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Necessary and Sufficient Conditions of Asymptotic Mean Square Stability for Stochastic Linear Difference Equations

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Abstract—Many processes in automatic regulation, physics, mechanics, biology, economy, ecology, etc. can be modelled by hereditary systems (see, e.g., [1–4]). One of the main problems for the theory of such systems and their applications is connected with stability (see, e.g., [2–4]). Many stability results were obtained by the construction of appropriate Lyapunov functionals. At present, the method is proposed which allows us, in some sense, to formalize the procedure of the corresponding Lyapunov functionals construction [5–10]. In this work, by virtue of the proposed procedure, the necessary and sufficient conditions of asymptotic mean square stability for stochastic linear difference equations are obtained.

1. MAIN RESULT

Consider the scalar difference equation

$$x_{i+1} = \sum_{j=0}^{k} a_j x_{i-j} + \sigma x_{i-l}\xi_i, \qquad i \in Z, \qquad x_i = \varphi_i, \qquad i \in Z_0. \tag{1}$$

Here i is the discrete time, $i \in Z \cup Z_0$, $Z_0 = \{-h, \dots, 0\}$, $Z = \{0, 1, \dots\}$, $h = \max(k, l)$.

Let $\{\Omega, \sigma, \mathbf{P}\}$ be a probability space, $\{f_i \in \sigma\}$, $i \in Z$, be a sequence of σ-algebras, $\xi_0, \xi_1, \dots$ be mutually independent random values, ξ_i be f_{i+1}-adapted and independent from f_i, $\mathbf{E}\xi_i = 0$, $\mathbf{E}\xi_i^2 = 1$.

DEFINITION. *Zero solution of equation (1) is called mean square stable if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}x_i^2 < \epsilon$, $i \in Z$, if $\|\varphi\|^2 = \sup_{i\in Z_0} \mathbf{E}\varphi_i^2 < \delta$. If, besides, $\lim_{i\to\infty} \mathbf{E}x_i^2 = 0$, then equation (1) zero solution is called asymptotic mean square stable.*

THEOREM 1. *[9]. Let there exist the nonnegative functional $V_i = V(i, x_{-h}, \dots, x_i)$, $i \in Z$, for which the conditions*

$$\mathbf{E}V(0, x_{-h}, \dots, x_0) \le c_1 \|\varphi\|^2,$$
$$\mathbf{E}\Delta V_i \le -c_2 \mathbf{E}x_i^2, \qquad i \in Z,$$

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX *where $\Delta V_i = V_{i+1} - V_i$, $c_1 > 0$, $c_2 > 0$ hold. Then equation (1) zero solution is asymptotic mean square stable.*

Consider the vectors $x(i) = (x_{i-k}, \dots, x_{i-1}, x_i)'$ and $b = (0, \dots, \sigma)'$ of dimension $k+1$ and the square matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ a_k & a_{k-1} & a_{k-2} & \dots & a_0 \end{pmatrix}.$$

Then equation (1) can be described in the form

$$x(i+1) = Ax(i) + bx_{i-l}\xi_i. \tag{2}$$

Consider the matrix equation

$$A'DA - D = -U, \tag{3}$$

in which the square matrix $U = \|u_{ij}\|$ of dimension $k+1$ has all zero elements except for $u_{k+1,k+1} = 1$.

THEOREM 2. *Let equation (3) have a positive semidefinite solution D. Then, for asymptotic mean square stability of equation (1) zero solution, it is necessary and sufficient that the inequality*

$$\sigma^2 d_{k+1,k+1} < 1 \tag{4}$$

hold.

PROOF. Consider the functional

$$V_i = x'(i)Dx(i) + \sigma^2 d_{k+1,k+1} \sum_{j=1}^{l} x_{i-j}^2. \tag{5}$$

Calculating $\mathbf{E}\Delta V_i$ by virtue of (5),(2), we obtain

$$\begin{aligned} \mathbf{E}\Delta V_i &= \mathbf{E}\left[x'(i+1)Dx(i+1) + \sigma^2 d_{k+1,k+1}\sum_{j=1}^{l} x_{i+1-j}^2 - x'(i)Dx(i) - \sigma^2 d_{k+1,k+1}\sum_{j=1}^{l} x_{i-j}^2\right] \\ &= \mathbf{E}[(Ax(i) + bx_{i-l}\xi_i)'D(Ax(i) + bx_{i-l}\xi_i) - x'(i)Dx(i)] + \sigma^2 d_{k+1,k+1}\mathbf{E}\left(x_i^2 - x_{i-l}^2\right) \\ &= \mathbf{E}\left[x'(i)(A'DA - D)x(i) + b'Dbx_{i-l}^2\right] + \sigma^2 d_{k+1,k+1}\mathbf{E}\left(x_i^2 - x_{i-l}^2\right) \\ &= \left(\sigma^2 d_{k+1,k+1} - 1\right)\mathbf{E}x_i^2. \end{aligned}$$

Let condition (4) hold. Then the functional (5) satisfies the conditions of Theorem 1. It means that equation (1) zero solution is asymptotic mean square stable. It follows that condition (4) is sufficient for asymptotic mean square stability of equation (1) zero solution.

Let condition (4) not hold, i.e., $\sigma^2 d_{k+1,k+1} \geq 1$. Then $\mathbf{E}\Delta V_i \geq 0$. From here it follows that

$$\sum_{j=0}^{i-1} \mathbf{E}\Delta V_j = \mathbf{E}V_i - \mathbf{E}V_0 \geq 0,$$

i.e., $\mathbf{E}V_i \geq \mathbf{E}V_0 > 0$. It means that equation (1) zero solution cannot be mean square stable. Therefore, condition (4) is necessary for asymptotic mean square stability of equation (1) zero solution. Theorem 2 is proved.

2. EXAMPLES

Remark that for every k, equation (3) is the system of $(k+1)(k+2)/2$ equations. Consider the different particular cases of equation (3).

EXAMPLE 1. Let $k = 0$, $a_0 = a$. In this case, equation (3) has the form $d_{11}(a^2 - 1) = -1$. The necessary and sufficient condition of asymptotic mean square stability is

$$0 < d_{11} = \frac{1}{1-a^2} < \sigma^{-2}$$

or $a^2 + \sigma^2 < 1$. If $\sigma = 0$, this condition takes the form $|a| < 1$.

EXAMPLE 2. Let $k = 1$, $a_0 = a$, $a_1 = b$. In this case, equation (3) is the system of equations

$$b^2 d_{22} - d_{11} = 0, \qquad (b-1)d_{12} + abd_{22} = 0, \qquad d_{11} + 2ad_{12} + (a^2-1)d_{22} = -1.$$

Solving this system, we obtain the necessary and sufficient condition of asymptotic mean square stability

$$0 < d_{22} = \frac{1-b}{(1+b)((1-b)^2 - a^2)} < \sigma^{-2}.$$

If $\sigma = 0$, this condition has the form $|b| < 1$, $|a| < 1 - b$. In Figure 1, the region of stability is shown by (1) $\sigma^2 = 0$, (2) $\sigma^2 = 0.4$, (3) $\sigma^2 = 0.8$.

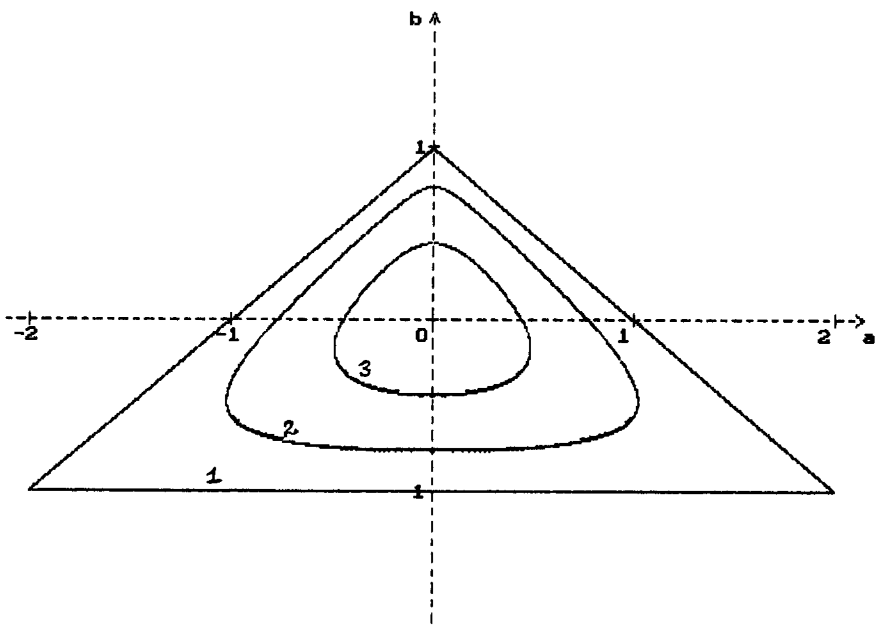

Figure 1.

EXAMPLE 3. Let $k = 2$, $a_0 = a$, $a_1 = 0$, $a_2 = b$. In this case, equation (3) is the system of equations

$$b^2 d_{33} - d_{11} = 0, \qquad bd_{13} - d_{12} = 0, \qquad d_{11} - d_{22} = 0, \qquad bd_{23} + abd_{33} - d_{13} = 0,$$
$$d_{12} + ad_{13} - d_{23} = 0, \qquad d_{22} + 2ad_{23} + (a^2-1)d_{33} = -1.$$

Solving this system, we obtain the necessary and sufficient condition of asymptotic mean square stability

$$0 < d_{33} = \left(1 - b^2 - a^2 \frac{1+(a+b)b}{1-(a+b)b}\right)^{-1} < \sigma^{-2}.$$

In Figure 2, the region of stability is shown by (1) $\sigma^2 = 0$, (2) $\sigma^2 = 0.4$, (3) $\sigma^2 = 0.8$.

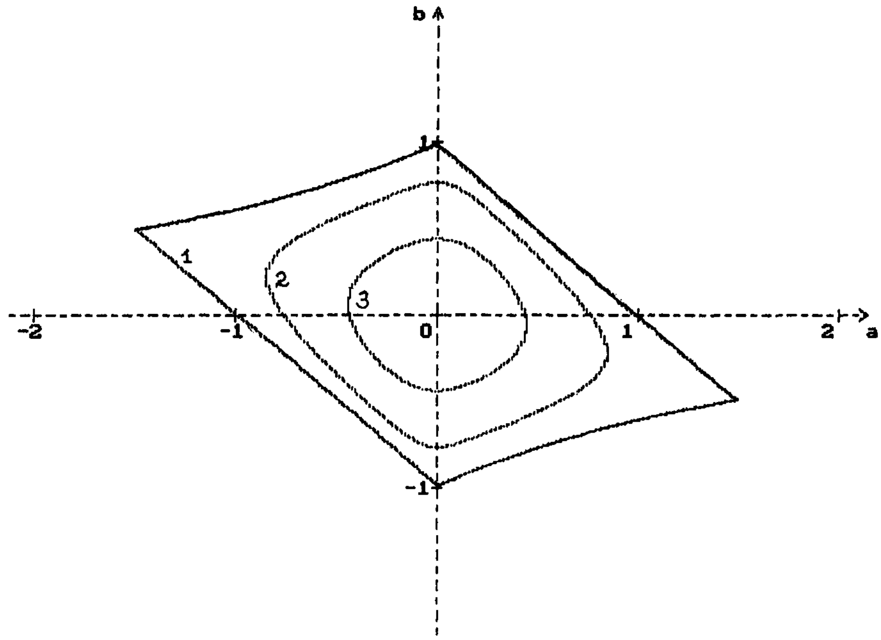

Figure 2.

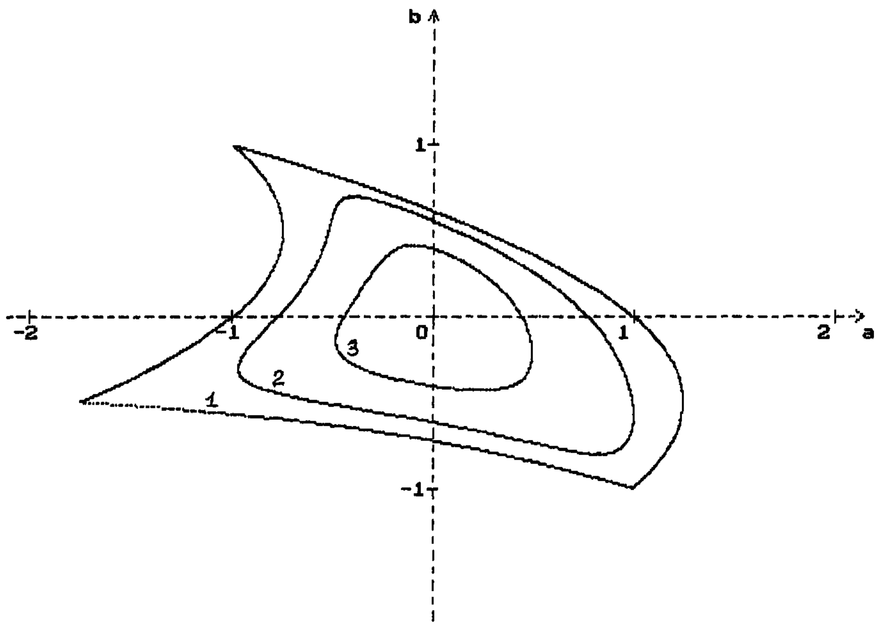

Figure 3.

EXAMPLE 4. Let $k = 2$, $a_0 = a$, $a_1 = b$, $a_2 = b^2$. In this case, equation (3) is the system of equations

$$\begin{aligned} b^4 d_{33} - d_{11} &= 0, & b^2 d_{13} + b^3 d_{33} - d_{12} &= 0, \\ b^2 d_{23} + ab^2 d_{33} - d_{13} &= 0, & d_{11} + 2bd_{13} + b^2 d_{33} - d_{22} &= 0, \\ d_{12} + ad_{13} + abd_{33} + (b-1)d_{23} &= 0, & d_{22} + 2ad_{23} + (a^2 - 1)d_{33} &= -1. \end{aligned}$$

Solving this system, we obtain the necessary and sufficient condition of asymptotic mean square stability

$$0 < d_{33} = \left(1 - a^2 - b^2 - b^4 - 2ab^3 - \frac{2b(1+ab)\left(a+b^2\right)\left(a+b^3\right)}{1 - b - b^4 - ab^2}\right)^{-1} < \sigma^{-2}.$$

In Figure 3, the region of stability is shown by (1) $\sigma^2 = 0$, (2) $\sigma^2 = 0.4$, (3) $\sigma^2 = 0.8$.

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