$$

DEFINITION 2. *The zero solution of equation (1) is called stable in probability if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\mathbf{E}|x_i|^2 < \epsilon$, $i \in Z$, if $\|\varphi\|_1^2 < \delta$. If besides, $\lim_{i \rightarrow \infty} \mathbf{E}|x_i|^2 = 0$ for all initial functions, then the zero solution of equation (1) is called asymptotically mean square stable.*

THEOREM 1. *Let there exists a functional $V_i = V(i, x_{-h}, \dots, x_i)$, which satisfies the conditions*

$$V(i, x_{-h}, \dots, x_i) \geq c_0 |x_i|^2, \quad (6)$$

$$V(0, x_{-h}, \dots, x_0) \leq c_1 \|\varphi\|_0^2, \quad (7)$$

$$\mathbf{E}_i \Delta V_i \leq 0, \quad x_j \in U_\epsilon, \quad -h \leq j \leq i, \quad i \in Z, \quad (8)$$

where $\epsilon > 0$, $c_0 > 0$, $c_1 > 0$. Then the zero solution of equation (1) with the initial condition (2) is stable in probability.

PROOF. We will show that for any positive numbers ϵ and ϵ_1 , there exists a positive number δ such that the solution of equation (1) satisfies condition (4) if the initial function (2) satisfies condition (5).

Let x_i be a solution of equation (1). Consider the random variable τ such that

$$\tau = \inf \{ i \in Z : |x_i| \leq \epsilon, |x_{i+1}| > \epsilon \} \quad (9)$$

and two events: $\{ \sup_{i \in Z} |x_i| > \epsilon \}$ and $\{ |x_{\tau+1}| > \epsilon \}$. It is easy to see that

$$\left\{ \sup_{i \in Z} |x_i| > \epsilon \right\} \subset \{ |x_{\tau+1}| > \epsilon \}. \quad (10)$$

From (8), we have

$$\mathbf{E}_0 \sum_{i=0}^{\tau} \mathbf{E}_i \Delta V_i = \mathbf{E}_0 V_{\tau+1} - V_0 \leq 0. \quad (11)$$

Using (10), Chebyshev inequality, (5)–(7) and (11), we get

$$\begin{aligned} \mathbf{P} \left\{ \sup_{i \in Z} |x_i| > \epsilon / f_0 \right\} &\leq \mathbf{P} \{ |x_{\tau+1}| > \epsilon / f_0 \} \\ &\leq \frac{\mathbf{E}_0 |x_{\tau+1}|^2}{\epsilon^2} \leq \frac{\mathbf{E}_0 V_{\tau+1}}{c_0 \epsilon^2} \leq \frac{V_0}{c_0 \epsilon^2} \leq \frac{c_1 \|\varphi\|_0^2}{c_0 \epsilon^2} < \frac{c_1 \delta^2}{c_0 \epsilon^2}. \end{aligned}$$

Choosing δ such that $\delta = \epsilon \sqrt{\epsilon_1 c_0 / c_1}$, we get (4). The theorem is proved.

REMARK 1. It is easy to see that if $\epsilon \geq \epsilon_0$, then $\mathbf{P} \{ \sup_{i \in Z} |x_i| > \epsilon / f_0 \} \leq \mathbf{P} \{ \sup_{i \in Z} |x_i| > \epsilon_0 / f_0 \}$. It means that if condition (4) holds for enough small $\epsilon > 0$, then it holds for every $\epsilon > 0$. Thus, for stability in probability of the zero solution of equation (1), it is sufficient to prove condition (4) for enough small $\epsilon > 0$.

THEOREM 2. (See [5].) Let there exists a nonnegative functional $V_i = V(i, x_{-h}, \dots, x_i)$, which satisfies the conditions

$$\mathbf{E}V(0, x_{-h}, \dots, x_0) \leq c_1 \|\varphi\|_1^2,$$

$$\mathbf{E}\Delta V_i \leq -c_2 \mathbf{E}|x_i|^2, \quad i \in Z,$$

where $c_1 > 0$, $c_2 > 0$. Then the zero solution of equation (1) is asymptotically mean square stable.

From these theorems, it follows that the stability investigation of stochastic equations can be reduced to the construction of appropriate Lyapunov functionals.

2. CONDITIONS OF STABILITY IN PROBABILITY

Denote

$$a = \sum_{j=0}^{\infty} |a_j|, \quad \sigma = \sum_{j=0}^{\infty} |\sigma_j|, \quad \gamma = \sum_{j=0}^{\infty} \gamma_j. \quad (12)$$

THEOREM 3. Let $\gamma < \infty$ and

$$a^2 + \sigma^2 < 1. \quad (13)$$

Then the zero solution of equation (1) is stable in probability.

PROOF. To prove, it is sufficient to construct a functional satisfying the Theorem 1 conditions. Following the general method of Lyapunov functionals construction [5], we will construct a functional V_i in the form $V_i = V_{i1} + V_{i2}$.

Let $V_{i1} = x_i^2$. Then

$$\mathbf{E}_i \Delta V_{i1} = \left(\sum_{j=0}^{i+h} a_j x_{i-j} \right)^2 + \left(\sum_{j=0}^{i+h} \sigma_j x_{i-j} \right)^2 + g_i^2 + 2 \sum_{j=0}^{i+h} a_j x_{i-j} g_i - x_i^2.$$

Put

$$\mu_k(\epsilon) = \sum_{j=0}^{\infty} \gamma_j \epsilon^{k(\nu_j-1)}, \quad k = 1, 2. \quad (14)$$

It is easy to see that if $\epsilon \leq 1$, then $\mu_k(\epsilon) \leq \gamma < \infty$.

Using (3), (12) and assuming that $x_j \in U_\epsilon$, $j \leq i$, we obtain

$$\left(\sum_{j=0}^{i+h} a_j x_{i-j} \right)^2 \leq a \sum_{j=0}^{i+h} |a_j| x_{i-j}^2, \quad \left(\sum_{j=0}^{i+h} \sigma_j x_{i-j} \right)^2 \leq \sigma \sum_{j=0}^{i+h} |\sigma_j| x_{i-j}^2, \quad (15)$$

$$g_i^2(x_{-h}, \dots, x_i) \leq \gamma \sum_{j=0}^{i+h} \gamma_j |x_{i-j}|^{2\nu_j} \leq \gamma \sum_{j=0}^{i+h} \gamma_j \epsilon^{2(\nu_j-1)} x_{i-j}^2, \quad (16)$$

$$2 \sum_{j=0}^{i+h} a_j x_{i-j} g_i(x_{-h}, \dots, x_i) \leq \sum_{j=0}^{i+h} (a \gamma_j \epsilon^{\nu_j-1} + \mu_1(\epsilon) |a_j|) x_{i-j}^2.$$

Therefore,

$$\mathbf{E}_i \Delta V_{1i} \leq (A_0 - 1) x_i^2 + \sum_{j=1}^{i+h} A_j x_{i-j}^2, \quad (17)$$

where

$$A_j = (a + \mu_1(\epsilon)) |a_j| + \sigma |\sigma_j| + (a + \gamma \epsilon^{\nu_j-1}) \gamma_j \epsilon^{\nu_j-1}. \quad (18)$$

Choosing the functional V_{2i} in the form

$$V_{2i} = \sum_{j=1}^{i+h} x_{i-j}^2 \sum_{l=j}^{\infty} A_l, \quad (19)$$

we get

$$\mathbf{E}_i \Delta V_{2i} = \sum_{j=0}^{i+h} x_{i-j}^2 \sum_{l=j+1}^{\infty} A_l - \sum_{j=1}^{i+h} x_{i-j}^2 \sum_{l=j}^{\infty} A_l = x_i^2 \sum_{l=1}^{\infty} A_l - \sum_{j=1}^{i+h} A_j x_{i-j}^2. \quad (20)$$

From (17),(18),(20) for $V_i = V_{1i} + V_{2i}$, we have

$$\mathbf{E}_i \Delta V_i \leq (a^2 + \sigma^2 + 2a\mu_1(\epsilon) + \gamma\mu_2(\epsilon) - 1) x_i^2.$$

From (13), it follows that $\mathbf{E}_i \Delta V_i \leq 0$ for enough small ϵ . It is easy to see that the functional V_i which was constructed satisfies the conditions $V_i \geq x_i^2$ and $V_0 \leq (1+h)\|\varphi\|_0^2$. Thus, for the functional V_i , the Theorem 1 conditions hold. Therefore, using Remark 1, we get that the zero solution of equation (1) is stable in probability. Theorem is proved.

Denote

$$\alpha = \sum_{l=1}^{\infty} \left| \sum_{j=l}^{\infty} a_j \right|, \quad \beta = \sum_{j=0}^{\infty} a_j. \quad (21)$$

THEOREM 4. *Let $\gamma < \infty$ and*

$$\beta^2 + 2\alpha |1 - \beta| + \sigma^2 < 1. \quad (22)$$

Then the zero solution of equation (1) is stable in probability.

PROOF. It is easy to see that equation (1) can be represented in the form

$$x_{i+1} = \beta x_i + \Delta F_i + \sum_{j=0}^{i+h} \sigma_j x_{i-j} \xi_i + g_i(x_{-h}, \dots, x_i), \quad (23)$$

where

$$F_i = - \sum_{l=1}^{i+h} x_{i-l} \sum_{j=l}^{\infty} a_j. \quad (24)$$

Using the general method of Lyapunov functionals construction [5], we will construct the Lyapunov functional V_i in the form $V_i = V_{1i} + V_{2i}$ again, but now let $V_{1i} = (x_i - F_i)^2$ and V_{2i} has the form (19).

Calculating $\mathbf{E}_i \Delta V_{1i}$, similarly as Theorem 3, we will get

$$\mathbf{E}_i \Delta V_i \leq [\beta^2 + 2\alpha |1 - \beta| + \sigma^2 - 1 + 2(\alpha + |\beta|) \mu_1(\epsilon) + \gamma\mu_2(\epsilon)] x_i^2.$$

From here and (22), it follows that for enough small ϵ , there exists $c_2 > 0$ such that

$$\mathbf{E}_i \Delta V_i \leq -c_2 x_i^2. \quad (25)$$

It means that the functional V_i which was constructed here satisfies condition (8). It is easy to see that condition (7) for some $c_1 > 0$ hold too. But the functional V_i doesn't satisfy condition (6). Therefore, we cannot use Theorem 1 and must find another way to prove.

Let us consider the random variable τ which is described by (9). Using (10) and Chebyshev inequality, we get

$$\mathbf{P} \left\{ \sup_{i \in \mathbb{Z}} |x_i| > \epsilon / f_0 \right\} \leq \mathbf{P} \{ |x_{\tau+1}| > \epsilon / f_0 \} \leq \frac{\mathbf{E}_0 x_{\tau+1}^2}{\epsilon^2}. \quad (26)$$

Let us estimate $\mathbf{E}_0 x_{\tau+1}^2$. Let $0 \leq k \leq i \leq \tau$. Then from (25), it follows

$$\mathbf{E}_k \sum_{j=k}^i \mathbf{E}_j \Delta V_j = \mathbf{E}_k V_{i+1} - V_k \leq -c_2 \sum_{j=k}^i x_j^2 \leq -c_2 x_k^2 \leq 0.$$

From here we get

$$V_k \geq c_2 x_k^2, \quad 0 \leq k \leq \tau, \quad (27)$$

$$\mathbf{E}_0 V_{i+1} \leq V_0, \quad 0 \leq i \leq \tau. \quad (28)$$

From the form of the functional V_i , it follows that

$$V_{\tau+1} \geq (x_{\tau+1} - F_{\tau+1})^2 \geq x_{\tau+1}^2 - 2x_{\tau+1}F_{\tau+1}. \quad (29)$$

From (21),(24), it follows that

$$\begin{aligned} 2x_{\tau+1}F_{\tau+1} &\leq \sum_{l=1}^{\tau+1+h} \left| \sum_{j=l}^{\infty} a_j \right| (x_{\tau+1}^2 + x_{\tau+1-l}^2) \leq \\ &= \alpha x_{\tau+1}^2 + \sum_{l=1}^{\tau} \left| \sum_{j=l}^{\infty} a_j \right| x_{\tau+1-l}^2 + \sum_{l=\tau+1}^{\tau+1+h} \left| \sum_{j=l}^{\infty} a_j \right| x_{\tau+1-l}^2. \end{aligned}$$

Using (5), (21), and (29), we get

$$2x_{\tau+1}F_{\tau+1} \leq \alpha x_{\tau+1}^2 + \sum_{l=1}^{\tau} \left| \sum_{j=l}^{\infty} a_j \right| \frac{V_{\tau+1-l}}{c_2} + \alpha \delta^2. \quad (30)$$

Note that in (22) $|\beta| < 1$ and $\alpha < 1$. Therefore, substituting (30) into (29), we get

$$x_{\tau+1}^2 \leq (1 - \alpha)^{-1} \left(\alpha \delta^2 + \sum_{l=1}^{\tau} \left| \sum_{j=l}^{\infty} a_j \right| \frac{V_{\tau+1-l}}{c_2} + V_{\tau+1} \right).$$

Calculating \mathbf{E}_0 by virtue of (7), (21), (28), which holds as was noted above, and (5), we get

$$\mathbf{E}_0 x_{\tau+1}^2 < C \delta^2, \quad C = (1 - \alpha)^{-1} \left(\alpha + \left(\frac{\alpha}{c_2} + 1 \right) c_1 \right).$$

From here and (28) by $\delta = \epsilon \sqrt{\epsilon_1 / C}$, (4) follows. Therefore, the zero solution of equation (1) is stable in probability. The theorem is proved.

REMARK 2. It is easy to see that if in equation (1) $g_i = 0$, then the functionals which were constructed in Theorems 3 and 4 are satisfying the conditions of Theorem 2. It means that conditions (13) and (22) are sufficient for asymptotic mean square stability of the zero solution of equation (1) by $g_i = 0$. Thus, it is shown that investigation of stability in probability of nonlinear stochastic difference equations with order of nonlinearity more than one can be reduced to the investigation of asymptotic mean square stability of the linear part of these equations.

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