BUBBLING WITH L^2-ALMOST CONSTANT MEAN CURVATURE AND AN ALEXANDROV-TYPE THEOREM FOR CRYSTALS

M. G. DELGADINO, F. MAGGI, C. MIHAILA, AND R. NEUMAYER

ABSTRACT. A compactness theorem for volume-constrained almost-critical points of elliptic integrands is proven. The result is new even for the area functional, as almost-criticality is measured in an integral rather than in a uniform sense. Two main applications of the compactness theorem are discussed. First, we obtain a description of critical points/local minimizers of elliptic energies interacting with a confinement potential. Second, we prove an Alexandrov-type theorem for crystalline isoperimetric problems.

1. INTRODUCTION

1.1. **Overview.** The study of critical points in geometric variational problems often calls for the understanding of bubbling/concentration phenomena. Classical examples are discussed in the seminal papers of Brezis-Coron [BC84] and Struwe [Str84], where the authors investigate immersed disks with almost-constant mean curvature and conformally flat metrics with almost-constant scalar curvature. As illustrated by the monographs [Str00, Heb14], this kind of result plays an important role in various contexts.

Here we are interested in sets with almost-constant mean curvature, that is to say, in sets that are close to being critical in isoperimetric problems. Such sets arise in various contexts of physical and geometric importance, like capillarity theory and mean curvature flows. Depending on the application one has in mind, different ways of measuring almost-criticality are appropriate. For example, in the study of capillarity problems, one is naturally led to consider surfaces whose mean curvature is uniformly close to a constant. In geometric applications, we have a more complicated situation, as uniform proximity to constant mean curvature should be replaced by L^2-proximity.

Our starting point is the paper [CM17], where, as illustrated in more detail below (Section 1.2), Ciraolo and the second author obtained a compactness result for boundaries whose mean curvature is uniformly close to a constant. In our main result, Theorem 1.1 below, we obtain two critical improvements of the compactness theorem from [CM17], which require a substantial rethinking of many technical aspects of the original argument.

A first improvement consists of replacing uniform proximity with L^2-proximity. The main difficulty here is of course that uniform proximity to constant mean curvature, unlike L^2-proximity, carries information on the size of the mean curvature oscillation at *every* boundary point, and thus allows one to exploit powerful sliding/maximum principle arguments.

A second major improvement consists of replacing the area functional with a surface energy for a generic elliptic integrand. Elliptic integrands model anisotropic surface tensions and are thus of importance in numerous applications. From the mathematical viewpoint, the area functional is quite exceptional among elliptic integrands, and there are various steps in the argument of [CM17] where this fact was exploited.

Referring to Section 1.3 for more comments on the proof of Theorem 1.1, we now discuss some applications of physical and geometric interest.

We start with Theorem 1.3, where we obtain a description of critical points and local minimizers in elliptic capillarity problems. This theorem is stated in Section 1.4, where we also provide additional context on this type of problem.

In Section 1.5, we state Theorem 1.4, an extension of Theorem 1.1 to sequences of almost-critical points corresponding to elliptic integrands with degenerating ellipticity. The latter property allows these elliptic integrands to converge to an arbitrary (i.e., possibly non-smooth and non-elliptic) convex integrand, and our result proves convergence of almost-critical points (with sufficiently fast convergence of the first variation to a constant) to (possibly multiple copies of) the Wulff shape of the limit integrand. We propose an interpretation of this result as a suitable formulation of Alexandrov's theorem for generic anisotropic energies. Let us recall that for smooth, elliptic anisotropic energies one has a pointwise notion of mean curvature for which an exact analog of the classical Alexandrov's theorem holds [HLMG09]. The situation is quite different for generic anisotropic energies, like crystalline energies, as in those cases the first variation of the energy does not even define a linear functional on the space of variations. In the proposed interpretation, we circumvent these difficulties by defining critical points of generic anisotropic problems as the accumulation points of almost-critical points of smooth elliptic anisotropic problems.

1.2. **Compactness in the Euclidean case.** We start by recalling the situation in the basic Euclidean case. The starting point is *Alexandrov's theorem*: if Ω is a smooth bounded connected open set with constant mean curvature, then Ω is a ball of radius $(n+1)|\Omega|/P(\Omega) = n/H_\Omega$. Here, $|\Omega|$ and $P(\Omega) = \mathcal{H}^n(\partial\Omega)$ are the volume and the perimeter of Ω, while H_Ω is the scalar mean curvature of Ω with respect to the outer unit normal ν_Ω to Ω, with the convention that $H_{B_r(x)} = n/r$ if $B_r(x)$ is the ball of radius r in $\mathbb{R}^{n+1}$ centered at a point x. In [CM17] the *Alexandrov's deficit* $\delta^*(\Omega)$ of Ω

$$\delta^*(\Omega) = \left\|\frac{H_\Omega}{H_\Omega^0} - 1\right\|_{C^0(\partial\Omega)} \qquad H_\Omega^0 = \frac{n\,P(\Omega)}{(n+1)|\Omega|}\,. \tag{1.1}$$

is introduced as a measure of how far Ω is from being a critical point in the Euclidean isoperimetric problem. It is then proven that, if $\{\Omega_h\}_{h\in\mathbb{N}}$ is a sequence of smooth bounded open sets in $\mathbb{R}^{n+1}$ normalized to have $H_\Omega^0 = n$ and satisfying, for some $L \in \mathbb{N}$ and $\sigma \in (0,1)$,

$$P(\Omega_h) \le (L+\sigma)\,P(B_1) \qquad \forall h \in \mathbb{N}$$

and if

$$\lim_{h\to\infty} \delta^*(\Omega_h) = 0\,, \tag{1.2}$$

then there exists an open set Ω consisting of the union of at most L disjoint balls of radius one such that

$$\lim_{h\to\infty} \big|P(\Omega_h) - P(\Omega)\big| + |\Omega_h \Delta \Omega| = 0\,. \tag{1.3}$$

Allard's monotonicity formula [Sim83, Section 17] can then be exploited to deduce Hausdorff convergence (therefore, if the sets Ω_h are connected, then the balls in Ω are mutually tangent). Then, by exploiting Allard's regularity theorem [Sim83, Section 23] and a calibration-type argument, one obtains the $C^{1,\alpha}$-convergence of $\partial\Omega_h$ to $\partial\Omega$ away from the tangency points of the limit balls. A quantitative analysis is also possible, both in the bubbling case (with non-sharp decay rates, see [CM17]) and in the one-bubble case $L = 1$ (with sharp decay rates, see [KM17]).

This type of compactness result is crucial for addressing the shape of critical points in capillarity problems. To illustrate this point, consider the capillarity energy

$$P(\Omega) + \int_\Omega g(x)\,dx\,, \qquad |\Omega| = v\,, \tag{1.4}$$

of a liquid droplet occupying a region $\Omega \subset \mathbb{R}^{n+1}$ of fixed volume v under the action of a confinement potential g. If v is suitably small with respect to g, then the surface energy $P(\Omega) = \mathrm{O}(v^{n/(n+1)})$ dominates over the potential energy term (of order $\mathrm{O}(v)$). By direct comparison with balls and by quantitative isoperimetry [FMP08, FMP10, CL12] global minimizers are seen to be L^1-close to balls (quantitatively in terms of the size of v), and then a variational analysis proves they are actually C^2-close and thus convex [FM11]. But direct comparison with balls is not available for addressing the shape of critical points, or even of local minimizers, and this is why a compactness theorem like the one proved in [CM17] is needed to get this analysis started. And indeed, in [CM17, Corollary 1.4] it is shown that critical points of (1.4) are quantitatively close close to compounds of mutually tangent balls with same radii, and that local minimizers are close to single balls, for m small.

1.3. **The anisotropic setting and the elliptic compactness theorem.** In various situations of physical and geometric interest (see, e.g., the survey paper [Tay78]) one is led to consider energies like (1.4) with $P(\Omega)$ replaced by an anisotropic surface energy of the form

$$\mathcal{F}(\Omega) = \int_{\partial\Omega} F(\nu_\Omega)\,d\mathcal{H}^n\,.$$

Here $F : \mathbb{S}^n \to (0, \infty)$ is a *convex integrand*: namely, the one-homogenous extension of F is convex on $\mathbb{R}^{n+1}$. As was proven in [Tay74, Tay75, Fon91, FM91, BM94], the isoperimetric problem for $\mathcal{F}$ is uniquely solved by translations and scalings of the *Wulff shape K_F of F*.

$$K_F = \bigcap_{\nu \in \mathbb{S}^n} \left\{ x \in \mathbb{R}^{n+1} : x \cdot \nu < F(\nu) \right\}. \tag{1.5}$$

This translates into the *Wulff inequality*,

$$\mathcal{F}(\Omega) \geq (n+1)\,|K_F|^{1/(n+1)}\,|\Omega|^{n/(n+1)} \qquad 0 < |\Omega| < \infty\,, \tag{1.6}$$

where the right-hand side equals $\mathcal{F}(r\,K_F)$ for $r = (|\Omega|/|K_F|)^{1/(n+1)}$ and where equality holds if and only if $\Omega = x + r\,K_F$ for some $x \in \mathbb{R}^{n+1}$. The Wulff shape is always a bounded open convex set containing the origin; and, conversely, every bounded open convex set containing the origin is the Wulff shape of some F. Of particular interest is the case when F is a *smooth elliptic integrand*, that is, $F \in C^\infty(\mathbb{S}^n; (0, \infty))$, and there exist constants $0 < \lambda \leq \Lambda < \infty$ such that, for every $\nu \in \mathbb{S}^n$,

$$\lambda\,\mathrm{Id} \leq \nabla^2 F(\nu) \leq \Lambda\,\mathrm{Id} \qquad \text{on } \nu^\perp = T_\nu\mathbb{S}^n\,. \tag{1.7}$$

In this setting one has a natural anisotropic extension of the notion of mean curvature. More precisely, the anisotropic mean curvature $H_\Omega^F : \partial\Omega \to \mathbb{R}$ of a set $\Omega \subset \mathbb{R}^{n+1}$ with boundary of class C^2 is defined by

$$H_\Omega^F = \mathrm{div}^{\partial\Omega}\big(\nabla F \circ \nu_\Omega\big) = \mathrm{tr}(\nabla^2 F(\nu_\Omega)\,\nabla\nu_\Omega)\,, \tag{1.8}$$

where $\mathrm{div}^{\partial\Omega}$ denotes the tangential divergence along $\partial\Omega$, and $\nabla\nu_\Omega$ is the second fundamental form of $\partial\Omega$ with respect to ν_Ω. For the Wulff shape, K_F one has

$$H_{K_F}^F = n\,.$$

An anisotropic version of Alexandrov's theorem was shown in [HLMG09]: if Ω is a bounded smooth connected open set in $\mathbb{R}^{n+1}$ with constant anisotropic mean curvature, then $\Omega = x + r\, K_F$ for some $x \in \mathbb{R}^{n+1}$ and $r > 0$. In order to formulate a compactness theorem in this setting we introduce the scale invariant quantity

$$\delta_F(\Omega) = \left(\frac{1}{\mathcal{F}(\Omega)} \int_{\partial\Omega} \left|\frac{H_\Omega^F}{H_\Omega^{F,0}} - 1\right|^2 F(\nu_\Omega)\, d\mathcal{H}^n\right)^{1/2} \qquad H_\Omega^{F,0} = \frac{n\,\mathcal{F}(\Omega)}{(n+1)|\Omega|} \qquad (1.9)$$

as an anisotropic generalization of (1.1), and then state the following theorem.

Theorem 1.1. *Let F be a smooth elliptic integrand; see (1.7), and let $\{\Omega_h\}_{h\in\mathbb{N}}$ be a sequence of bounded open sets with smooth boundary normalized to have*

$$H_{\Omega_h}^{F,0} = n\,.$$

If, for some $L \in \mathbb{N}$, $\sigma \in (0,1)$, and $\kappa \in (0,1)$

$$\sup_{h\in\mathbb{N}} \mathrm{diam}(\Omega_h) < \infty\,, \qquad \sup_{h\in\mathbb{N}} \mathcal{F}(\Omega_h) \leq (L+\sigma)\,\mathcal{F}(K_F)\,, \qquad H_{\Omega_h}^F \geq \kappa\,n \ \text{on } \partial\Omega_h\,,$$

and if

$$\lim_{h\to\infty} \delta_F(\Omega_h) = 0\,, \qquad (1.10)$$

then there exists an open set Ω consisting of the union of at most L-many disjoint translations of K_F, such that, up to translations and up to extracting subsequences,

$$\lim_{h\to\infty} \left|\mathcal{F}(\Omega_h) - \mathcal{F}(\Omega)\right| + |\Omega_h \Delta \Omega| = 0\,. \qquad (1.11)$$

Remark 1.2. Setting $\delta(\Omega) = \delta_F(\Omega)$ for $F \equiv 1$, and recalling the definition (1.1) of $\delta^*(\Omega)$, we obviously have that $\delta(\Omega) \leq \delta^*(\Omega)$, with $H_\Omega \geq \kappa\, H_\Omega^0$ provided $\delta^*(\Omega) \leq 1 - \kappa$. In particular, Theorem 1.1 contains the fact that (1.2) implies (1.3), i.e., the key conclusion of [CM17, Theorem 2.4]. As explained in Section 1.1, passing from the C^0 to the L^2 deficit is non-trivial because a key argument in the proof of [CM17, Theorem 2.4] is a sliding argument based on the maximum principle for the mean curvature operator (see the argument right after [CM17, Equation (2.48)]). For this kind of argument to work, it is crucial that whichever the contact point produced in the sliding argument is, the constant mean curvature deficit contains information at that point. This works naturally when using the C^0-deficit $\delta^*(\Omega)$, but it is clearly more delicate for the L^2-deficit $\delta(\Omega)$. We bypass this problem by exploiting the vanishing deficit assumption for obtaining a family of Pohozaev's-type identities of different homogeneities; see in particular step six in the proof of Theorem 1.4, Section 3.6. We also notice that considering the L^2-deficit in this kind of problem is not merely done for the sake of generality, but is particularly significant in view of possible applications to the analysis of mean curvature flows.

We now describe some additional aspects of the proof of Theorem 1.1. As we are going to see, a key tool will be a new potential theoretic proof of the anisotropic Heintze-Karcher inequality, done in the spirit of [Ros87].

Let us first recall that the classical Heintze-Karcher inequality states that if Ω is an open bounded connected set with smooth boundary and positive mean curvature, then

$$\int_{\partial\Omega} \frac{n}{H_\Omega}\, d\mathcal{H}^n \geq (n+1)|\Omega|\,, \qquad (1.12)$$

with equality if and only if Ω is a ball. This inequality has been exploited by many authors [Ros87, MR91, Bre13] as an effective starting point for proving the classical Alexandrov's theorem and its generalizations to higher order curvatures and to non-Euclidean ambient spaces. The relation is seen, also in the context compactness problems, if one considers the simple inequality

$$\eta(\Omega) \leq \delta^*(\Omega) \tag{1.13}$$

between the Alexandrov's deficit (1.1) and the *Heintze-Karcher deficit*

$$\eta(\Omega) = 1 - \frac{(n+1)|\Omega|}{\int_{\partial\Omega} (n/H_\Omega)\, d\mathcal{H}^n}\, .$$

In particular, (1.13) says that if Ω has constant mean curvature, then Ω is an equality case in (1.12) (and thus a ball).

The compactness result obtained in [CM17] starts from Ros' proof [Ros87] of (1.12), which exploits the celebrated Reilly's identity [Rei77] in order to relate the Heintze-Karcher deficit and the torsion potential u of Ω, i.e., the unique solution of

$$\begin{cases} \Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

For instance, a key estimate on u in terms of $\eta(\Omega)$ implied by Ros' argument takes the form

$$C(n)\,|\Omega|\,\eta(\Omega) \geq \int_\Omega \left| \nabla^2 u - \frac{\mathrm{Id}}{n+1} \right|^2 \tag{1.14}$$

where $|T| = (\sum_{ij} T_{ij}^2)^{1/2}$ for a matrix T. This inequality expresses, in a rather direct way, the proximity of the torsion potential of Ω to the torsion potential $(|x - x_0|^2 - r^2)/2(n+1)$ of a suitable ball $B_r(x_0)$. The other known proofs of (1.12), namely [MR91, Bre13], provide control of the almost-umbilicality of $\partial\Omega$ in terms of $\eta(\Omega)$. As explained in detail in [CM17, Appendix], almost-umbilicality is more difficult to exploit than direct information on the torsion potential to get compactness results in this setting. This is reflected in the fact that the current results concerning almost-umbilicality do not describe bubbling phenomena; see [DLM05, DLM06, Per11].

The proof of Alexandrov's theorem for elliptic integrands in [HLMG09] is based on an anisotropic version of (1.12), which states that if F is smooth and elliptic and Ω is an open bounded connected set with smooth boundary and positive H_Ω^F, then

$$\int_{\partial\Omega} \frac{n}{H_\Omega^F} F(\nu_\Omega)\, d\mathcal{H}^n \geq (n+1)|\Omega|\, , \tag{1.15}$$

with equality if and only if $\Omega = x + r\, K_F$ for some $x \in \mathbb{R}^{n+1}$ and $r > 0$. Alternative proofs of (1.15) are obtained in [MX13] and [XZ]. These arguments provide a control on the anisotropic almost-umbilicality of $\partial\Omega$ in terms of an anisotropic Heintze-Karcher deficit. Anisotropic almost-umbilicality has been recently addressed in [DRG17] under a convexity assumption, which of course prevents the possibility of bubbling into multiple Wulff shapes.

Our approach to Theorem 1.4 will pass through a new proof of (1.15), based on an adaptation of Ros' argument that allows us to control the suitable anisotropic variant of the torsion potential in terms of the *anisotropic Heintze-Karcher deficit* η_F of Ω,

$$\eta_F(\Omega) = 1 - \frac{(n+1)|\Omega|}{\int_{\partial\Omega} (n\, F(\nu_\Omega)/H_\Omega^F)}\, .$$

(We consider $\eta_F(\Omega)$ only for sets with $H_\Omega^F > 0$ on $\partial\Omega$, and we also notice that $\eta_F(\Omega) \leq \delta_F(\Omega)/\kappa$ provided $H_\Omega^F \geq \kappa H_\Omega^{F,0}$; see Lemma 3.6 below.) The right notion of anisotropic torsion potential is found by solving

$$\begin{cases} \Delta_F u = 1 & \text{in } \Omega \,, \\ u = 0 & \text{on } \partial\Omega \,. \end{cases}$$

where Δ_F denotes the Finslerian Laplace operator, defined by

$$\Delta_F u = \text{div}\left(\nabla(F^2/2)(\nabla u)\right).$$

The operator is non-smooth at $\{\nabla u = 0\}$, and particular attention must be paid in many parts of the argument in managing the critical set of u.

We now make some more technical comments on the proof, which should also be interesting in connection with our subsequent discussion of the role of ellipticity in the compactness theorem (see Section 1.5 below). As already mentioned, Ros' proof of (1.14) crucially exploits Reilly's identity [Rei77]. Similarly, in Proposition 3.3 below, we prove the following anisotropic version of Reilly's identity (which, apparently, has not been previously stated in the literature):

$$\int_\Omega (\Delta_F u)^2 - \text{tr}\left(((\nabla(\nabla_F u))^2\right) = \int_{\partial\Omega} H_\Omega^F \, F(\nabla u)^2 F(\nu_\Omega) \, d\mathcal{H}^n \,, \qquad (1.16)$$

where $\nabla_F u = \nabla(F^2/2)(\nabla u)$. Armed with (1.16) and with a global Lipschitz estimate for u *independent of the ellipticity constants of F* (see Proposition 3.5), we generalize (1.14) to

$$C(n) \, |\Omega| \, \eta_F(\Omega) \geq \int_\Omega \left\| \nabla(\nabla_F u) - \frac{\text{Id}}{n+1} \right\|^2 \,. \qquad (1.17)$$

Here we set $\|A\|^2 = \sum_{i=1}^{n+1} \mu_i^2$ where μ_i are the eigenvalues of and A. Unless we are in the isotropic case, the matrix $A = \nabla(\nabla_F u)$ (which satisfies $A = \nabla^2 F(\nabla u) \nabla^2 u$ on $\{\nabla u \neq 0\}$) is not symmetric, but is real diagonalizable. In particular (1.17) does not provide direct control on the norm of A. However, such an estimate can be expressed if one allows the ellipticity constants of F to appear in the estimate, which then takes the form

$$C(n) \, |\Omega| \, \frac{\Lambda}{\lambda} \, \eta_F(\Omega) \geq \int_\Omega \left| \nabla(\nabla_F u) - \frac{\text{Id}}{n+1} \right|^2 \,. \qquad (1.18)$$

This simple but delicate point is the only step of the proof of Theorem 1.1 where the ellipticity assumption is used.

1.4. Critical points and local minimizers of anisotropic energies.

A natural and important application of Theorem 1.1 is the quantitative characterization of critical points and local minimizers of anisotropic surface energies under the action of a confining potential and with a small volume constraint. More precisely, the kind of energy we consider takes the form

$$\mathcal{E}(\Omega) = \mathcal{F}(\Omega) + \int_\Omega g \,,$$

for a convex integrand F and a smooth potential $g : \mathbb{R}^{n+1} \to \mathbb{R}$. When $g(x) \to \infty$ as $|x| \to \infty$ volume-constrained minimizers of $\mathcal{E}$ exist for every volume v. Of particular relevance in capillarity and phase transition models is the case when v is small, and thus the surface energy $\mathcal{F}$ dominates the minimization.

We say that a bounded set of finite perimeter Ω is a *volume-constrained critical point* of $\mathcal{E}$ if

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{E}(f_t(\Omega)) = 0 \tag{1.19}$$

whenever f_t is a curve of diffeomorphisms such that $f_0 = \mathrm{Id}$ and $|f_t(\Omega)| = |\Omega|$ for every t in a neighborhood of $t = 0$. We say that Ω is a *volume-constrained r_0-local minimizer of $\mathcal{E}$* if there exists $r_0 > 0$ such that

$$\mathcal{E}(\Omega) \le \mathcal{E}(\Omega') \qquad \text{whenever } |\Omega'| = |\Omega| \text{ and } \Omega \Delta \Omega' \subset\subset I_{|\Omega|^{1/(n+1)} r_0}(\partial\Omega). \tag{1.20}$$

Here, $I_r(E) = \{x \in \mathbb{R}^{n+1} : \mathrm{dist}(x, E) < r\}$ denotes the r-neighborhood of a set $E \subset \mathbb{R}^{n+1}$. Obviously, local minimizers are critical points.

As an application of Theorem 1.1, we analyze volume-constrained local minimizers and critical points of $\mathcal{E}$.

Theorem 1.3. *Let F be a smooth elliptic integrand, $g : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function, $M > 0$, and let $\Omega \subset B_M$ be an open connected set with smooth boundary such that*

$$\frac{\mathcal{F}(\Omega)^{n+1}}{|\Omega|^n} \le M, \qquad \frac{\mathrm{diam}(\Omega)}{|\Omega|^{1/(n+1)}} \le M.$$

Then the following holds:

(i) for every $\varepsilon > 0$ there exists $v_\varepsilon = v_\varepsilon(\varepsilon, n, M, F, g) > 0$ such that if Ω is a volume-constrained critical point of $\mathcal{E}$ with

$$|\Omega| < v_\varepsilon,$$

then

$$\left| \Omega^* \Delta \bigcup_{i=1}^{L}(x_i + K_F) \right| \le \varepsilon \qquad \text{where} \quad \Omega^* = \frac{H_\Omega^{F,0}}{n}\Omega,$$

for some $L \ge 1$ (which is a priori bounded from above in terms of F and M) and $\{x_i\}_{i=1}^{L} \subset \mathbb{R}^{n+1}$ such that the sets $\{x_i + K_F\}_{i=1}^{L}$ are mutually disjoint.

(ii) given $r_0 > 0$, there exists $v_0 = v_0(r_0, n, M, F, g) > 0$ such that, if Ω is a volume-constrained r_0-local minimizer of $\mathcal{E}$ with

$$|\Omega| < v_0,$$

then there exists $x_0 \in \mathbb{R}^{n+1}$ such that, setting $K = x + s\,K_F$ with $s = (|\Omega|/|K_F|)^{1/(n+1)}$, one has

$$\frac{|\Omega \Delta K|}{|\Omega|} \le C(n, F, g, M)\,|\Omega|^{1/(n+1)}, \tag{1.21}$$

$$\frac{\mathrm{hd}(\partial\Omega, \partial K)}{|\Omega|^{1/(n+1)}} \le C(n, F, g, M)\,|\Omega|^{1/(n+1)^2}. \tag{1.22}$$

This result was shown for volume-constrained global minimizers of $\mathcal{E}$ in [FM11, Theorem 1]. The key difference between the case of global minimizers and the case of critical points/local minimizers addressed here is that (1.19) and (1.20) do not immediately allow the comparison of the energy of Ω with that of a Wulff shape with same volume. This direct comparison was the key step in [FM11] to exploit the quantitative Wulff inequality from [FMP10] and to deduce the L^1-proximity of any global minimizer to a Wulff shape.

Volume-constrained critical points of $\mathcal{E}$ turn out be volume-constrained almost-critical points of $\mathcal{F}$ thanks to a variational argument, so Theorem 1.1 allows one to deduce that Ω is close in volume to a finite family of Wulff shapes. In the case of local minimizers, using volume density estimates we can show this family consists of a single Wulff shape, and then $\partial\Omega$ is close to the boundary of this Wulff shape in Hausdorff distance. But this implies that a Wulff shape of the appropriate volume is an admissible competitor in (1.20), which in turn enables us to exploit the quantitative Wulff inequality and obtain (1.21) and (1.22); see Section 4.

In [FM11, Theorem 2] it was proven that a volume-constrained global minimizer Ω of $\mathcal{E}$ is actually convex with

$$\left\|\nabla^2 F(\nu_{\Omega^*})\nabla\nu_{\Omega^*} - \mathrm{Id}\right\|_{C^0(\partial\Omega^*)} \leq C(n, F, g)\,|\Omega|^{2/(n+3)}\,. \tag{1.23}$$

(Let us recall here that $\nabla^2 F(\nu_{K_F})\nabla\nu_{K_F} = \mathrm{Id}$ on ∂K_F.) In our setting of connected volume-constrained local minimizers, once (1.22) is proven, one can also repeat all the remaining analysis from [FM11]. In particular, arguing as in [FM11, Appendix C], one can show that $\partial\Omega$ is a C^1-small normal deformation of ∂K with quantitative bounds on the C^1-norm of the normal deformation in terms of explicit powers of $|\Omega|$. Then, one can also repeat the argument from [FM11, Theorem 13] to show that (1.23) holds and thus that Ω is convex. As this part of the argument would be identical to that of [FM11], we omit it and simply remark that, thanks to Theorem 1.1, all the small-volume regime properties of volume-constrained global minimizers of $\mathcal{E}$ proved in [FM11] hold for connected local minimizers as well.

1.5. **Weak Alexandrov's theorem for convex integrands.**

A way to look at the classical Alexandrov's theorem is to consider it as a *convexity property* of the volume-constrained perimeter functional; Alexandrov's theorem says that the only critical points of the volume-constrained perimeter are its global minimizers. The property that global minimizers are the only critical points is a characteristic consequence of convexity.

The same can be said about the anisotropic Alexandrov's theorem for smooth elliptic integrands [HLMG09]. Once again, we have an energy functional whose only critical points are its global minimizers. A natural question is whether this property depends on the assumption that the anisotropy is smooth and elliptic. Indeed, anisotropic energies that fail to be either smooth or elliptic (or both) are of great interest in applications, and the anisotropic isoperimetric problem is totally unaffected by the lack of smoothness or of ellipticity (recall that Wulff shapes are the unique isoperimetric sets of every convex integrand). Therefore one conjectures that Alexandrov's theorem should hold, in some proper formulation, for every anisotropic energy. This conjecture seems open in ambient space dimension $n + 1 \geq 3$. A result of Morgan [Mor94], which proves that Wulff shapes are the only anisotropic critical immersion of a curve in the plane, is an indication in favor of this conjecture. Of course, there is a substantial difference between the planar case and higher dimensions, because in the planar case criticality implies convexity of the connected components.

With this premise in mind, we now discuss a generalization of Theorem 1.1, namely Theorem 1.4 below, which serves as a sort of *weak version of Alexandrov's theorem valid for arbitrary convex integrands*.

Theorem 1.4 (Weak anisotropic Alexandrov theorem). *Let $\{F_h\}_{h\in\mathbb{N}}$ be a sequence of smooth elliptic integrands that converges pointwise to a limit function F, and assume that, for some positive*

constants m, M, λ_h and Λ_h (m and M independent of h)

$$m \leq F_h \leq M \qquad \text{on } \mathbb{S}^n \,,$$
$$\lambda_h \, \text{Id} \leq \nabla^2 F_h(\nu) \leq \Lambda_h \, \text{Id} \qquad \text{on } \nu^\perp, \, \forall \nu \in \mathbb{S}^n \,. \tag{1.24}$$

Then the following holds: If $\{\Omega_h\}_{h \in \mathbb{N}}$ is a sequence of bounded open sets with smooth boundary, normalized so that $H_{\Omega_h}^{F_h,0} = n$ with

$$H_{\Omega_h}^{F_h} \geq \kappa \, n \text{ on } \partial\Omega_h\,, \qquad \sup_{h \in \mathbb{N}} \text{diam}(\Omega_h) < \infty\,, \qquad \sup_{h \in \mathbb{N}} \frac{\mathcal{F}_h(\Omega_h)}{\mathcal{F}_h(K_{F_h})} \leq L + \sigma \tag{1.25}$$

for some $\kappa > 0$, $L \in \mathbb{N}$ and $\sigma \in (0, 1)$, and if

$$\lim_{h \to \infty} \max\left\{ \frac{1}{\lambda_h^2}\,, \frac{\Lambda_h}{\lambda_h} \right\} \eta_{F_h}(\Omega_h) + \delta_{F_h}(\Omega_h) = 0\,, \tag{1.26}$$

then there exists an open set Ω, which is the disjoint union of at most L-many translations of K_F such that, up to translations and up to extracting subsequences,

$$\lim_{h \to \infty} \left| \mathcal{F}_h(\Omega_h) - \mathcal{F}(\Omega) \right| + |\Omega_h \Delta \Omega| = 0\,. \tag{1.27}$$

Remark 1.5. The assumption (1.26) is implied by $\max\{1/\lambda_h^2, \Lambda_h/\lambda_h\} \, \delta_{F_h}(\Omega_h) \to 0$ because $\eta_F \leq \delta_F/\kappa$ (see Lemma 3.6 below). In particular, Theorem 1.1 is a corollary of Theorem 1.4 if one takes F to be smooth and elliptic and $F_h \equiv F$.

The interpretation of Theorem 1.4 as a weak version of Alexandrov's theorem for general convex integrands is the following. Clearly, every convex integrand F can be approximated by smooth elliptic integrands F_h as in (1.24), as the ratio $\max\{1/\lambda_h^2, \lambda_h/\Lambda_h\}$ is allowed to vanish in the limit as $h \to \infty$. Theorem 1.4 thus asserts that (unions of) Wulff shapes of F are the only possible accumulation points of sequences of almost-critical points of the approximating $\mathcal{F}_h$ with anisotropic Heintze-Karcher deficit vanishing faster than the rate of degeneracy of the ellipticity constants. This fast convergence assumption may be purely technical in nature. In our argument, it arises exactly in the derivation of (1.18) from (1.17), as previously explained.

Theorem 1.4 immediately leads to the following delicate question: given a convex integrand F and a bounded open set Ω with Lipschitz boundary that is a critical point for the volume-constrained anisotropic energy $\mathcal{F}$, is it possible to construct a sequence of smooth elliptic integrands F_h and smooth open sets Ω_h such that F_h converges to F as in (1.24) and Ω_h satisfies (1.25), (1.26) and $|\Omega_h \Delta \Omega| \to 0$? Whenever this is possible, of course, Theorem 1.4 implies that finite unions of Wulff shapes are the only critical points for the convex integrand F.

2. Basic facts on integrands

We recall without proof some standard facts about convex integrands and their gauge functions. By *convex integrand* we refer to a positive function F on $\mathbb{S}^n$ with a convex one-homogeneous extension to $\mathbb{R}^{n+1}$. The *gauge function of F* is defined by

$$F_*(x) = \sup\left\{ x \cdot \nu : F(\nu) < 1 \right\} \qquad x \in \mathbb{R}^{n+1} \tag{2.1}$$

and is itself a convex integrand. By convexity of F, one has $(F_*)_* = F$. Moreover, the *Fenchel inequality* holds:

$$x \cdot \nu \leq F_*(x) \, F(\nu) \qquad \forall x, \nu \in \mathbb{R}^{n+1} \,. \tag{2.2}$$

If we denote the minimum and the maximum values of F on $\mathbb{S}^n$ by

$$m_F = \min_{\mathbb{S}^n} F, \qquad M_F = \max_{\mathbb{S}^n} F, \tag{2.3}$$

then we have

$$m_F \leq |\nabla F(\nu)| \leq M_F \tag{2.4}$$

whenever F is differentiable at ν (that is to say, at almost every $\nu \in \mathbb{R}^{n+1}$), and moreover

$$m_{F_*} = \frac{1}{M_F} \qquad M_{F_*} = \frac{1}{m_F}. \tag{2.5}$$

By (1.5), (2.1) and since $(F_*)_* = F$,

$$K_F = \{F_* < 1\} \qquad K_{F_*} = \{F < 1\}. \tag{2.6}$$

Recall that the subdifferential of F at $\nu \in \mathbb{R}^{n+1}$ is defined by

$$\partial F(\nu) = \left\{ x \in \mathbb{R}^{n+1} : F(w) \geq F(\nu) + x \cdot (w - \nu) \quad \forall w \in \mathbb{R}^{n+1} \right\},$$

so $\partial F(\nu) = \{\nabla F(\nu)\}$ if F is differentiable at ν. We immediately see from (1.5) that

$$K_F = \partial F(0), \qquad K_{F_*} = \partial F_*(0)$$

and hence, by one-homogeneity of F,

$$\partial K_F = \{F_* = 1\} = \bigcup_{\nu \neq 0} \partial F(\nu), \qquad \partial K_{F_*} = \{F = 1\} = \bigcup_{x \neq 0} \partial F_*(x). \tag{2.7}$$

(Notice that we use the same symbol ∂ both for topological boundaries and for subdifferentials.)

If $F \in C^1(\mathbb{S}^n)$ and is strictly convex, then $F_* \in C^1(\mathbb{S}^n)$ and is strictly convex, and therefore $\partial F(\nu) = \{\nabla F(\nu)\}$ and $\partial F_*(x) = \{\nabla F_*(x)\}$ for every $\nu, x \neq 0$. Under these assumptions, using (2.7), one deduces the useful properties

$$\begin{aligned} F(\nabla F_*(x)) &= 1, \qquad F_*(x)\,\nabla F(\nabla F_*(x)) = x \qquad \forall x \neq 0, \\ F_*(\nabla F(\nu)) &= 1, \qquad F(\nu)\nabla F_*(\nabla F(\nu)) = \nu \qquad \forall \nu \neq 0. \end{aligned} \tag{2.8}$$

A short explanation of the vector identities in (2.8) is as follows. By $\{F_* < 1\}$, the normal to K_F at $\nabla F(\nu) \in \partial K_F$ (for $\nu \neq 0$) is parallel to $\nabla F_*(\nabla F(\nu))$. At the same time, by exploiting (1.5) and the one-homogeneity of F, we see that the normal to K_F at $\nabla F(\nu)$ is parallel to ν itself. Thus $\nabla F_*(\nabla F(\nu)) = \alpha\,\nu$ and then one finds $\alpha = 1/F(\nu)$ thanks to $F(\nabla F_*(x)) = 1$ for $x \neq 0$.

We conclude with some remarks about the functions $F^2/2$ and $F_*^2/2$. First, an important consequence of (2.8) is that

$$\nabla(F_*^2/2) = [\nabla(F^2/2)]^{-1} \qquad \text{on } \mathbb{R}^{n+1} \setminus \{0\}. \tag{2.9}$$

Indeed, for every $z \in \mathbb{R}^{n+1}$,

$$\begin{aligned} \nabla(F_*^2/2)(\nabla(F^2/2)(z)) &= F_*\big(F(z)\nabla F(z)\big)\,\nabla F_*\big(F(z)\nabla F(z)\big) \\ &= F(z)\,F_*(\nabla F(z))\,\nabla F_*(\nabla F(z)) = F(z)\,\frac{z}{F(z)} = z. \end{aligned}$$

Moreover, (2.9) holds for generic convex integrands in the sense that

$$z \in \partial(F^2/2)(x) \iff x \in \partial(F_*^2/2)(z). \tag{2.10}$$

When $F \in C^\infty(\mathbb{S}^n)$ is an elliptic integrand (so that (1.7) holds with constants λ and Λ), one has that $F^2/2 \in C^{1,1}(\mathbb{R}^{n+1}) \cap C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ with

$$\nabla(F^2/2)(\nu) = \begin{cases} F(\nu)\,\nabla F(\nu)\,, & \text{if } \nu \neq 0\,, \\ 0\,, & \text{if } \nu = 0\,, \end{cases} \tag{2.11}$$

$$\nabla^2(F^2/2)(\nu) = \nabla F(\nu) \otimes \nabla F(\nu) + F(\nu)\nabla^2 F(\nu) \qquad \text{if } \nu \neq 0\,,$$

and

$$\lambda_* \,\mathrm{Id} \leq \nabla^2(F^2/2)(\nu) \leq \Lambda_* \,\mathrm{Id} \qquad \text{on } \mathbb{R}^{n+1} \setminus \{0\} \tag{2.12}$$

for positive constants λ_* and Λ_*. By (2.11), we see that we can take

$$\lambda_* = m_F \min\{\lambda, m_F\} \qquad \Lambda_* = 2M_F \max\{M_F, \Lambda\}\,.$$

We have the identity

$$\nabla^2(F^2/2)(\nu) \circ \nabla^2(F_*^2/2)(\nabla F(\nu)) = \mathrm{Id} \qquad \forall \nu \in \mathbb{S}^n\,. \tag{2.13}$$

To conclude this section, we note that will sometimes use the shorthand

$$\nabla_F u(x) = \nabla(F^2/2)(\nabla u(x)) \tag{2.14}$$

3. Proof of the main theorem

This section contains the proof of Theorem 1.4. Section 3.1 and Section 3.2 serve to introduce, the anisotropic signed distance function and the anisotropic torsion potential on a smooth bounded open set respectively. In Section 3.3 we prove an anisotropic version of Reilly's identity, while in Section 3.4 we obtain a Lipschitz estimate on the anisotropic torsion potential that is interestingly independent of ellipticity. In Section 3.5 we exploit the torsion potential to give a proof of the anisotropic Heintze-Karcher inequality of [HLMG09] in the spirit of Ros' argument. The key result is identity (3.23), which relates the gap in the anisotropic Heintze-Karcher inequality to the properties of the torsion potential. Together with the Lipschitz estimate, this is a key fact to obtain compactness. This is finally done in Section 3.6, where we prove Theorem 1.4.

3.1. **The anisotropic signed distance function.** In the next proposition we collect some useful facts on the anisotropic signed distance function from a bounded open smooth set.

Proposition 3.1. *Let F be an elliptic integrand, let Ω be a bounded open smooth set, and define the F-anisotropic signed distance function of Ω by*

$$\gamma(x) = \inf\left\{F_*(x - y) : y \in \Omega\right\} - \inf\left\{F_*(y - x) : y \in \Omega^c\right\}, \qquad x \in \mathbb{R}^{n+1}\,. \tag{3.1}$$

Then there exists an open neighborhood N of $\partial\Omega$ such that γ is smooth in N and

$$\nabla\gamma(y) = \frac{\nu_\Omega(y)}{F(\nu_\Omega(y))} \qquad \forall y \in \partial\Omega\,. \tag{3.2}$$

Proof. Note that $\gamma < 0$ in Ω, $\gamma > 0$ in Ω^c. Since Ω has smooth boundary, it satisfies uniform exterior and interior balls conditions. Since F is a smooth elliptic integrand, the Wulff shape $K_F = \{F_* < 1\}$ is uniformly convex, and $K_F^- = \{x \in \mathbb{R}^{n+1} : F_*(-x) < 1\}$ is uniformly convex as well. Combining these facts, we see that there exists $r > 0$ with the following property: for every $y \in \partial\Omega$, there exist $x_y \in \Omega$ and $z_y \in \Omega^c$ such that

$$\left\{w \in \mathbb{R}^{n+1} : F_*(w - x_y) < r\right\} \subset \Omega\,, \qquad F_*(y - x_y) = r\,, \tag{3.3}$$

$$\left\{w \in \mathbb{R}^{n+1} : F_*(-(w - z_y)) < r\right\} \subset \Omega^c\,, \qquad F_*(-(y - z_y)) = r\,; \tag{3.4}$$
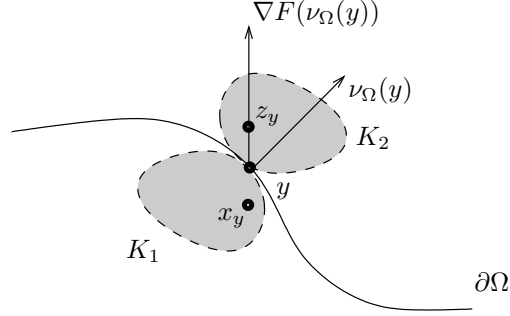
FIGURE 1. The uniformly convex sets $K_1 = \{F_*(\cdot - x_y) < r\}$ and $K_2 = \{F_*(z_y - \cdot) < r\}$ touch Ω at y, respectively, from inside and from outside. The point y lies on the segment joining centers x_y and z_y, which is parallel to $\nabla F(\nu_\Omega(y))$.

see Figure 1. We claim that $y - x_y$ and $z_y - y$ are parallel to $\nabla F(\nu_\Omega(y))$. Indeed, by (3.3), $\nabla F_*(y - x_y) = \alpha\,\nu_\Omega(y)$ with $\alpha > 0$; applying F to both sides and exploiting (2.8), we find $\alpha\,F(\nu_\Omega(y)) = 1$ and so

$$\nabla F_*(y - x_y) = \frac{\nu_\Omega(y)}{F(\nu_\Omega(y))}\,. \tag{3.5}$$

An analogous argument shows that $-\nabla F_*(z_y - y) = -\beta\,\nu_\Omega(y)$ for some $\beta > 0$, which satisfies $\beta = 1/F(\nu_\Omega(y))$ thanks again to (2.8). Thus

$$\nabla F_*(z_y - y) = \frac{\nu_\Omega(y)}{F(\nu_\Omega(y))}\,.$$

Applying ∇F to both sides of (3.5) and (3.1) and taking (2.8) into account, we find

$$\frac{y - x_y}{F_*(y - x_y)} = \nabla F(\nu_\Omega(y)) = \frac{z_y - y}{F_*(z_y - y)}\,, \tag{3.6}$$

as claimed. Now consider the projection map $p : \mathbb{R}^{n+1} \to \partial\Omega$ defined by

$$p(x) = \begin{cases} \{y \in \partial\Omega : F_*(y - x) = -\gamma(x)\} & \text{if } x \in \Omega\,, \\ \{y \in \partial\Omega : F_*(x - y) = \gamma(x)\} & \text{if } x \in \Omega^c\,. \end{cases} \tag{3.7}$$

Fix $y \in \partial\Omega$. By the first claim, if $x \in [x_y, y]$ or $x \in [y, z_y]$, then $p(x) = \{y\}$. So there exists a constant $c > 0$, depending only on m_F, M_F and the C^2 norm of F, such that

$$c \leq \min\left\{|x_y - y|, |z_y - y|\right\}\,.$$

Hence there exists r_0 such that if we set

$$N = \bigcup_{y \in \partial\Omega} \left\{y + t\,\nabla F(\nu_\Omega(y)) : |t| < r_0\right\}$$

then for every $x \in N$ there exists a unique $y \in \partial\Omega$ such that $p(x) = \{y\}$, with

$$x = y + \gamma(x)\nabla F(\nu_\Omega(y))\,.$$

Exploiting the implicit function theorem as in [GT83, Lemma 14.16], we find that γ is smooth on N. Now, if $x_0 \in \Omega$, then

$$\gamma(x) \geq -F_*(p(x_0) - x) \qquad \forall x \in \Omega\,, \qquad\qquad \gamma(x_0) = -F_*(p(x_0) - x_0)\,,$$

so that $\nabla\gamma(x_0) = \nabla F_*(p(x_0) - x_0)$ for every $x_0 \in \Omega$. By (3.5),

$$\nabla\gamma(x_0) = \frac{\nu_\Omega(p(x_0))}{F(\nu_\Omega(p(x_0)))}\,.$$

Letting $x_0 \to y \in \partial\Omega$, we obtain (3.2). $\qquad\square$

3.2. **The anisotropic torsion potential.** Let F be an elliptic functional and Ω a bounded open set with smooth boundary. The *F-anisotropic torsion potential of Ω* is the unique minimizer $u \in H_0^1(\Omega)$ of the strictly convex functional

$$\int_\Omega \frac{F^2(\nabla u)}{2} + u\,.$$

The Euler-Lagrange equation

$$\int_\Omega \nabla(F^2/2)(\nabla u) \cdot \nabla\varphi = -\int_\Omega \varphi \qquad \forall\varphi \in C_c^1(\Omega) \tag{3.8}$$

holds, that is to say, u is a distributional solution of

$$\begin{cases} \Delta_F u = 1\,, & \text{on } \Omega \\ \quad u = 0\,, & \text{on } \partial\Omega\,. \end{cases}$$

Here, we have introduced the *Finslerian Laplace operator*

$$\Delta_F u = \operatorname{div}\left(\nabla(F^2/2)(\nabla u)\right).$$

Notice that the Finslerian Laplace operator satisfies the classical *comparison principle*:

$$\begin{cases} v_1, v_2 \in H_0^1(\Omega) \\ -\Delta_F v_1 \geq -\Delta_F v_2 & \text{on } \Omega \\ \quad v_1 \geq v_2 & \text{on } \partial\Omega \end{cases} \Rightarrow \quad v_1 \geq v_2 \text{ on } \Omega. \tag{3.9}$$

Indeed, setting $w = \max\{v_2 - v_1, 0\}$ we find

$$\begin{aligned} 0 &\geq \int_\Omega w\left(-\Delta_F v_2 + \Delta_F v_1\right) = \int_\Omega \nabla w \cdot \left(\nabla(F^2/2)(\nabla v_2) - \nabla(F^2/2)(\nabla v_1)\right) \\ &\geq \lambda_* \int_{\{v_2 > v_1\}} |\nabla v_2 - \nabla v_1|^2 \end{aligned}$$

thanks to (2.12). Based on classical regularity arguments, one can prove that $u \in C^{1,\alpha}(\overline{\Omega}) \cap H^2(\Omega)$ for some $\alpha \in (0,1)$, see, e.g. [CS09, Proposition 2.3]. The critical set

$$\mathcal{C} = \left\{x \in \overline{\Omega} : \nabla u(x) = 0\right\}$$

is thus closed in $\overline{\Omega}$ and, since $F^2/2 \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$, u is smooth in a neighborhood of each $x \in \overline{\Omega} \setminus \mathcal{C}$. The following proposition collects some further properties of u.

Proposition 3.2. *The critical set $\mathcal{C}$ of the anisotropic torsion potential has Lebesgue measure zero. The torsion potential u is negative in Ω with positive outer normal derivative along $\partial\Omega$. In particular, u is smooth in a neighborhood of $\partial\Omega$.*

Proof. Step one: The function $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ defined by $f(\nu) = \nabla(F^2/2)(\nu)$ is Lipschitz with $f(0) = 0$. Setting $v = \nabla u \in H^1(\Omega; \mathbb{R}^{n+1})$, we have $f \circ v \in H^1(\Omega; \mathbb{R}^{n+1})$. In particular, (3.8) implies that

$$\mathrm{tr}\big(\nabla(f \circ v)\big) = 1 \qquad \text{a.e. on } \Omega. \tag{3.10}$$

Let us denote by L_x the affine n-dimensional space $L_x = v(x) + \nabla v(x)[\mathbb{R}^{n+1}]$. By the Sobolev chain rule for vector valued Lipschitz functions [ADM90] (see also [LM07, Theorem 1.1]), for almost every $x \in \Omega$ we have that $(f \circ v)|_{L_x}$ is differentiable at $v(x)$, with

$$\nabla(f \circ v)(x) = \nabla\big((f \circ v)|_{L_x}\big)(v(x)) \circ \nabla v(x).$$

Since $\nabla v = 0$ almost everywhere on $\mathcal{C}$, we conclude that $\nabla(f \circ v) = 0$ almost everywhere on $\mathcal{C}$. This fact, combined with (3.10), implies that $|\mathcal{C}| = 0$.

Step two: We show that

$$h(x) = \frac{F_*(x)^2}{2(n+1)} \qquad x \in \mathbb{R}^{n+1},$$

satisfies $\Delta_F h = 1$ on $\mathbb{R}^{n+1}$. Indeed $\nabla h(x) = F_*(x) \nabla F_*(x)/(n+1)$, so, by one-homogeneity of F and by (2.8), we have

$$F(\nabla h(x)) = \frac{F\big(F_*(x) \nabla F_*(x)\big)}{n+1} = \frac{F_*(x)}{n+1} F(\nabla F_*(x)) = \frac{F_*(x)}{n+1}.$$

At the same time, by zero-homogeneity of ∇F and again by (2.8),

$$\nabla F(\nabla h(x)) = \nabla F(\nabla F_*(x)) = \frac{x}{F_*(x)} \qquad \forall x \neq 0.$$

Thus $\nabla(F^2/2)(\nabla h(x)) = x/(n+1)$ for $x \in \mathbb{R}^{n+1} \setminus \{0\}$, and the desired result follows.

Step three: We use translations of h as barriers to prove Hopf's lemma and the negativity of u in Ω. As seen in Proposition 3.1, for every $y_0 \in \partial\Omega$ we can find $x_0 \in \Omega$ such that

$$\inf_{y \in \Omega^c} F_*(y - x_0) = F_*(y_0 - x_0), \qquad \nabla F_*(y_0 - x_0) = \frac{\nu_\Omega(y_0)}{F(\nu_\Omega(y_0))}.$$

Corresponding to this choice of x_0, we have that

$$\beta = \inf\Big\{h(y - x_0) : y \in \Omega^c\Big\} = h(y_0 - x_0), \qquad \nabla h(y_0 - x_0) \cdot \nu_\Omega(y_0) > 0.$$

If we set $v(x) = h(x - x_0) - \beta$ for $x \in \Omega$, then $v \geq 0$ on $\partial\Omega$, and $\Delta_F v = 1$ in Ω. By the comparison principle (3.9), we find $v \geq u$ in Ω, and since $v(y_0) = u(y_0) = 0$ with $\nabla v(y_0) \cdot \nu_\Omega(y_0) > 0$ we conclude that

$$-\nu_\Omega(y_0) \cdot \nabla u(y_0) = \lim_{t \to 0^+} \frac{u(y_0 - t\nu_\Omega(y_0)) - u(y_0)}{t} \leq \lim_{t \to 0^+} \frac{v(y_0 - t\nu_\Omega(y_0)) - v(y_0)}{t} < 0.$$

This proves that $\nu_\Omega \cdot \nabla u > 0$ on $\partial\Omega$. Since $\partial\Omega = \{u = 0\}$, we have that $|\nabla u| > 0$ on $\partial\Omega$. Therefore, because $u \in C^{1,\alpha}(\overline{\Omega})$, we find $\mathrm{dist}(\mathcal{C}, \partial\Omega) > 0$. Since u is smooth on $\overline{\Omega} \setminus \mathcal{C}$, we conclude that u is smooth in a neighborhood of $\partial\Omega$. Thus, u is strictly negative in a neighborhood of $\partial\Omega$, and therefore on the rest of Ω by the comparison principle. $\square$

3.3. **The anisotropic Reilly's identity.** The goal of this section is to prove an anisotropic variant of Reilly's identity. Let us recall that if Ω is an open set with smooth boundary and F is an elliptic integrand, then the F-anisotropic mean curvature of Ω (with respect to the outer unit normal ν_Ω) is defined as

$$H_\Omega^F = \operatorname{div}^{\partial\Omega}(\nabla F(\nu_\Omega)) \qquad \text{on } \partial\Omega \,.$$

The anisotropic mean curvature shares many basic properties of its isotropic counterpart. Of particular importance to us will be the anisotropic variant of the classical identity

$$\Delta u - \nabla^2 u[\nu_\Omega, \nu_\Omega] = |\nabla u| \, H_\Omega \qquad \text{on } \partial\Omega \,,$$

which holds when $\partial\Omega$ is the level set of a smooth function u with non-vanishing gradient on $\partial\Omega$. The anisotropic counterpart of this formula involves the Finslerian Laplace operator and takes the form

$$\Delta_F u - \nabla^2 u[\nabla F(\nabla u), \nabla F(\nabla u)] = F(\nabla u) \, H_\Omega^F \qquad \text{on } \partial\Omega \,. \tag{3.11}$$

This formula is derived, under different conventions, in [WX11, Theorem 3]. For the sake of clarity we recall the short proof. First, we claim that

$$H_\Omega^F = \operatorname{div}\left(\nabla F(\nabla u)\right) . \tag{3.12}$$

Indeed by $\nu_\Omega = \nabla u / |\nabla u|$ and the zero-homogeneity of ∇F, we have

$$H_\Omega^F = \operatorname{div}^{\partial\Omega}(\nabla F(\nu_\Omega)) = \operatorname{div}^{\partial\Omega}(\nabla F(\nabla u)) = \operatorname{div}\left(\nabla F(\nabla u)\right) ,$$

provided that

$$\nabla u \cdot \nabla\big(\nabla F(\nabla u)\big)[\nabla u] = 0 \,.$$

To prove this last identity, we denote by superscripts components and by subscripts partial derivatives, and compute

$$\big(\nabla(\nabla F(\nabla u))\big)^{ij} = (F_{\xi_j}(\nabla u))_{x_i} = \sum_k F_{\xi_j \xi_k}(\nabla u) u_{x_k x_i} \,,$$

$$\big(\nabla\big(\nabla F(\nabla u)\big)[\nabla u]\big)^j = \sum_{ik} F_{\xi_j \xi_k}(\nabla u) u_{x_k x_i} u_{x_i} \,,$$

$$\nabla u \cdot \nabla\big(\nabla F(\nabla u)\big)[\nabla u] = \sum_{ijk} F_{\xi_j \xi_k}(\nabla u) u_{x_k x_i} u_{x_i} u_{x_j}$$

$$= \sum_{ik} u_{x_k x_i} u_{x_i} \sum_j F_{\xi_j \xi_k}(\nabla u) u_{x_j} \,.$$

The last sum over j is equal to zero because $\nabla^2 F(\nu)[\nu] = 0$ for every $\nu \in \mathbb{S}^n$. Indeed, $F_{\xi_k}(t\nu) = F_{\xi_k}(\nu)$ for every $t > 0$ and $\nu \in \mathbb{S}^n$. This proves (3.12). Then,

$$\Delta_F u = \operatorname{div}\left(F(\nabla u) \nabla F(\nabla u)\right) = \sum_i \big(F(\nabla u) F_{\xi_i}(\nabla u)\big)_{x_i}$$

$$= \sum_{ij} F_{\xi_j}(\nabla u) F_{\xi_i}(\nabla u) u_{x_j x_i} + F(\nabla u) F_{\xi_i \xi_j}(\nabla u) u_{x_j x_i}$$

$$= \nabla^2 u[\nabla F(\nabla u), \nabla F(\nabla u)] + F(\nabla u) \operatorname{div}\left(\nabla F(\nabla u)\right) ,$$

and thus (3.11) holds thanks to (3.12). We now exploit (3.11) in the proof of the following anisotropic version of Reilly's identity.

Proposition 3.3 (Anisotropic Reilly's identity)**.** *If Ω is a bounded open set with smooth boundary, F is an elliptic integrand and u is the F-anisotropic torsion potential of Ω, then*

$$\int_\Omega (\Delta_F u)^2 - \operatorname{tr}\left((\nabla(\nabla_F u))^2\right) = \int_{\partial\Omega} H_\Omega^F \, F(\nabla u)^2 F(\nu_\Omega) \, d\mathcal{H}^n \,, \qquad (3.13)$$

In (3.13) we use notation introduced in (2.14). Notice there are no regularity issues in (3.13) as $u \in H^2(\Omega)$ and is smooth in a neighborhood of $\partial\Omega$ (Proposition 3.2). We first prove the following lemma:

Lemma 3.4. *If $V \in H^1(\Omega; \mathbb{R}^{n+1})$ is such that $\operatorname{div} V = c_0$, then $S = (\operatorname{div} V)V - \nabla V[V] \in L^1(\Omega; \mathbb{R}^{n+1})$ with $\operatorname{div} S \in L^1(\Omega)$ given by*

$$\operatorname{div} S = (\operatorname{div} V)^2 - \operatorname{tr}((\nabla V)^2) \,. \qquad (3.14)$$

Proof of Lemma 3.4. Since $\operatorname{div} V$ is constant we immediately find $\operatorname{div}((\operatorname{div} V)V) = (\operatorname{div} V)^2$. First, we assume that V is smooth. If $\varphi \in C_c^\infty(\Omega)$, then

$$\int_\Omega \nabla V[V] \cdot \nabla\varphi = \sum_{ij} \int_\Omega V_{x_j}^i V^j \varphi_{x_i} = \sum_{ij} \int_\Omega V_{x_j}^i \, (V^j \varphi)_{x_i} - \int_\Omega \varphi \operatorname{tr}((\nabla V)^2) \,,$$

where

$$\sum_{ij} \int_\Omega V_{x_j}^i \, (V^j \varphi)_{x_i} = -\sum_{ij} \int_\Omega V^i \, (V^j \varphi)_{x_i x_j} = \sum_{ij} \int_\Omega V_{x_i}^i \, (V^j \varphi)_{x_j}$$

$$= c_0 \sum_j \int_\Omega (V^j \varphi)_{x_j} = 0$$

as $V^j \varphi = 0$ on $\partial\Omega$. This proves that if $V \in H^1(\Omega) \cap C^\infty(\Omega)$, and $\operatorname{div} V$ is constant, then

$$\int_\Omega \nabla V[V] \cdot \nabla\varphi = -\int_\Omega \varphi \operatorname{tr}((\nabla V)^2) \,, \qquad \forall \varphi \in C_c^\infty(\Omega) \,. \qquad (3.15)$$

Now let $V \in H^1(\Omega)$ with $\operatorname{div} V = c_0$. Fix $\varphi \in C_c^\infty(\Omega)$ and consider an ε-regularization $V_\varepsilon = V \star \rho_\varepsilon$ of V, so that V_ε is smooth on $\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$. Since $\operatorname{div} V_\varepsilon = (\operatorname{div} V) \star \rho_\varepsilon$ is constant on Ω_ε, we can apply (3.15) to V_ε if ε is small enough to have $\operatorname{spt}\varphi \subset\subset \Omega_\varepsilon$. Since $V_\varepsilon \to V$ in H^1 on an open neighborhood of $\operatorname{spt}\varphi$, we can pass to the limit as $\varepsilon \to 0$ and deduce that (3.15) holds without the smoothness assumption on V. $\qquad\square$

Proof of Proposition 3.3. Since $u \in H^2(\Omega)$, we apply Lemma 3.4 with $V = \nabla_F u$ to find

$$\int_\Omega (\Delta_F u)^2 - \operatorname{tr}\left((\nabla(\nabla_F u))^2\right) = \int_\Omega \operatorname{div}\left(\Delta_F u \nabla_F u - \nabla(\nabla_F u)[\nabla_F u]\right) \,.$$

Since u is smooth in a neighborhood of $\partial\Omega$ with $\nu_\Omega = \nabla u / |\nabla u|$ we can apply the divergence theorem to get

$$\int_\Omega (\Delta_F u)^2 - \operatorname{tr}\left((\nabla(\nabla_F u))^2\right) = \int_{\partial\Omega} \left(\nabla(\nabla_F u)[\nabla_F u] - \Delta_F u \nabla_F u\right) \cdot \frac{\nabla u}{|\nabla u|} \,.$$

By $\nabla\frac{F^2}{2}(\nabla u) \cdot \nabla u = F(\nabla u)^2$ and (3.11), we have

$$-\int_{\partial\Omega} \Delta_F u \, \nabla_F u \cdot \frac{\nabla u}{|\nabla u|} = \int_{\partial\Omega} H_\Omega^F \, F(\nabla u)^2 F(\nu_\Omega) - \nabla^2 u[\nabla F(\nabla u), \nabla F(\nabla u)] \frac{F(\nabla u)^2}{|\nabla u|} \,.$$

Thus, in order to complete the proof, it suffices to show that

$$\nabla(\nabla_F u)[\nabla_F u] \cdot \nabla u = F(\nabla u)^2 \, \nabla^2 u[\nabla F(\nabla u), \nabla F(\nabla u)]. \tag{3.16}$$

We have that

$$[\nabla(\nabla_F u)]^{ij} = (F(\nabla u)F_{\xi_i}(\nabla u))_{x_j} = \sum_k F(\nabla u)F_{\xi_i \xi_k}(\nabla u) \, u_{x_k x_j} + F_{\xi_i}(\nabla u)F_{\xi_k}(\nabla u)u_{x_j x_k}.$$

Hence

$$
\begin{aligned}
\nabla(\nabla_F u)[\nabla_F u] \cdot \nabla u &= \sum_i \left(\nabla(\nabla_F u)[\nabla_F u]\right)^i u_{x_i} \\
&= F(\nabla u) \sum_{ij} [\nabla(\nabla_F u)]^{ij} F_{\xi_j}(\nabla u) u_{x_i} \\
&= F(\nabla u)^2 \sum_{kj} F_{\xi_j}(\nabla u) u_{x_k x_j} \sum_i F_{\xi_i \xi_k}(\nabla u) \, u_{x_i} \\
&\quad + F(\nabla u) \sum_{kj} F_{\xi_k}(\nabla u) F_{\xi_j}(\nabla u) \, u_{x_j x_k} \sum_i F_{\xi_i}(\nabla u) \, u_{x_i}.
\end{aligned}
$$

Again, $\nabla F(\nu)[\nu] = 0$ by homogeneity, so

$$\sum_i F_{\xi_i \xi_k}(\nabla u) \, u_{x_i} = \nabla^2 F(\nabla u)[\nabla u, e_k] = 0.$$

At the same time, $\sum_i F_{\xi_i}(\nabla u) \, u_{x_i} = \nabla F(\nabla u) \cdot \nabla u = F(\nabla u)$, so (3.16) is proven. $\qquad\square$

3.4. **Universal Lipschitz estimate.** We now turn to the proof of the following gradient bound for u, which is notably independent of the ellipticity constants of F.

Proposition 3.5. *If F is an elliptic integrand, Ω is a bounded open smooth set with $H_\Omega^F > 0$ on $\partial\Omega$ and u is the F-anisotropic torsion potential of Ω, then*

$$\sup_\Omega |\nabla u| \le \frac{1}{m_F \, H_{\Omega,\inf}^F} \tag{3.17}$$

where $H_{\Omega,\inf}^F$ denotes the infimum of H_Ω^F over $\partial\Omega$.

Proof. Recall from Proposition 3.1 that the F-anisotropic signed distance function γ of Ω, defined in (3.1), is smooth in a neighborhood of $\partial\Omega$ with $\gamma < 0$ in Ω and

$$\nabla\gamma(y) = \frac{\nu_\Omega(y)}{F(\nu_\Omega(y))} \qquad \forall y \in \partial\Omega.$$

Hence, $F(\nabla\gamma) = 1$ on $\partial\Omega$, and so

$$\Delta_F \gamma = \operatorname{div}\left(F(\nabla\gamma)\nabla F(\nabla\gamma)\right) = \operatorname{div}\left(\nabla F(\nu_\Omega)\right) = H_\Omega^F \qquad \text{on } \partial\Omega$$

by the zero-homogeneity of ∇F and (3.12).

Step one: We show that γ satisfies $\Delta_F \gamma \ge H_{\Omega,\inf}^F$ in the viscosity sense in Ω. Indeed, let P denote a second order polynomial touching γ from above at some $x \in \Omega$, i.e., for some $r > 0$, $P \ge \gamma$ on $B_r(x) \subset \Omega$ with $P(x) = \gamma(x)$. We want to prove that

$$\Delta_F P(x) \ge H_{\Omega,\inf}^F.$$

Let $y \in p(x)$, where p is the projection map as defined in (3.7), and note that $x_t = x + t\,(y - x) \in \Omega$ for all $t \in (0, 1)$. In particular, let $r_t \in (0, r)$ be such that $B_{r_t}(x_t) \subset \Omega$. We claim that the second order polynomial Q_t defined by

$$Q_t(z) = P(z - x_t + x) + F_*(x_t - x)\,, \qquad z \in \mathbb{R}^{n+1}$$

is such that

$$Q_t(x_t) = \gamma(x_t)\,, \qquad\qquad Q_t(z) \geq \gamma(z) \qquad \forall z \in B_{r_t}(x_t)\,. \tag{3.18}$$

Indeed,

$$Q_t(x_t) = P(x) + F_*(x_t - x) = \gamma(x) + t\,F_*(y - x) = (1 - t)\gamma(x) = \gamma(x_t)\,,$$

while if $z \in B_{r_t}(x_t)$ then $z - x_t + x \in B_r(x) \subset \Omega$, and thus

$$
\begin{aligned}
Q_t(z) &\geq \gamma(z - x_t + x) + F_*(x_t - x) = -\inf\left\{F_*(w - (z - x_t + x)) : w \in \Omega^c\right\} + F_*(x_t - x)\\
&\geq -\inf\left\{F_*(w - z) : w \in \Omega^c\right\} - F_*(x_t - x) + F_*(x_t - x) = \gamma(z)
\end{aligned}
$$

as $z \in \Omega$. Now, if t is close enough to 1, then x_t lies in the neighborhood N of $\partial\Omega$ in which γ is smooth. Hence, for t close enough to 1, (3.18) implies that

$$\Delta_F P(x) = \Delta_F Q_t(x_t) \geq \Delta_F \gamma(x_t)\,.$$

Letting $t \to 1^-$, we find $\Delta_F P(x) \geq \Delta_F \gamma(y) = H_F^\Omega(y) \geq H_{F,\inf}^\Omega$, as required.

Step two: We claim that

$$\gamma(x) \leq H_{\Omega,\inf}^F\, u(x) \qquad \forall x \in \Omega\,. \tag{3.19}$$

Both functions are equal to zero on $\partial\Omega$ and are strictly negative in Ω, so the claim follows by showing that, for every $\alpha > 0$,

$$v(x) = (H_{\Omega,\inf}^F - \alpha)u(x) - \gamma(x) \geq 0 \qquad \forall x \in \Omega\,.$$

In turn, since v is continuous in $\overline{\Omega}$ and $v = 0$ on $\partial\Omega$, it suffices to show that v has no interior minimum points in Ω.

Arguing by contradiction, let $x_0 \in \Omega$ be a local minimum point of v in Ω. If $\nabla u(x_0) \neq 0$, then u is smooth nearby x_0. Moreover, since x_0 is a local minimum of v, we have that

$$(H_{\Omega,\inf}^F - \alpha)u - [(H_{\Omega,\inf}^F - \alpha)u(x_0) - \gamma(x_0)]$$

touches γ from above at x_0, and thus, by Step one,

$$H_{\Omega,\inf}^F \leq \Delta_F[(H_{\Omega,\inf}^F - \alpha)u](x_0) = (H_{\Omega,\inf}^F - \alpha)\,\Delta_F u(x_0) = H_{\Omega,\inf}^F - \alpha\,,$$

a contradiction to $\alpha > 0$. This implies that $\nabla u(x_0) = 0$. Hence γ cannot be differentiable at x_0: otherwise $\nabla v(x_0) = 0$ would imply $\nabla\gamma(x_0) = 0$, whereas $\nabla\gamma = (\nu_\Omega \circ p)/F(\nu_\Omega \circ p) \neq 0$ at every differentiability point of γ.

We are thus left to consider the possibility of a local minimum point $x_0 \in \Omega$ of v, where $\nabla u(x_0) = 0$ and γ is not differentiable at x_0. By local minimality, for every $\nu \in \mathbb{S}^n$ and $t > 0$ small enough, we have

$$(H_{\Omega,\inf}^F - \alpha)\frac{u(x_0 + t\,\nu) - u(x_0)}{t} - \frac{\gamma(x_0 + t\,\nu) - \gamma(x_0)}{t} \geq 0\,.$$

So $\nabla u(x_0) = 0$ implies

$$\liminf_{t \to 0^+} \frac{\gamma(x_0 + t\,\nu) - \gamma(x_0)}{t} \leq 0\,.$$

Since γ is not differentiable at x_0, there exists $\nu_0 \in \mathbb{S}^n$ such that

$$\liminf_{t \to 0^+} \frac{\gamma(x_0 + t\,\nu_0) - \gamma(x_0)}{t} < 0\,,$$

which in turn implies

$$\liminf_{t \to 0^+} \frac{\gamma(x_0 + t\,\nu_0) + \gamma(x_0 - t\nu_0) - 2\gamma(x_0)}{t^2} = -\infty\,. \tag{3.20}$$

We will now obtain a contradiction to (3.20) by proving that for every $x \in \Omega$ there exists a second order polynomial P touching γ from below at x.

Indeed, since γ is smooth in a neighborhood N of $\partial\Omega$, there exists $C > 0$ such that for every $z \in N$ there exists a second order polynomial P_z with the properties

$$P_z(z) = \gamma(z)\,, \qquad \nabla^2 P_z \geq -C\,\mathrm{Id}\,, \qquad \gamma(w) \geq P_z(w) \text{ for every } w \text{ near } z\,.$$

Choose $x \in \Omega$ with $B_r(x) \subset \Omega$, let $y \in p(x)$, and set $x_t = x + t(y - x)$. For t sufficiently close to 1, $x_t \in N$ and we can find $r_t \in (0, r)$ such that $B_{r_t}(x_t) \subset \Omega$ and, setting $P_t = P_{x_t}$,

$$P_t(x_t) = \gamma(x_t)\,, \qquad \nabla^2 P_t \geq -C\,\mathrm{Id}\,, \qquad \gamma(w) \geq P_t(w) \text{ for every } w \in B_{r_t}(x_t)\,.$$

Set

$$P(w) = P_t(w - (x - x_t)) + \gamma(x) - \gamma(x_t)\,.$$

Clearly $P(x) = \gamma(x)$. If $w \in B_{r_t}(x) \subset \Omega$, then $w - (x - x_t) \in B_{r_t}(x_t)$ and thus

$$P(w) \leq \gamma(w - (x - x_t)) + \gamma(x) - \gamma(x_t)$$
$$= \gamma(w - (x - x_t)) - t\gamma(x) = \gamma(w - (x - x_t)) - t\,F_*(y - x)\,.$$

Now, since $w - (x - x_t) \in B_{r_t}(x_t) \subset \Omega$, by definition of γ there exists $\bar{w} \in \partial\Omega$ such that

$$\gamma(w - (x - x_t)) = -F_*(\bar{w} - (w - (x - x_t)))$$

so that, by the subadditivity of F_*,

$$P(w) \leq -F_*(\bar{w} - (w - (x - x_t))) - F_*(t(y - x))$$
$$= -F_*(\bar{w} - (w + t(y - x))) - F_*(t(y - x))$$
$$\leq -F_*(\bar{w} - w) \leq \gamma(w)\,,$$

again by definition of γ in Ω, and thanks to $w \in \Omega$. This completes the proof of step two.

Step three: We claim that if u is the F-anisotropic torsion potential of Ω, then

$$\sup_\Omega |\nabla u| \leq \frac{1}{m_F\, H^F_{\Omega,\inf}}\,. \tag{3.21}$$

First, let us pick $x \in \Omega$ and $y \in \partial\Omega$. By (3.19), $-u(x) \leq -\gamma(x)/H^F_{\Omega,\inf}$, thus

$$\frac{|u(x) - u(y)|}{|x - y|} = \frac{-u(x)}{|x - y|} \leq \frac{1}{H^F_{\Omega,\inf}} \frac{-\gamma(x)}{|x - y|}\,,$$

so, first using $|x - y|/m_F \geq F_*(y - x) = -\gamma(x)$ and then letting $x \to y$, we have proven

$$\sup_{\partial\Omega} |\nabla u| \leq \frac{1}{m_F\, H^F_{\Omega,\inf}}\,. \tag{3.22}$$

We now prove that $|\nabla u|$ (which is a Hölder continuous function on $\overline{\Omega}$) achieves its maximum on $\partial\Omega$. Recall that u is smooth and solves $1 = \Delta_F u$ pointwise on $\Omega \setminus \mathcal{C}$, with

$$\Delta_F u = \sum_{ij} \left(F(\nabla u)\, F_{\xi_i \xi_j}(\nabla u) + F_{\xi_i}(\nabla u) F_{\xi_j}(\nabla u) \right) u_{x_i x_j}\,.$$

Since the operator Δ_F is smooth on $\Omega \setminus \mathcal{C}$, by [GT83, Theorem 15.1], the maximum of $|\nabla u|$ is attained on $\partial(\Omega \setminus \mathcal{C})$. Then, since $|\nabla u| = 0$ on $\mathcal{C}$,

$$\sup_{\Omega} |\nabla u| \le \sup_{\partial\Omega} |\nabla u| \le \frac{1}{m_F\, H^F_{\Omega,\inf}}\,.$$

$\square$

3.5. **The anisotropic Heintze-Karcher and bounds on the torsion potential.** In this section we give a proof of the anisotropic Heintze-Karcher inequality (1.15) that uses the properties of the anisotropic torsion potential, in the spirit of Ros' argument [Ros87]; see Proposition 3.7 below. We recall from the introduction the definitions, for an open set with smooth boundary Ω,

$$\eta_F(\Omega) \;=\; 1 - \frac{(n+1)|\Omega|}{\int_{\partial\Omega} \left(n\, F(\nu_\Omega)/H^F_\Omega \right)}\,, \qquad \text{if } H^F_\Omega > 0\,,$$

$$\delta_F(\Omega) \;=\; \left(\frac{1}{\mathcal{F}(\Omega)} \int_{\partial\Omega} \left| \frac{H^F_\Omega}{H^{F,0}_\Omega} - 1 \right|^2 F(\nu_\Omega) \right)^{1/2}\,,$$

where $H^{F,0}_\Omega = n\, \mathcal{F}(\Omega)/(n+1)|\Omega|$. We have the following relation between η_F and δ_F:

Lemma 3.6. *If $\kappa > 0$ and Ω is a smooth bounded open set with $H^F_\Omega \ge \kappa H^{F,0}_\Omega$ on $\partial\Omega$, then*

$$\eta_F(\Omega) \le \frac{1}{\kappa\, \mathcal{F}(\Omega)} \int_{\partial\Omega} \left| \frac{H^F_\Omega}{H^{F,0}_\Omega} - 1 \right| F(\nu_\Omega)\,.$$

In particular,

$$\eta_F(\Omega) \le \delta_F(\Omega)/\kappa.$$

Proof. Since $(n+1)|\Omega| = \int_\Omega n\, F(\nu_\Omega)/H^{F,0}_\Omega$, we find

$$
\begin{aligned}
\eta(\Omega) \;&=\; \frac{1}{\int_{\partial\Omega}(n\,F(\nu_\Omega)/H^F_\Omega)} \left(\int_{\partial\Omega} \frac{n\,F(\nu_\Omega)}{H^F_\Omega} - \int_{\partial\Omega} \frac{n\,F(\nu_\Omega)}{H^{F,0}_\Omega} \right) \\
&=\; \frac{1}{\int_{\partial\Omega}(F(\nu_\Omega)/H^F_\Omega)} \int_{\partial\Omega} \left(1 - \frac{H^F_\Omega}{H^{F,0}_\Omega} \right) \frac{F(\nu_\Omega)}{H^F_\Omega} \\
&\le\; \frac{1}{\kappa\, H^{F,0}_\Omega \int_{\partial\Omega}(F(\nu_\Omega)/H^F_\Omega)} \int_{\partial\Omega} \left| 1 - \frac{H^F_\Omega}{H^{F,0}_\Omega} \right| F(\nu_\Omega) \\
&=\; \frac{(n+1)|\Omega|}{\int_{\partial\Omega}(n\,F(\nu_\Omega)/H^F_\Omega)} \frac{1}{\kappa\, \mathcal{F}(\Omega)} \int_{\partial\Omega} \left| 1 - \frac{H^F_\Omega}{H^{F,0}_\Omega} \right| F(\nu_\Omega)
\end{aligned}
$$

where the first factor is less than 1 exactly by the anisotropic Heintze-Karcher inequality. The proposition now follows from Hölder's inequality. $\square$

Proposition 3.7. *If $F \in C^\infty(\mathbb{S}^n)$ is an elliptic integrand, Ω is a bounded open smooth set with $H_\Omega^F > 0$, and u is the F-torsion potential of Ω, then*

$$\frac{|\Omega|}{n+1}\left(\int_{\partial\Omega} \frac{n\,F(\nu_\Omega)}{H_\Omega^F} - (n+1)|\Omega|\right) = \int_{\partial\Omega} \frac{F(\nu_\Omega)}{H_\Omega^F}\left(\int_\Omega \operatorname{tr}((\nabla(\nabla_F u))^2) - \frac{(\Delta_F u)^2}{n+1}\right) \qquad (3.23)$$

$$+ \int_{\partial\Omega} \frac{F(\nu_\Omega)}{H_\Omega^F}\int_{\partial\Omega} H_\Omega^F F(\nabla u)^2 F(\nu_\Omega) - \left(\int_{\partial\Omega} F(\nabla u)F(\nu_\Omega)\right)^2.$$

Moreover:

(a) *The right-hand side of (3.23) is non-negative, so (3.23) implies the anisotropic Heintze-Karcher inequality (1.15);*

(b) *If the right-hand side is equal to zero and Ω is connected, then $\Omega = x + r\,K_F$ for some $r > 0$;*

(c) *We have*

$$C(n)\frac{\Lambda_*}{\lambda_*}|\Omega|\,\eta_F(\Omega) \geq \int_\Omega \left|\nabla(\nabla_F u) - \frac{\operatorname{Id}}{n+1}\right|^2, \qquad (3.24)$$

$$\frac{C(n)}{\lambda_*^2}|\Omega|\,\eta_F(\Omega) \geq \int_\Omega \left|\nabla^2 u - \frac{\nabla^2(F_*^2/2)(\nabla F(\nabla u))}{n+1}\right|^2, \qquad (3.25)$$

and, if $\delta_F(\Omega) < 1/2$ and $H_\Omega^F \geq \kappa H_\Omega^{F,0}$ for some $\kappa > 0$ on $\partial\Omega$, then

$$\frac{C(n)}{\kappa}\frac{|\Omega|^{(n+2)/(n+1)}}{|K_F|^{1/(n+1)}}\left\{\eta_F(\Omega) + \frac{\delta_F(\Omega)^2}{\kappa}\right\} \geq \int_{\partial\Omega}\left|\frac{(n/H_\Omega^{F,0})}{n+1} - F(\nabla u)\right|^2 F(\nu_\Omega). \qquad (3.26)$$

Proof. Step one: We prove (3.23), following Ros' argument in [Ros87]. By the divergence theorem, since $\nu_\Omega = \nabla u/|\nabla u|$ on $\partial\Omega$, we have

$$|\Omega|^2 = \left(\int_\Omega \Delta_F u\right)^2 = \left(\int_{\partial\Omega} F(\nabla u)\nabla F(\nabla u)\cdot\nu_\Omega\right)^2 = \left(\int_{\partial\Omega} \frac{F(\nabla u)^2}{|\nabla u|}\right)^2 = \left(\int_{\partial\Omega} F(\nabla u)\,F(\nu_\Omega)\right)^2$$

$$\leq \int_{\partial\Omega} H_\Omega^F\,F(\nabla u)^2\,F(\nu_\Omega)\int_{\partial\Omega}\frac{F(\nu_\Omega)}{H_\Omega^F}.$$

By the anisotropic Reilly's identity,

$$\int_{\partial\Omega} H_\Omega^F\,F(\nabla u)^2 F(\nu_\Omega)\,d\mathcal{H}^n = \int_\Omega (\Delta_F u)^2 - \operatorname{tr}\left((\nabla(\nabla_F u))^2\right)$$

$$= \frac{n}{n+1}\int_\Omega (\Delta_F u)^2 + \int_\Omega \frac{(\Delta_F u)^2}{n+1} - \operatorname{tr}\left((\nabla(\nabla_F u))^2\right)$$

$$\leq \frac{n}{n+1}\int_\Omega (\Delta_F u)^2 = \frac{n|\Omega|}{n+1},$$

where the last inequality follows from the following linear algebra consideration. Set $A = \nabla^2(F^2/2)(\nabla u)$ and $B = \nabla^2 u$, so $\nabla(\nabla_F u) = AB$ and therefore

$$\operatorname{tr}((\nabla(\nabla_F u))^2) = \operatorname{tr}((AB)^2).$$

Since A is positive definite and symmetric on $\Omega \setminus \mathcal{C}$, $A^{1/2}$ and $A^{-1/2}$ exist and

$$A^{1/2}BA^{1/2} = A^{-1/2}(AB)A^{1/2}$$

is symmetric. Hence, there exist an orthogonal matrix O and a diagonal matrix L such that

$$A^{1/2}BA^{1/2} = O^{-1}LO. \tag{3.27}$$

So,

$$AB = A^{1/2}O^{-1}LOA^{-1/2}, \qquad (AB)^2 = A^{1/2}O^{-1}L^2OA^{-1/2} \tag{3.28}$$

and

$$B - A^{-1} = A^{-1/2}O^{-1}(L - \mathrm{Id})OA^{-1/2}. \tag{3.29}$$

Because the trace operator is invariant under conjugation,

$$\mathrm{tr}(AB) = \mathrm{tr}(L), \qquad \mathrm{tr}((AB)^2) = \mathrm{tr}(L^2).$$

By Hölder's inequality, $(n+1)\,\mathrm{tr}(L^2) \geq \mathrm{tr}(L)^2$ so that

$$\mathrm{tr}((\nabla(\nabla_F u))^2) - \frac{(\Delta_F u)^2}{n+1} = \mathrm{tr}(L^2) - \frac{(\mathrm{tr}L)^2}{n+1} \geq 0,$$

We have now shown the anisotropic Heintze-Karcher inequality and the identity (3.23) for the anisotropic torsion potential u.

Step two: We prove (3.24), (3.25), and (3.26). Let us first notice that if v and w belong to a Hilbert space with norm $|\cdot|$ and $v \cdot w \geq 0$ then

$$
\begin{aligned}
|v|^2|w|^2 - (v \cdot w)^2 &\geq |v|\,|w|\left(|v||w| - (v \cdot w)\right) = \frac{|v|\,|w|}{2}\left|\sqrt{\frac{|w|}{|v|}}v - \sqrt{\frac{|v|}{|w|}}w\right|^2 \\
&= \frac{|v|^2}{2}\left|\frac{|w|}{|v|}v - w\right|^2.
\end{aligned} \tag{3.30}
$$

Given $(\lambda_i)_{i=1}^N$ such that $\sum_i \lambda_i = 1$, we can apply (3.30) to $w = \sum_i \lambda_i e_i$ and $v = \sum_i e_i$ to obtain

$$
\begin{aligned}
N \sum_i \lambda_i^2 - \left(\sum_i \lambda_i\right)^2 &\geq \frac{N}{2}\left|\frac{(\sum_i \lambda_i^2)^{1/2}}{N}\sum_i e_i - \sum_i \lambda_i e_i\right|^2 \\
&\geq \frac{N}{2}\inf_{c \in \mathbb{R}}\sum_i (c - \lambda_i)^2 = \frac{N}{2}\sum_i\left(\frac{\sum_j \lambda_j}{N} - \lambda_i\right)^2 = \frac{N}{2}\sum_i\left(\frac{1}{N} - \lambda_i\right)^2.
\end{aligned}
$$

Thanks to (2.12),

$$|A^{1/2}| \leq \sqrt{(n+1)\,\Lambda_*} \qquad |A^{-1/2}| \leq \sqrt{\frac{n+1}{\lambda_*}}.$$

By (3.28), if we set $N = n + 1$ and denote by λ_i the eigenvalues of L, then

$$
\begin{aligned}
\left|\nabla(\nabla_F u) - \frac{\mathrm{Id}}{n+1}\right|^2 &\leq C(n)\frac{\Lambda_*}{\lambda_*}\left|L - \frac{\mathrm{Id}}{n+1}\right|^2 \\
&\leq C(n)\frac{\Lambda_*}{\lambda_*}\sum_i\left(\frac{1}{n+1} - \lambda_i\right)^2 \leq C(n)\frac{\Lambda_*}{\lambda_*}\left(\mathrm{tr}(L^2) - \frac{\mathrm{tr}(L)^2}{n+1}\right).
\end{aligned}
$$

Hence, by (3.23),

$$\frac{n\,|\Omega|}{n+1}\eta_F(\Omega) \geq \int_\Omega \mathrm{tr}((\nabla(\nabla_F u))^2) - \frac{(\Delta_F u)^2}{n+1} \geq \frac{1}{C(n)}\frac{\lambda_*}{\Lambda_*}\int_\Omega\left|\nabla(\nabla_F u) - \frac{\mathrm{Id}}{n+1}\right|^2.$$

This proves (3.24). In the same way, using (3.29) and recalling (2.13) to express A^{-1}, we have

$$\Big|\nabla^2 u - \frac{\nabla^2(F_*^2/2)(\nabla F(\nabla u))}{n+1}\Big|^2 \leq \frac{C(n)}{\lambda_*^2}\Big|L - \frac{\mathrm{Id}}{n+1}\Big|^2$$

$$\leq \frac{C(n)}{\lambda_*^2}\sum_i\Big(\frac{1}{n+1} - \lambda_i\Big)^2 \leq \frac{C(n)}{\lambda_*^2}\Big(\mathrm{tr}(L^2) - \frac{\mathrm{tr}(L)^2}{n+1}\Big).$$

So, again by (3.23),

$$\frac{n\,|\Omega|}{n+1}\,\eta_F(\Omega) \geq \int_\Omega \mathrm{tr}((\nabla(\nabla_F u))^2) - \frac{(\Delta_F u)^2}{n+1} \geq \frac{\lambda_*^2}{C(n)}\int_\Omega\Big|\nabla^2 u - \frac{\nabla^2(F_*^2/2)(\nabla F(\nabla u))}{n+1}\Big|^2,$$

proving (3.25).

Similarly, (3.23) implies

$$\frac{|\Omega|}{n+1}\,\eta_F(\Omega)\int_{\partial\Omega}\frac{n\,F(\nu_\Omega)}{H_\Omega^F} \geq |v|^2|w|^2 - (v\cdot w)^2$$

where $v = (H_\Omega^F)^{-1/2}$ and $w = (H_\Omega^F)^{1/2}F(\nabla u)$ are seen as vectors of $L^2(F(\nu_\Omega)\mathcal{H}^n\llcorner\partial\Omega)$. Writing down (3.30) we find

$$\frac{|\Omega|}{n+1}\,\eta_F(\Omega)\int_{\partial\Omega}\frac{n\,F(\nu_\Omega)}{H_\Omega^F} \geq \frac{1}{2}\int_{\partial\Omega}\frac{F(\nu_\Omega)}{H_\Omega^F}\int_{\partial\Omega}\Big|\frac{|w|}{|v|}\frac{1}{(H_\Omega^F)^{1/2}} - (H_\Omega^F)^{1/2}F(\nabla u)\Big|^2 F(\nu_\Omega).$$

After simplifying we have,

$$C(n)\,|\Omega|\,\eta_F(\Omega) \geq \int_{\partial\Omega}\Big|\frac{|w|}{|v|}\frac{1}{H_\Omega^F} - F(\nabla u)\Big|^2 H_\Omega^F\,F(\nu_\Omega)$$

and hence

$$C(n)\Big\{|\Omega|\,\eta_F(\Omega) + \frac{|w|^2}{|v|^2}\int_{\partial\Omega}\Big|\frac{1}{H_\Omega^{F,0}} - \frac{1}{H_\Omega^F}\Big|^2 H_\Omega^F\,F(\nu_\Omega)\Big\} \geq \int_{\partial\Omega}\Big|\frac{|w|}{|v|}\frac{1}{H_\Omega^{F,0}} - F(\nabla u)\Big|^2 H_\Omega^F\,F(\nu_\Omega).$$

Thanks to (3.13) we have $|w|^2 \leq |\Omega|$, so that by $1/H_\Omega^F \leq 1/\kappa\,H_\Omega^{F,0}$,

$$\frac{|w|^2}{|v|^2}\int_{\partial\Omega}\Big|\frac{1}{H_\Omega^{F,0}} - \frac{1}{H_\Omega^F}\Big|^2 H_\Omega^F\,F(\nu_\Omega) \;\leq\; \frac{|\Omega|}{\int_{\partial\Omega}F(\nu_\Omega)/H_\Omega^F}\int_{\partial\Omega}\Big|\frac{H_\Omega^F}{H_\Omega^{F,0}} - 1\Big|^2\frac{F(\nu_\Omega)}{H_\Omega^F}$$

$$\leq\; \frac{|\Omega|}{\kappa\,H_\Omega^{F,0}\int_{\partial\Omega}F(\nu_\Omega)/H_\Omega^F}\int_{\partial\Omega}\Big|\frac{H_\Omega^F}{H_\Omega^{F,0}} - 1\Big|^2 F(\nu_\Omega)$$

$$=\; \frac{|\Omega|}{\kappa}\frac{(n+1)\,|\Omega|}{\int_{\partial\Omega}n\,F(\nu_\Omega)/H_\Omega^F}\delta_F(\Omega)^2 \leq \frac{|\Omega|}{\kappa}\delta_F(\Omega)^2$$

where in the last step we have used the anisotropic Heintze-Karcher inequality. Thus,

$$C(n)\,|\Omega|\Big\{\eta_F(\Omega) + \frac{\delta_F(\Omega)^2}{\kappa}\Big\} \geq \int_{\partial\Omega}\Big|\frac{|w|}{|v|}\frac{1}{H_\Omega^{F,0}} - F(\nabla u)\Big|^2 H_\Omega^F\,F(\nu_\Omega). \qquad (3.31)$$

Exploiting again $H_\Omega^F \geq \kappa H_\Omega^{F,0}$

$$\int_{\partial\Omega}\Big|\frac{|w|}{|v|}\frac{1}{H_\Omega^{F,0}} - F(\nabla u)\Big|^2 H_\Omega^F\,F(\nu_\Omega) \geq \kappa H_\Omega^{F,0}\inf_{c\in\mathbb{R}}\int_{\partial\Omega}\Big|c - F(\nabla u)\Big|^2 F(\nu_\Omega). \qquad (3.32)$$

By the divergence theorem, the optimal c is given by the average

$$c = \frac{\int_{\partial\Omega} F(\nabla u) F(\nu_\Omega)}{\int_{\partial\Omega} F(\nu_\Omega)} = \frac{\int_\Omega \Delta_F u}{\mathcal{F}(\Omega)} = \frac{|\Omega|}{\mathcal{F}(\Omega)} = \frac{(n/H_\Omega^{F,0})}{n+1}.$$

Thus by combining (3.31), (3.32), and the last identity, we have

$$C(n) \frac{|\Omega|}{\kappa\, H_\Omega^{F,0}} \left\{ \eta_F(\Omega) + \frac{\delta_F(\Omega)^2}{\kappa} \right\} \geq \int_{\partial\Omega} \left| \frac{n/H_\Omega^{F,0}}{n+1} - F(\nabla u) \right|^2 F(\nu_\Omega), \tag{3.33}$$

where by the Wulff inequality (1.6),

$$\frac{|\Omega|}{H_\Omega^{F,0}} \leq C(n) \frac{|\Omega|^2}{\mathcal{F}(\Omega)} \leq C(n) \frac{|\Omega|^{(n+2)/(n+1)}}{|K_F|^{1/(n+1)}}.$$

We thus conclude the proof of (3.26).

Step three: We finally notice that if the right-hand side of (3.23) is equal to zero and Ω is connected, then $\Omega = x + r\, K_F$ for some $x \in \mathbb{R}^{n+1}$ and $r > 0$. Indeed, in this case, (3.24) gives

$$\nabla(\nabla_F u) = \frac{\mathrm{Id}}{n+1} \qquad \text{on } \Omega$$

so that, for some $x_0 \in \mathbb{R}^{n+1}$,

$$\nabla_F u(x) = \frac{x - x_0}{n+1} \qquad \forall x \in \Omega.$$

Since $\nabla_F u(x) = \nabla(F^2/2)(\nabla u)$, by (2.9)

$$\nabla u(x) = \nabla(F_*^2/2)\Big(\frac{x - x_0}{n+1}\Big) = \frac{\nabla(F_*)^2(x - x_0)}{2(n+1)} \qquad \forall x \in \Omega.$$

In particular, for some $c \in \mathbb{R}$ we have

$$u(x) = c + \frac{F_*(x - x_0)^2}{2(n+1)} \qquad \forall x \in \Omega,$$

so that $u = 0$ on $\partial\Omega$ implies that $\partial\Omega$ is a level set of $F_*(\cdot - x_0)$, as claimed. $\qquad\square$

3.6. **Proof of Theorem 1.4.** *Step one*: We first recall our setting. We consider a convex integrand F that is the pointwise limit of a sequence $\{F_h\}_{h\in\mathbb{N}}$ of smooth elliptic integrands with

$$m \leq F_h \leq M \qquad \text{on } \mathbb{S}^n,$$
$$\lambda_h\, \mathrm{Id} \leq \nabla^2 F_h(\nu) \leq \Lambda_h\, \mathrm{Id} \qquad \text{on } \nu^\perp, \, \forall \nu \in \mathbb{S}^n, \tag{3.34}$$

where m, M, λ_h, and Λ_h are positive constants (m and M independent of h). Notice that, by convexity,

$$F_h \to F \quad \text{uniformly on compact subsets of } \mathbb{R}^{n+1}. \tag{3.35}$$

We also consider a sequence $\{\Omega_h\}_{h\in\mathbb{N}}$ of bounded open sets with smooth boundary with

$$H_{\Omega_h}^{F_h,0} = n, \qquad H_{\Omega_h}^{F_h} \geq \kappa\, n \quad \text{on } \partial\Omega_h \tag{3.36}$$

and

$$\sup_{h\in\mathbb{N}} \mathrm{diam}(\Omega_h) < \infty, \qquad \mathcal{F}_h(\Omega_h) \leq (L + \sigma)\, \mathcal{F}_h(K_{F_h}) \tag{3.37}$$

for some $\kappa > 0$, $L \in \mathbb{N}$ and $\sigma \in (0,1)$. Moreover, we assume that

$$\lim_{h \to \infty} \max \left\{ \frac{1}{\lambda_h^2}, \frac{\Lambda_h}{\lambda_h} \right\} \eta_{F_h}(\Omega_h) + \delta_{F_h}(\Omega_h) = 0 \,. \tag{3.38}$$

We want to prove the existence of an open set Ω, which is the disjoint union of at most L-many translations of K_F, such that, up to translations and up to extracting subsequences,

$$\lim_{h \to \infty} \left| \mathcal{F}_h(\Omega_h) - \mathcal{F}(\Omega) \right| + |\Omega_h \Delta \Omega| = 0 \,. \tag{3.39}$$

colorblueWe now turn to the proof of (3.39).

Step two: Next colorForestGreen

Step two: We now turn to the proof of (3.39). Next we obtain some immediate compactness properties for the sets Ω_h. Since (3.34) implies $\mathcal{F}_h(E) \geq mP(E)$ for any set of finite perimeter E, (3.37) implies

$$\sup_{h \in \mathbb{N}} |\Omega_h| + P(\Omega_h) < \infty \,. \tag{3.40}$$

By (3.37), up to translations, we may assume $\Omega_h \subset B_R$ for some $R > 0$, and since $\sup_{h \in \mathbb{N}} P(\Omega_h) < \infty$, the compactness theorem for sets of finite perimeter implies that, up to subsequences, there exists a bounded set of finite perimeter Ω such that $|\Omega_h \Delta \Omega| \to 0$ as $h \to \infty$.

Step three: We show that, if $u_h : \mathbb{R}^{n+1} \to (-\infty, 0]$ denotes the F_h-torsion potential of Ω_h extended by zero outside of Ω_h, then, up to extracting further subsequences,

$$u_h \to u \text{ uniformly on } \mathbb{R}^{n+1}$$

where $|\{u < 0\} \setminus \Omega| = 0$ and

$$u(x) = \sum_{j \in J} \min \left\{ \frac{F_*(x - x_j)^2 - s_j^2}{2(n+1)}, 0 \right\} \qquad x \in \mathbb{R}^{n+1} \,, \tag{3.41}$$

where J is an at most countable set, $x_j \in \mathbb{R}^{n+1}$ and $s_j > 0$. Indeed, by (3.24), (3.26) and (3.38) we have that

$$\lim_{h \to \infty} \int_{\Omega_h} \left| \nabla(\nabla_{F_h} u_h) - \frac{\mathrm{Id}}{n+1} \right|^2 = 0 \,, \tag{3.42}$$

$$\lim_{h \to \infty} \int_{\partial \Omega_h} \left| \frac{1}{n+1} - F_h(\nabla u_h) \right|^2 F_h(\nu_{\Omega_h}) = 0 \,, \tag{3.43}$$

$$\lim_{h \to \infty} \int_{\Omega_h} \left| \nabla^2 u_h - \frac{\nabla^2((F_h)_*/2)^2 (\nabla F_h(\nabla u_h))}{n+1} \right|^2 = 0 \,. \tag{3.44}$$

By the universal Lipschitz estimate of Proposition 3.5 and by (3.36),

$$\mathrm{Lip}(u_h) \leq \frac{1}{m\, H^{F_h}_{\Omega_h, \inf}} \leq C(n, m, \kappa) \,. \tag{3.45}$$

Since $\{u_h > 0\} = \Omega_h \subset B_R$, it follows that $u_h \to u$ uniformly on $\mathbb{R}^{n+1}$ for a non-positive Lipschitz function u. We notice that

$$\left| \{u < 0\} \setminus \Omega \right| = 0 \,. \tag{3.46}$$

Indeed, by uniform convergence, for every $\varepsilon > 0$ we have $\{u < -\varepsilon\} \subset \Omega_h$ if h is large enough. This last fact, combined with (3.42), implies that $\{\nabla(\nabla_{F_h} u_h)\}_{h \in \mathbb{N}}$ is bounded in $L^2(\{u < 0\})$. In

addition, $|\nabla_{F_h} u_h| \le C$ on $\mathbb{R}^{n+1}$ thanks to (2.4), (3.34), and (3.45). Since $\{u < -\varepsilon\}$ is bounded and $\varepsilon > 0$ is arbitrary, we find that

$$\nabla_{F_h} u_h \to v \qquad \text{in } L^2(\{u < 0\}) \tag{3.47}$$

for some $v \in H^1(\{u < 0\}; \mathbb{R}^{n+1})$. Now, $f(M) = |M - \mathrm{Id}|^2$ is a convex function on $\mathbb{R}^{n+1} \otimes \mathbb{R}^{n+1}$, so that

$$\int_A f(\nabla w) \le \liminf_{h \to \infty} \int_A f(\nabla w_h)$$

whenever $A \subset \mathbb{R}^{n+1}$ is open and $w_h \to w$ in $L^1(A; \mathbb{R}^{n+1})$ for some $w \in W^{1,1}(A; \mathbb{R}^{n+1})$. Applying this to $w_h = \nabla_{F_h} u_h$, and thanks to (3.42), we find that

$$\int_{\{u < -\varepsilon\}} \left| \nabla v - \frac{\mathrm{Id}}{n+1} \right|^2 = 0 \qquad \forall \varepsilon > 0 \,,$$

and thus

$$\nabla v = \frac{\mathrm{Id}}{n+1} \qquad \text{on } \{u < 0\}.$$

Therefore, if $\{A_j\}_{j \in J}$ are the connected components of the open set $\{u < 0\}$ (here J is an at most countable set), then there exists $x_j \in \mathbb{R}^{n+1}$ such that

$$v(x) = \frac{x - x_j}{n+1} \qquad \forall x \in A_j \,. \tag{3.48}$$

We now need to translate (3.48) in terms of u. To this end, we claim that

$$v(x) \in \partial(F^2/2)(\nabla u(x)) \qquad \text{for a.e. } x \in \mathbb{R}^{n+1} \,. \tag{3.49}$$

Indeed, by the convexity of F_h^2 and since $2 \nabla_{F_h} u_h(x) = \nabla(F_h^2)(\nabla u_h(x))$, we have that

$$F_h(z)^2 \ge F_h(\nabla u_h(x))^2 + 2 \nabla_{F_h} u_h(x) \cdot \big(z - \nabla u_h(x)\big) \qquad \forall z \in \mathbb{R}^{n+1} \,. \tag{3.50}$$

By (3.35), (3.47) and the uniform Lipschitz bound on the potentials u_h, (3.50) implies that, for almost every $x \in \mathbb{R}^{n+1}$ and for every $z \in \mathbb{R}^{n+1}$,

$$F(z)^2 \ge F(\nabla u_h(x))^2 + 2 \, v(x) \cdot \big(z - \nabla u_h(x)\big) + o(1) \qquad \text{as } h \to \infty \tag{3.51}$$

where $o(1) \to 0$ as $h \to \infty$. Handling ∇u_h is more delicate, because we only have $\nabla u_h \rightharpoonup \nabla u$ in $L^2(\mathbb{R}^{n+1})$. However, by Mazur's lemma, for every $h \in \mathbb{N}$ there exist $N_h \ge h$ and $\{\alpha_l^h\}_{l=h}^{N_h} \subset [0, 1]$ with $\sum_{l=h}^{N_h} \alpha_l^h = 1$ such that

$$\sum_{l=h}^{N_h} \alpha_l^h \nabla u_l \to \nabla u \qquad \text{in } L^2(\mathbb{R}^{n+1}) \text{ and a.e. on } \mathbb{R}^{n+1} \,. \tag{3.52}$$

In particular, thanks to (3.52), this implies that by convexity of F^2 and by (3.51), for almost every $x \in \mathbb{R}^{n+1}$ and for every $z \in \mathbb{R}^{n+1}$ we have

$$F(z)^2 = \sum_{l=h}^{N_h} \alpha_l^h F(z)^2 \ge \sum_{l=h}^{N_h} \alpha_l^h F(\nabla u_h(x))^2 + 2 \sum_{l=h}^{N_h} \alpha_l^h \, v(x) \cdot (z - \nabla u_h(x)) + \mathrm{O}(1)$$

$$\ge F\Big(\sum_{l=h}^{N_h} \alpha_l^h \nabla u_h(x) \Big)^2 + 2 \, v(x) \cdot \Big(z - \sum_{l=h}^{N_h} \alpha_l^h \nabla u_h(x) \Big) + \mathrm{O}(1)$$

$$= F(\nabla u(x))^2 + 2 \, v(x) \cdot (z - \nabla u(x)) \,.$$

This proves (3.49). Inverting (3.49) using (2.10), we have that

$$\nabla u(x) \in \partial(F_*^2/2)(v(x)) \qquad \text{for a.e. } x \in \mathbb{R}^{n+1}.$$

By (3.48), and since F_* is differentiable almost everywhere on $\mathbb{R}^{n+1}$, this implies that

$$\nabla u(x) = \nabla(F_*^2/2)(x - x_j) \qquad \text{for a.e. } x \in A_j.$$

We thus conclude that there exist $c_j \in \mathbb{R}$ such that

$$u(x) = \frac{F_*(x - x_j)^2}{2(n+1)} + c_j \qquad \text{for a.e. } x \in A_j.$$

Since $u < 0$ in A_j and $u = 0$ on ∂A_j, we deduce that $c_j > 0$ and $A_j = x_j + s_j K_F$. Hence (3.41) follows.

Step four: The next few steps of the argument are devoted to showing that $s_j = 1$ for every $j \in J$. To begin, we prove that

$$\lim_{h \to \infty} \int_{\Omega_h} F_h(\nabla u_h)^2 = \int_{\{u < 0\}} F(\nabla u)^2. \tag{3.53}$$

First note that by the convergence of F_h to F (3.35) and the Lipschitz bound on u_h (3.45), we have

$$\int_{\Omega_h} F_h(\nabla u_h)^2 = \int_{\Omega_h} F(\nabla u_h)^2 + o(1) \qquad \text{as } h \to \infty \tag{3.54}$$

where $o(1)$ denotes a vanishing sequence. Since $\{u < -\varepsilon\} \subset \Omega_h$ for h large enough, by $\nabla u_h \rightharpoonup \nabla u_h$ in L^2 and the convexity of F, for every $\varepsilon > 0$ we have

$$\liminf_{h \to \infty} \int_{\Omega_h} F(\nabla u_h)^2 \geq \liminf_{h \to \infty} \int_{\{u < -\varepsilon\}} F(\nabla u_h)^2 \geq \int_{\{u < -\varepsilon\}} F(\nabla u)^2.$$

Letting $\varepsilon \to 0$ we see that, by (3.54),

$$\liminf_{h \to \infty} \int_{\Omega_h} F_h(\nabla u_h)^2 \geq \int_{\{u < 0\}} F(\nabla u)^2. \tag{3.55}$$

To prove the opposite inequality, we recall that u_h is the unique minimizer of $\int_\Omega F_h(\nabla v)^2$ among $v \in W_0^{1,2}(\Omega_h)$. Setting

$$\varepsilon_h = -\inf_{\mathbb{R}^{n+1} \setminus \Omega_h} u, \qquad v_h = (u + \varepsilon_h)_- \in W_0^{1,2}(\Omega_h),$$

we find that

$$\int_{\Omega_h} F(\nabla u_h)^2 \leq \int_{\Omega_h} F_h(\nabla v_h)^2 = \int_{\Omega_h} F(\nabla v_h)^2 + o(1) = \int_{\{u < -\varepsilon_h\}} F(\nabla u)^2 + o(1) \tag{3.56}$$

where in the first identity we have used the convergence of F_h to F (3.35). Combining (3.56) with (3.55), we prove (3.53).

Step five: We show that

$$\lim_{h \to \infty} \int_{\Omega_h} F_h(\nabla u_h)^\alpha = \int_{\{u < 0\}} F(\nabla u)^\alpha \qquad \forall \alpha > 0. \tag{3.57}$$

Notice that this would be obvious if we had the strong convergence of ∇u_h to ∇u, and that in that case, one could actually assert (3.57) with any non-negative locally bounded function H replacing

F^α. However, we only have that $\nabla u_h \overset{*}{\rightharpoonup} \nabla u$ in $L^\infty(\mathbb{R}^{n+1})$. To obtain (3.57), we will exploit the strict convexity of $t \in (-1, \infty) \mapsto F((1+t)\nu)^2$ through the theory of Young measures.

Let us recall that, by the uniform Lipschitz bound (3.45) and since spt $u_h \subset B_R$ for R independent of h, we can apply the fundamental theorem of Young measures [Eva90, Chapter 1, Theorem 11] to find a measurable family of probability measures $\{\nu_x\}_{x \in \mathbb{R}^{n+1}}$ such that

$$\int_{\mathbb{R}^{n+1}} \varphi(x) \psi(\nabla u_h(x)) \, dx \rightarrow \int_{\mathbb{R}^{n+1}} \varphi(x) \int_{\mathbb{R}^{n+1}} \psi(\xi) \, d\nu_x(\xi) \qquad \forall \varphi \in L^1(B_R), \psi \in C_c^0(\mathbb{R}^{n+1}).$$

In particular, we easily deduce that

$$\nabla u(x) = \int_{\mathbb{R}^{n+1}} \xi \, d\nu_x(\xi) \qquad \text{for a.e. } x \in \mathbb{R}^{n+1}. \tag{3.58}$$

By (3.53) and by convexity of F^2, if p denotes a measurable selection of $\partial F^2(\nabla u)$ on $\mathbb{R}^{n+1}$ (that is, $p : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ is a measurable map with $p(x) \in \partial F^2(\nabla u(x))$ for almost every $x \in \mathbb{R}^{n+1}$), then

$$
\begin{aligned}
\int_{\mathbb{R}^{n+1}} F(\nabla u)^2 &= \lim_{h \to \infty} \int_{\mathbb{R}^{n+1}} F(\nabla u_h)^2 \, dx = \int_{B_R} dx \int_{\mathbb{R}^{n+1}} F(\xi)^2 \, d\nu_x(\xi) \\
&\geq \int_{B_R} dx \int_{\mathbb{R}^{n+1}} F(\nabla u(x))^2 + p(x) \cdot (\xi - \nabla u(x)) \, d\nu_x(\xi) \\
&= \int_{\mathbb{R}^{n+1}} F(\nabla u)^2 + \int_{B_R} dx \int_{\mathbb{R}^{n+1}} p(x) \cdot (\xi - \nabla u(x)) \, d\nu_x(\xi) = \int_{\mathbb{R}^{n+1}} F(\nabla u)^2,
\end{aligned}
$$

where in the last identity we have used (3.58). Hence, for every measurable selection p of $\partial F^2(\nabla u)$, almost every $x \in \mathbb{R}^{n+1}$, and ν_x-almost every $\xi \in \mathbb{R}^{n+1}$, we have

$$F(\xi)^2 = F(\nabla u(x))^2 + p(x) \cdot (\nabla u(x) - \xi) \qquad \text{for a.e. } x \in \mathbb{R}^{n+1} \text{ and } .$$

As $t \mapsto F(\nu + t\,z)^2$ is strictly convex and increasing on $t > 0$ whenever z is an outer normal direction to $\{F < F(\nu)\}$ at ν, this in turn implies that for almost every $x \in \mathbb{R}^{n+1}$

$$\text{spt}(\nu_x) \subset \{F = F(\nabla u(x))\}.$$

Consequently, for any $g \in C^0(\mathbb{R}^{n+1})$ with $g(0) = 0$, we have

$$\lim_{h \to \infty} \int_{\mathbb{R}^{n+1}} g(\nabla u_h) \, dx = \int_{\mathbb{R}^{n+1}} dx \int_{\mathbb{R}^{n+1}} g(\xi) \, d\nu_x(\xi) = \int_{B_R} g(\nabla u).$$

Recalling that

$$\int_{B_R} F_h(\nabla u_h)^\alpha = \int_{B_R} F(\nabla u_h)^\alpha + o(1) \qquad \text{as } h \to \infty$$

and setting $g = F^\alpha$, the claim follows.

Step six: We prove that if $\alpha > 0$, then

$$\frac{n+1+\alpha}{(n+1)^2} \int_{\mathbb{R}^{n+1}} F(\nabla u)^\alpha = \frac{n+2+\alpha}{n+1} \int_{\mathbb{R}^{n+1}} F(\nabla u)^{\alpha+1}. \tag{3.59}$$

Indeed, let us consider the vector field

$$\theta_{\alpha,h} = F_h(\nabla u_h)^\alpha \nabla_{F_h} u_h.$$

Recalling that $\nabla_{F_h} u_h = F_h(\nabla u_h) \nabla F_h(\nabla u_h)$, we have

$$\nabla\big(F_h(\nabla u_h)^{\alpha+1}\big) \cdot \big(\nabla_{F_h} u_h\big) = (\alpha+1) F_h(\nabla u_h)^{\alpha+1} \nabla^2 u_h[\nabla F_h(\nabla u_h), \nabla F_h(\nabla u_h)]$$

so that $\operatorname{div}(\nabla_{F_h} u_h) = 1$ yields

$$
\begin{aligned}
\operatorname{div}\theta_{\alpha+1,h} &= F_h(\nabla u_h)^{\alpha+1} + (\alpha+1) F_h(\nabla u_h)^{\alpha+1} \nabla^2 u_h[\nabla F_h(\nabla u_h), \nabla F_h(\nabla u_h)] \\
&= \left(1 + \frac{\alpha+1}{n+1}\right) F_h(\nabla u_h)^{\alpha+1} \\
&\quad + (\alpha+1)\, F_h(\nabla u_h)^{\alpha+1} \left(\nabla^2 u_h - \frac{\nabla^2[(F_h)_*/2]^2(\nabla F_h(\nabla u_h))}{n+1}\right)[\nabla F_h(\nabla u_h), \nabla F_h(\nabla u_h)],
\end{aligned}
$$

where we have used the identity

$$\nabla^2[H^2/2](\nu)[\nu,\nu] = 1 \qquad \text{with } H = (F_h)_* \text{ and } \nu = \nabla F_h(\nabla u_h)\,.$$

By (3.44), and taking into account the Lipschitz bound (3.45), we find

$$\int_{\Omega_h} \operatorname{div}\theta_{\alpha+1,h} = \frac{n+2+\alpha}{n+1} \int_{\Omega_h} F_h(\nabla u_h)^{\alpha+1} + o(1) \qquad \text{as } h \to \infty\,. \tag{3.60}$$

At the same time, by the divergence theorem, the Lipschitz bound, and (3.43), we find

$$
\begin{aligned}
\int_{\Omega_h} \operatorname{div}\theta_{\alpha+1,h} &= \int_{\partial\Omega_h} F_h(\nabla u)^{\alpha+1} \nabla_{F_h} u \cdot \nu_{\Omega_h} \\
&= \frac{1}{n+1} \int_{\partial\Omega_h} F_h(\nabla u)^{\alpha} \nabla_{F_h} u \cdot \nu_{\Omega_h} \\
&\quad + \int_{\partial\Omega_h} \left(F_h(\nabla u) - \frac{1}{n+1}\right) F_h(\nabla u)^{\alpha} \nabla_{F_h} u \cdot \nu_{\Omega_h} \\
&= \frac{1}{n+1} \int_{\partial\Omega_h} F_h(\nabla u)^{\alpha} \nabla_{F_h} u \cdot \nu_{\Omega_h} + o(1) \\
&= \frac{1}{n+1} \int_{\Omega_h} \operatorname{div}\theta_{\alpha,h} + o(1)\,, \qquad \text{as } h \to \infty\,.
\end{aligned}
$$

By using (3.60) with α and $\alpha+1$ we thus conclude that

$$\frac{n+2+\alpha}{n+1} \int_{\Omega_h} F_h(\nabla u_h)^{\alpha+1} = \frac{n+1+\alpha}{(n+1)^2} \int_{\Omega_h} F_h(\nabla u_h)^{\alpha} + o(1) \qquad \text{as } h \to \infty \tag{3.61}$$

which implies (3.59) thanks to (3.57) if $\alpha > 0$. Notice that (3.61) holds also when $\alpha = 0$ and that, in this case, it implies

$$\frac{n+2}{n+1} \int_{\mathbb{R}^{n+1}} F(\nabla u) = \frac{|\Omega|}{n+1} \geq \frac{|\{u<0\}|}{n+1}\,, \tag{3.62}$$

where the limit on the left-hand side is computed by (3.57), the limit on the right-hand side follows by $|\Omega_h \Delta \Omega| \to 0$, and the inequality is a consequence of (3.46).

Step seven: We finally complete the proof. We recall from (3.41) that $\{u < 0\}$ is decomposed in at most countably many open connected components $A_j = x_j + s_j K_F$, $j \in J$, with $u(x) = (F_*(x - x_j)^2 - s_j^2)/2(n+1)$ for $x \in A_j$. Hence, by (2.8),

$$F(\nabla u(x)) = \frac{F(F_*(x - x_j)\nabla F_*(x - x_j))}{n+1} = \frac{F_*(x - x_j)}{n+1} \qquad \forall x \in A_j\,,$$

and by scaling, we have that

$$\int_{\mathbb{R}^{n+1}} F(\nabla u)^\alpha = \frac{1}{(n+1)^\alpha} \sum_{j \in J} s_j^{n+1+\alpha} \int_{K_F} F_*^\alpha \,. \tag{3.63}$$

Moreover, thanks to $K_F = \{F_* < 1\}$, the coarea formula, and the zero-homogeneity of ∇F_*,

$$\int_{K_F} F_*^\alpha = \int_0^1 t^\alpha \, dt \int_{\{F_* = t\}} \frac{d\mathcal{H}^n}{|\nabla F_*|} = \int_0^1 t^{n+\alpha} \, dt \int_{\partial K_F} \frac{d\mathcal{H}^n}{|\nabla F_*|} = \frac{C_K}{n+1+\alpha}$$

provided we set

$$C_K = \int_{\partial K_F} \frac{d\mathcal{H}^n}{|\nabla F_*|} \,.$$

Hence, (3.63) becomes

$$\int_{\mathbb{R}^{n+1}} F(\nabla u)^\alpha = \frac{C_K}{(n+1)^\alpha (n+1+\alpha)} \sum_{j \in J} s_j^{n+1+\alpha} \,. \tag{3.64}$$

Combining (3.59) with (3.64), we find that

$$\frac{C_K}{(n+1)^{\alpha+2}} \sum_{j \in J} s_j^{n+1+\alpha} = \frac{n+1+\alpha}{(n+1)^2} \int_{\mathbb{R}^{n+1}} F(\nabla u)^\alpha = \frac{n+2+\alpha}{n+1} \int_{\mathbb{R}^{n+1}} F(\nabla u)^{\alpha+1}$$

$$= \frac{C_K}{(n+1)^{\alpha+2}} \sum_{j \in J} s_j^{n+2+\alpha} \,,$$

that is

$$\sum_{j \in J} s_j^{n+1+\alpha} = \sum_{j \in J} s_j^{n+2+\alpha} \qquad \forall \alpha > 0 \,.$$

In particular, by the arbitrariness of α,

$$\sum_{j \in J} s_j^{n+3+\alpha} = \sum_{j \in J} s_j^{n+2+\alpha} = \sum_{j \in J} s_j^{n+1+\alpha} \,, \qquad \forall \alpha > 0 \,,$$

and thus

$$\sum_{j \in J} s_j^{n+1+\alpha} (s_j - 1)^2 = \sum_{j \in J} s_j^{n+3+\alpha} + s_j^{n+1+\alpha} - 2 \, s_j^{n+2+\alpha} = 0 \,.$$

This implies that $s_j = 1$ for every $j \in J$. Using (3.62), we get

$$\frac{C_K}{(n+1)^2} \sum_{j \in J} s_j^{n+2} = \frac{n+2}{n+1} \int_{\mathbb{R}^{n+1}} F(\nabla u) = \frac{|\Omega|}{n+1} \geq \frac{|\{u < 0\}|}{n+1} = \frac{C_K}{(n+1)^2} \sum_{j \in J} s_j^{n+1} \,, \tag{3.65}$$

which, combined with $s_j = 1$ for every $j \in J$, implies the finiteness of J as well as $|\Omega| = |\{u < 0\}|$. In particular, (3.46) implies

$$\Omega = \{u < 0\}$$

up to modifying Ω in a set of Lebesgue measure zero, which proves the conclusion that $|\Omega_h \Delta \Omega| \to 0$ for an open set Ω consisting of a union of finitely many disjoint translations of K_F. In turn, from this last property, we have $(n+1)|\Omega| = \mathcal{F}(\Omega)$, so $(n+1)|\Omega_h| = \mathcal{F}_h(\Omega_h)$ implies $\mathcal{F}_h(\Omega_h) \to \mathcal{F}(\Omega)$ as $h \to \infty$. Finally, $\mathcal{F}_h(\Omega_h) \leq (L+\sigma) \mathcal{F}_h(K_{F_h})$ gives

$$\# J \, \mathcal{F}(K_F) = \mathcal{F}(\Omega) \leq (L+\sigma)\mathcal{F}(K_F)$$

so $\# J \le L$. This completes the proof of Theorem 1.4.

4. PROOF OF THEOREM 1.3

We start by recalling our assumptions. We consider a smooth elliptic integrand F and a smooth potential g. We let $M > 0$ and consider an open connected set with smooth boundary Ω such that

$$\frac{\mathcal{F}(\Omega)^{n+1}}{|\Omega|^n} \le M, \qquad \frac{\operatorname{diam}(\Omega)}{|\Omega|^{1/(n+1)}} \le M, \qquad \Omega \subset B_M. \tag{4.1}$$

In case (i) we assume that Ω is a volume-constrained critical point of $\mathcal{E}$, so that, by smoothness and by the area formula, there exists a constant ℓ such that

$$H_\Omega^F + g = \ell \qquad \text{on } \partial\Omega. \tag{4.2}$$

Then a first variation argument allows one to compute that

$$\ell = H_\Omega^{F,0} + \frac{1}{(n+1)|\Omega|} \int_\Omega \operatorname{div}(g(x)\,x)\,dx, \tag{4.3}$$

see e.g. [FM11, Appendix A.1]. Let us now set

$$\Omega^* = t\,\Omega, \qquad t = \frac{H_\Omega^{F,0}}{n} = \frac{\mathcal{F}(\Omega)}{(n+1)|\Omega|}.$$

By (4.1) we easily find

$$H_{\Omega^*}^{F,0} = n, \qquad \mathcal{F}(\Omega^*) \le M, \qquad \operatorname{diam}(\Omega^*) \le M^{(n+2)/(n+1)}.$$

By the Wulff inequality (1.6),

$$\frac{1}{t} = \frac{(n+1)|\Omega|}{\mathcal{F}(\Omega)} \le C(n,F)\,|\Omega|^{1/(n+1)} \tag{4.4}$$

so that $\Omega \subset B_M$, (4.2) and (4.3) imply

$$\left\| \frac{H_\Omega^F}{H_\Omega^{F,0}} - 1 \right\|_{C^0(\partial\Omega)} \le C(n,F) \left(\|g\|_{C^0(B_M)} + M\,\|\nabla g\|_{C^0(B_M)} \right) |\Omega|^{1/(n+1)}, \tag{4.5}$$

and thus

$$\delta_F(\Omega^*) = \delta_F(\Omega) \le C(n,F,M,g)\,|\Omega|^{1/(n+1)}.$$

By Theorem 1.1, for every $\varepsilon > 0$ there exists v_ε depending on n, F, g, M, and ε, such that if $|\Omega| \le v_\varepsilon$, then

$$\left| \Omega^* \Delta \bigcup_{i=1}^L (x_i + K_F) \right| \le \varepsilon, \tag{4.6}$$

for some $L \ge 1$ (bounded from above in terms of F and M) and $\{x_i\}_{i=1}^L \subset \mathbb{R}^{n+1}$ such that the sets $\{x_i + K_F\}_{i=1}^L$ are mutually disjoint. This proves statement (i).

We now assume that Ω is a volume-constrained r_0-local minimizer of $\mathcal{E}$, that is

$$\mathcal{E}(\Omega) \le \mathcal{E}(A) \qquad \text{whenever } |A| = |\Omega| \text{ and } \Omega \Delta A \subset\subset I_{r_0\,|\Omega|^{1/(n+1)}}(\partial\Omega). \tag{4.7}$$

We shall apply statement (i) with a choice of ε depending on r_0, n, F, g and M, and then we shall assume $|\Omega| \le v_0$ for a suitable v_0 depending on r_0, n, F, g and M. We now divide the argument into steps.

Step one: We claim that there exist constants ρ_0 (depending on n, F, g, M and r_0) and S (depending on n, F, and M) such that Ω^* is a (S, ρ_0)-minimizer of $\mathcal{F}$ in $\mathbb{R}^{n+1}$, that is,

$$\mathcal{F}(\Omega^*) \leq \mathcal{F}(G) + S \, |G \Delta \Omega^*| \qquad \text{whenever } \operatorname{diam}(G \Delta \Omega^*) < \rho_0 \,. \tag{4.8}$$

(This will be used in the next step to get density estimates, see (4.16) below.) In proving (4.8) we can assume without loss of generality that

$$\mathcal{F}(G) \leq \mathcal{F}(\Omega^*) \,, \qquad G \Delta \Omega^* \subset\subset B_{\rho_0}(x_0) \qquad \text{for some } x_0 \in \partial \Omega^* \,. \tag{4.9}$$

We first show that, if we let

$$\mu = \left(\frac{|\Omega^*|}{|G|} \right)^{1/(n+1)}$$

then we have

$$|\mu - 1| \leq C(n) \frac{|\Omega^* \Delta G|}{|\Omega^*|} \qquad \text{and} \qquad |\mu - 1| \leq C(n, F) \, \rho_0^{n+1} \,. \tag{4.10}$$

Indeed, by (4.6) and provided $\varepsilon \leq |K_F|/2$, we have

$$\left(L - \frac{1}{2} \right) |K_F| \leq |\Omega^*| \leq (L+1) \, |K_F| \,. \tag{4.11}$$

So, if ρ_0 is small enough to have $|B_{\rho_0}| \leq |K_F|/4$, then (4.9) implies

$$|G| \geq |\Omega^*| - |B_{\rho_0}| \geq \left(L - \frac{3}{4} \right) |K_F| \geq \frac{|\Omega^*|}{8} \,, \qquad \text{and} \qquad |G| \leq |\Omega^*| + \frac{|K_F|}{4} \leq \frac{3}{2} \, |\Omega^*| \,,$$

hence

$$|\mu - 1| \leq C(n) \frac{|\Omega^* \Delta G|}{|G|} \leq C(n) \frac{|\Omega^* \Delta G|}{|\Omega^*|} \,,$$

that is, the first estimate in (4.10). The second estimate immediately follows by combining the first one with (4.11) and $|\Omega^* \Delta G| \leq |B_{\rho_0}|$. Now we set, for $x_0 \in \partial \Omega^*$ as in (4.9),

$$A^* = \mu \, (G - x_0) + x_0 \,, \qquad A = \frac{A^*}{t} \,,$$

and claim that A is admissible in (4.7). By definition of μ we have $|A^*| = |\Omega^*|$, so that $\Omega^* = t \, \Omega$ implies $|A| = |\Omega|$. We thus need to check that

$$\Omega \Delta A \subset\subset I_{r_0 \, |\Omega|^{1/(n+1)}}(\partial \Omega) \,.$$

We first claim that

$$\Omega^* \Delta A^* \subset\subset I_{2 \, \rho_0}(\partial \Omega^*) \,. \tag{4.12}$$

The argument is entirely elementary, but we include it for the sake of clarity. Let us first pick $y \in A^* \setminus \Omega^*$ and let $z \in G$ be such that $y = \mu(z - x_0) + x_0$. If $z \in B_{\rho_0}(x_0)$, then

$$\operatorname{dist}(y, \partial \Omega^*) \leq |y - x_0| \leq \mu \, |z - x_0| \leq \mu \, \rho_0 \leq 2 \rho_0 \,,$$

where we have used the second inequality in (4.10) and have assumed ρ_0 small enough to get $\mu \leq 2$. On the other hand, if $z \notin B_{\rho_0}(x_0)$, then $G \setminus B_{\rho_0}(x_0) = \Omega^* \setminus B_{\rho_0}(x_0)$ implies that $z \in \Omega^*$, so that a point on the segment joining y and z lies on $\partial \Omega^*$, and

$$\operatorname{dist}(y, \partial \Omega^*) \leq |z - y| = |\mu - 1| \, |z - x_0| \leq \operatorname{diam}(\Omega^*) \, |\mu - 1| \leq \rho_0 \,,$$

where we have used again the second inequality in (4.10) and the smallness of ρ_0. This proves that

$$\operatorname{dist}(A^* \setminus \Omega^*, \partial \Omega^*) \leq \rho_0 \,.$$

Now let us pick $y \in \Omega^* \setminus A^*$. If $y \notin G$, then, by (4.9), $y \in B_{\rho_0}(x_0)$ and thus $\text{dist}(y, \partial\Omega^*) \le \rho_0$. Otherwise, $y \in G \setminus A^*$, which means that the point z defined by $y = \mu(z - x_0) + x_0$ cannot lie in G. Thus the segment joining z and y meets a point in the boundary of G, so that, again by (4.10),

$$\text{dist}(y, \partial G) \le |y - z| = |\mu - 1| \, |z - x_0| = |\mu - 1| \, \frac{|y - x_0|}{\mu} \le 2 \, |\mu - 1| \, |y - x_0| \le 2 \, \text{diam}(\Omega^*) \, |\mu - 1| \le \rho_0 \,,$$

provided ρ_0 is small enough. By $\partial G \subset I_{\rho_0}(\partial\Omega^*)$ we get

$$\text{dist}(y, \partial\Omega^*) \le \rho_0 + \text{dist}(y, \partial G) \,,$$

and thus $\text{dist}(\Omega^* \setminus A^*, \partial\Omega^*) \le 2\rho_0$. The proof of (4.12) is complete.

By $|\Omega^*| = |A^*|$, (4.12), $\Omega^* = t\,\Omega$ and $A^* = t\,A$ we obtain that

$$|\Omega| = |A| \qquad \Omega \Delta A \subset\subset I_{2\rho_0/t}(\partial\Omega) \,. \tag{4.13}$$

By (4.4), if ρ_0 is small enough with respect to r_0, n, M and F, then (4.13) implies that A is admissible in (4.7). By $\mathcal{E}(\Omega) \le \mathcal{E}(A)$, and assuming without loss of generality that $r_0 \le M$, we deduce that

$$\mathcal{F}(\Omega) \le \mathcal{F}(A) + \|g\|_{C^0(B_{2M})} \, |\Omega \Delta A| \,. \tag{4.14}$$

Multiplying by t^n, this becomes

$$\begin{aligned} \mathcal{F}(\Omega^*) &\le& \mathcal{F}(A^*) + \frac{\|g\|_{C^0(B_{2M})}}{t} \, |\Omega^* \Delta A^*| \le \mathcal{F}(A^*) + C(n, F) \, \|g\|_{C^0(B_{2M})} \, v_0^{1/(n+1)} \, |\Omega^* \Delta A^*| \\ &\le& \mathcal{F}(A^*) + |\Omega^* \Delta A^*| \end{aligned}$$

where we have used first (4.4), and then the smallness of v_0. Now, $\mathcal{F}(A^*) = \mu^n \mathcal{F}(G)$ so that by the first estimate in (4.10) and by $\mathcal{F}(\Omega^*) = (n+1) |\Omega^*|$, we obtain

$$\begin{aligned} \mathcal{F}(A^*) &\le& \Big(1 + C(n, F, M) \frac{|\Omega^* \Delta G|}{|\Omega^*|}\Big) \mathcal{F}(G) \le \mathcal{F}(G) + C(n, F, M) \frac{\mathcal{F}(\Omega^*)}{|\Omega^*|} \, |\Omega^* \Delta G| \\ &\le& \mathcal{F}(G) + C(n, F, M) \, |\Omega^* \Delta G| \,. \end{aligned} \tag{4.15}$$

Similarly, by $\mathcal{F}(G) \le \mathcal{F}(\Omega^*) \le M$ and by [Mag12, Lemma 17.9],

$$|G \Delta A^*| \le C(n, M) \, |\mu - 1| \, P(G) \le C(n, F, M) \, |\mu - 1| \, \mathcal{F}(G) \le C(n, F, M) \, |\Omega^* \Delta G| \,,$$

so $|\Omega^* \Delta A^*| \le C(n, F, M) \, |\Omega^* \Delta G|$ by triangular inequality. Combining this last estimate with (4.4), (4.14) and (4.15) we conclude that (4.8) holds.

Step two: We show that $L = 1$ in (4.6). Starting from (4.8), and assuming without loss of generality that $\rho_0 \le 1/S$, a standard argument shows the existence of $\kappa \in (0, 1)$ depending on F and n only such that

$$\kappa \, |B_\rho(x)| \le |\Omega^* \cap B_\rho(x)| \le (1 - \kappa) \, |B_\rho(x)| \,, \qquad \forall x \in \partial\Omega^*, \rho < \rho_0 \,. \tag{4.16}$$

(See, e.g., [Mag12, Theorem 21.11] for the case when $\mathcal{F}$ is the classical perimeter and use the constants m_F and M_F bounding the anisotropy to adapt the proof to the anisotropic case.)

Let us introduce the shorthand $K_0 = \bigcup_{i=1}^{L} (x_i + K_F)$, and let $x \in \partial\Omega^*$ be such that

$$h_0 = \text{dist}(x, \partial K_0) = \sup_{z \in \partial\Omega^*} \text{dist}(z, \partial K_0) \,.$$

If $x \in K_0^c$, then by the lower density estimate in (4.16) and (4.6),

$$\kappa \, |B_{\min\{h_0, \rho_0\}}(x)| \le |\Omega^* \cap B_{\min\{h_0, \rho_0\}}(x)| \le |\Omega^* \setminus K_0| \le \varepsilon \,.$$

If instead $x \in K_0$, then the analogous argument using the upper density estimate in (4.16) shows that $\kappa \left| B_{\min\{h_0, \rho_0\}}(x) \right| \leq \varepsilon$ in this case as well. Taking ε small enough with respect to ρ_0 and κ, and therefore with respect to r_0, n, F, g and M, we find that $h_0 < \rho_0$ and in particular that

$$h_0^{n+1} \leq C(n, F)\, \varepsilon \,.$$

Furthermore, up to possibly decreasing v_0, we find that

$$h_1^{n+1} = \sup_{z \in \partial K_0} \operatorname{dist}(z, \partial \Omega) \leq C(n, F)\, \varepsilon$$

as well. Indeed, otherwise, we could find $r > 0$, a sequence $\{\Omega_h\}$ converging to K_0 in L^1 and $x \in K_0$ such that $B_r(x) \cap \partial \Omega_h$ is empty for h sufficiently large. Clearly, K_0 satisfies a lower perimeter density estimate. Pairing this estimate with the ellipticity of F and the lower semi-continuity of $\mathcal{F}$ implies that

$$cr^n \leq \mathcal{F}(K_0; B_r(x)) \leq \liminf_{h \to \infty} \mathcal{F}(\Omega_h; B_r(x)) = 0,$$

yielding a contradiction. We conclude that

$$h = \operatorname{hd}(\partial \Omega^*, \partial K_0) \leq C(n, F)\, \varepsilon^{1/(n+1)} \,. \tag{4.17}$$

Since Ω is connected, so is $\Omega^* \subset \bigcup_{i=1}^{L} I_h(x_i + K_F)$. Hence, if $L \geq 2$, then (up to relabeling the x_is) we have

$$I_h(x_1 + K_F) \cap I_h(x_2 + K_F) \neq \emptyset \,.$$

In particular, if $z_1 \in x_1 + \partial K_F$ and $z_2 \in x_2 + \partial K_F$ are such that $|z_1 - z_2| = \operatorname{dist}(x_1 + K_F, x_2 + K_F)$, then $|z_1 - z_2| < 2h \leq C(n, F)\, \varepsilon^{1/(n+1)}$. Moreover, by (4.17) there exists $x \in \partial \Omega^*$ such that $|z_1 - x| < C(n, F)\, \varepsilon^{1/(n+1)}$. Thus there exist w_1 and w_2 such that $w_1 + (x_1 + K_F)$ and $w_2 + (x_2 + K_F)$ are disjoint, with a common boundary point at x and

$$\max\{|w_1|, |w_2|\} \leq C(n, F)\, \varepsilon^{1/(n+1)} \,. \tag{4.18}$$

By the upper estimate in (4.16), we find that for every $\rho < \rho_0$

$$
\begin{aligned}
\kappa \left| B_{\rho/2}(x) \right| &\leq \left| B_{\rho/2}(x) \setminus \Omega^* \right| \leq \left| B_{\rho/2}(x) \setminus K_0 \right| + \varepsilon \\
&\leq \left| B_{\rho/2}(x) \setminus \left((x_1 + K_F) \cup (x_2 + K_F) \right) \right| + \varepsilon \\
&\leq \left| B_{\rho/2}(x) \setminus \left((w_1 + x_1 + K_F) \cup (w_2 + x_2 + K_F) \right) \right| + C(n, F)\, \varepsilon^{1/(n+1)} \,, \quad (4.19)
\end{aligned}
$$

where in the last step we have used (4.18). By definition of w_1 and w_2 and by smoothness of K_F,

$$\lim_{\rho_0 \to 0^+} \rho_0^{-1-n} \left| B_{\rho_0/2}(x) \setminus \left((w_1 + x_1 + K_F) \cup (w_2 + x_2 + K_F) \right) \right| = 0 \,.$$

Hence if ε is small enough with respect to ρ_0, (4.19) leads to a contradiction. This shows that $L = 1$. As a consequence, $K_0 = x_1 + K_F$ satisfies upper and lower volume density estimates, so arguing as above, we find that

$$\operatorname{hd}\left(\partial \Omega^*, x_1 + \partial K_F \right) \leq C(n, F) \left| \Omega^* \Delta (x_1 + K_F) \right|^{1/(n+1)} \leq C(n, F)\, \varepsilon^{1/(n+1)} \leq \frac{r_0}{2\, C_*(n, F)} \tag{4.20}$$

where the last inequality is obtained by further decreasing ε in terms of r_0, n and F, with $C_*(n, F)$ as in (4.4), that is to say, by taking $t^{-1} \leq C_*(n, F)\, |\Omega|^{1/(n+1)}$. In this way, we obtain

$$\operatorname{hd}\left(\partial \Omega, p_1 + \partial (K_F/t) \right) \leq \frac{r_0}{2\, C_*(n, F)\, t} \leq \frac{r_0}{2}\, |\Omega|^{1/(n+1)} \,, \tag{4.21}$$

where $p_1 = x_1/t$. This fact will be used in the next step.

Step three: We now prove (1.21) and (1.22) by comparing Ω^* with a scaling of the Wulff shape of the same volume. To this end, we introduce the scale invariant *Wulff deficit* of $\Omega \subset \mathbb{R}^{n+1}$, defined by

$$\delta_W(\Omega) = \frac{\mathcal{F}(\Omega)}{(n+1)|K_F|^{1/(n+1)}|\Omega|^{n/(n+1)}} - 1.$$

We first claim that

$$\delta_W(\Omega) \leq C(n,F)\,|\Omega^*\Delta(x_1 + K_F)| \leq C(n,F)\,\varepsilon. \tag{4.22}$$

Indeed, $\mathcal{F}(K_F) = (n+1)|K_F|$ and $\mathcal{F}(\Omega^*) = (n+1)|\Omega^*|$ (as $H^{F,0}_{\Omega^*} = n$), so that

$$\left|\mathcal{F}(\Omega^*) - \mathcal{F}(K_F)\right| \leq (n+1)\big||\Omega^*| - |K_F|\big| \leq (n+1)\,|\Omega^*\Delta(x_1 + K_F)|,$$

while

$$\begin{aligned}
\left|\mathcal{F}(K_F) - (n+1)|K_F|^{1/(n+1)}|\Omega^*|^{n/(n+1)}\right| &\leq& C(n,F)\,\big||K_F|^{n/(n+1)} - |\Omega^*|^{n/(n+1)}\big| \\
&\leq& C(n,F)\,\big||K_F| - |\Omega^*|\big|^{n/(n+1)} \\
&\leq& C(n,F)\,|\Omega^*\Delta(x_1 + K_F)|,
\end{aligned}$$

thanks also to (4.11). Again using (4.11) and the scaling invariance of $\delta_W(\Omega)$, we deduce (4.22).

This said, let $s = (|K_F|/|\Omega|)^{1/(n+1)}$. By (4.21),

$$\mathrm{hd}\Big(\partial\Omega, p_1 + \partial(K_F/s)\Big) \leq \frac{r_0}{2}\,|\Omega|^{1/(n+1)} + C(n,F)\left|\frac{1}{t} - \frac{1}{s}\right|$$

where by (4.4) and (4.22) and provided we take ε small enough in terms of r_0,

$$\begin{aligned}
\left|\frac{1}{t} - \frac{1}{s}\right| &=& \frac{\delta_W(\Omega)}{t} \leq C(n,F)\,|\Omega|^{1/(n+1)}\,\delta_W(\Omega) \\
&\leq& \frac{r_0}{2}\,|\Omega|^{1/(n+1)}.
\end{aligned} \tag{4.23}$$

We have thus shown that

$$\mathrm{hd}\Big(\partial\Omega, p_1 + \partial(K_F/s)\Big) \leq r_0\,|\Omega|^{1/(n+1)},$$

and since $|p_1 + (K_F/s)| = |\Omega|$, this implies that $A = p_1 + (K_F/s)$ is an admissible competitor in (4.7).

We can thus obtain (1.21) by the quantitative Wulff inequality of [FMP10] by direct comparison with the Wulff shape, as in the case of global minimizers [FM11]. We repeat the simple argument for the convenience of the reader. By (4.7), we get $\mathcal{E}(\Omega) \leq \mathcal{E}(K_F/s)$, which in turn implies

$$\begin{aligned}
\delta_W(\Omega) &=& \frac{\mathcal{F}(\Omega)}{\mathcal{F}(K_F/s)} - 1 \leq \frac{1}{\mathcal{F}(K_F/s)}\left|\int_\Omega g - \int_{p_1+(K_F/s)} g\right| \\
&\leq& \frac{C(n,F)}{|\Omega|^{n/(n+1)}}\,\|g\|_{C^0(B_{2M})}\,|\Omega\Delta(p_1 + (K_F/s))|.
\end{aligned} \tag{4.24}$$

Notice that $x_1 = t\,p_1$ could have actually been chosen so to satisfy

$$|\Omega\Delta(p_1 + (K_F/s))| = \min_{p\in\mathbb{R}^{n+1}} |\Omega\Delta(p + (K_F/s))|.$$

Correspondingly, by the quantitative Wulff inequality of [FMP10],

$$\delta_W(\Omega) \geq c(n) \left(\frac{|\Omega \Delta(p_1 + (K_F/s))|}{|\Omega|} \right)^2,$$

so that

$$\left(\frac{|\Omega \Delta(p_1 + (K_F/s))|}{|\Omega|} \right)^2 \leq C(n, F, g, M) \, \frac{|\Omega \Delta(p_1 + (K_F/s))|}{|\Omega|} \, |\Omega|^{1/(n+1)},$$

and (1.21) follows. Returning to (4.20) we have

$$\mathrm{hd}\Big(\partial\Omega, p_1 + \partial(K_F/t) \Big) \leq C(n, F) \left| \Omega \Delta(p_1 + K_F/t) \right|^{1/(n+1)}, \tag{4.25}$$

while by (4.23),(4.24), and (1.21),

$$
\begin{aligned}
\left| \frac{1}{t} - \frac{1}{s} \right| &\leq C(n, F) \, \delta_W(\Omega) \, |\Omega|^{1/(n+1)} \\
&\leq C(n, F, g, M) \, \frac{|\Omega \Delta(p_1 + (K_F/s))|}{|\Omega|} \, |\Omega|^{2/(n+1)} \leq C(n, F, g, M) \, |\Omega|^{3/(n+1)}
\end{aligned}
$$

By (4.4) and [Mag12, Lemma 17.9]

$$\left| (p_1 + K_F/s) \Delta(p_1 + K_F/t) \right| \leq C(F) \frac{1}{t^n} \left| \frac{1}{t} - \frac{1}{s} \right| \leq C(n, F, g, M) \, |\Omega|^{n/(n+1)} \, |\Omega|^{3/(n+1)},$$

while

$$\mathrm{hd}\Big(p_1 + \partial(K_F/t), p_1 + \partial(K_F/s) \Big) \leq C(F) \left| \frac{1}{t} - \frac{1}{s} \right| \leq C(n, F, g, M) \, |\Omega|^{3/(n+1)}.$$

By (4.25), we conclude

$$
\begin{aligned}
\frac{\mathrm{hd}\Big(\partial\Omega, p_1 + \partial(K_F/s) \Big)}{|\Omega|^{1/(n+1)}} &\leq \frac{\mathrm{hd}\Big(\partial\Omega, p_1 + \partial(K_F/t) \Big)}{|\Omega|^{1/(n+1)}} + \frac{\mathrm{hd}\Big(p_1 + \partial(K_F/t), p_1 + \partial(K_F/s) \Big)}{|\Omega|^{1/(n+1)}} \\
&\leq C(n, F) \left(\frac{|\Omega \Delta(p_1 + K_F/t)|}{|\Omega|} \right)^{1/(n+1)} + C(n, F, g, M) \frac{|\Omega|^{3/(n+1)}}{|\Omega|^{1/(n+1)}} \\
&\leq C(n, F, g, M) \, |\Omega|^{1/(n+1)^2},
\end{aligned}
$$

where in the last inequality we have used (1.21). This proves (1.22).

Conflict of Interest: The authors declare that they have no conflict of interest.

References

[ADM90] L. Ambrosio and G. Dal Maso. A general chain rule for distributional derivatives. *Proc. Amer. Math. Soc.*, 108(3):691–702, 1990.

[BC84] H. Brezis and J.-M. Coron. Multiple solutions of H-systems and Rellich's conjecture. *Comm. Pure Appl. Math.*, 37(2):149–187, 1984.

[BM94] J. E. Brothers and F. Morgan. The isoperimetric theorem for general integrands. *Mich. Math. J.*, 41(3):419–431, 1994.

[Bre13] S. Brendle. Constant mean curvature surfaces in warped product manifolds. *Publ. Math. Inst. Hautes Études Sci.*, 117:247–269, 2013.

[CL12] M. Cicalese and G. P. Leonardi. A selection principle for the sharp quantitative isoperimetric inequality. *Arch. Rat. Mech. Anal.*, 206(2):617–643, 2012.

[CM17] G. Ciraolo and F. Maggi. On the shape of compact hypersurfaces with almost-constant mean curvature. *Comm. Pure Appl. Math.*, 70(4):665–716, 2017.

[CS09] A. Cianchi and P. Salani. Overdetermined anisotropic elliptic problems. *Math. Ann.*, 345(4):859–881, 2009.

[DLM05] C. De Lellis and S. Müller. Optimal rigidity estimates for nearly umbilical surfaces. *J. Differential Geom.*, 69(1):75–110, 2005.

[DLM06] C. De Lellis and S. Müller. A C^0 estimate for nearly umbilical surfaces. *Calc. Var. Partial Differential Equations*, 26(3):283–296, 2006.

[DRG17] A. De Rosa and S. Gioffré. Quantitative stability for anisotropic nearly umbilical surfaces. 2017. Preprint available on arXiv.

[Eva90] L. C. Evans. *Weak convergence methods for nonlinear partial differential equations*, volume 74 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1990.

[FM91] I. Fonseca and S. Müller. A uniqueness proof for the Wulff theorem. *Proc. Roy. Soc. Edinburgh Sect. A*, 119(1-2):125–136, 1991.

[FM11] A. Figalli and F. Maggi. On the shape of liquid drops and crystals in the small mass regime. *Arch. Rat. Mech. Anal.*, 201:143–207, 2011.

[FMP08] N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative isoperimetric inequality. *Ann. Math.*, 168:941–980, 2008.

[FMP10] A Figalli, F. Maggi, and A. Pratelli. A mass transportation approach to quantitative isoperimetric inequalities. *Invent. Math.*, 182(1):167–211, 2010.

[Fon91] I. Fonseca. The Wulff theorem revisited. *Proc. Roy. Soc. London Ser. A*, 432(1884):125–145, 1991.

[GT83] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.

[Heb14] E. Hebey. *Compactness and stability for nonlinear elliptic equations.* Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2014.

[HLMG09] Y. He, H. Li, H. Ma, and J. Ge. Compact embedded hypersurfaces with constant higher order anisotropic mean curvatures. *Indiana Univ. Math. J.*, 58(2):853–868, 2009.

[KM17] B. Krummel and F. Maggi. Isoperimetry with upper mean curvature bounds and sharp stability estimates. *Calc. Var. Partial Differential Equations*, 56(2):56:53, 2017.

[LM07] G. Leoni and M. Morini. Necessary and sufficient conditions for the chain rule in $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^d)$ and $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^d)$. *J. Eur. Math. Soc. (JEMS)*, 9(2):219–252, 2007.

[Mag12] F. Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012.

[Mor94] F. Morgan. Soap bubbles in $\mathbb{R}^2$ and in surfaces. *Pacific J. Math.*, 165(2):347–361, 1994.

[MR91] S. Montiel and A. Ros. Compact hypersurfaces: the Alexandrov theorem for higher order mean curvatures. In *Differential geometry*, volume 52 of *Pitman Monogr. Surveys*

Pure Appl. Math., pages 279–296. Longman Sci. Tech., Harlow, 1991.

[MX13] H. Ma and C. Xiong. Hypersurfaces with constant anisotropic mean curvatures. *J. Math. Sci. Univ. Tokyo*, 20(3):335–347, 2013.

[Per11] D. Perez. On nearly umbilical surfaces. 2011. PhD. Thesis available at http://user.math.uzh.ch/delellis/uploads/media/Daniel.pdf.

[Rei77] R. C. Reilly. Applications of the Hessian operator in a Riemannian manifold. *Indiana Univ. Math. J.*, 26(3):459–472, 1977.

[Ros87] A. Ros. Compact hypersurfaces with constant higher order mean curvatures. *Rev. Mat. Iberoamericana*, 3(3-4):447–453, 1987.

[Sim83] L. Simon. *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis*. Australian National University, Centre for Mathematical Analysis, Canberra, 1983.

[Str84] M. Struwe. A global compactness result for elliptic boundary value problems involving limiting nonlinearities. *Math. Z.*, 187(4):511–517, 1984.

[Str00] M. Struwe. *Variational methods*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, third edition, 2000. Applications to nonlinear partial differential equations and Hamiltonian systems.

[Tay74] J. E. Taylor. Existence and structure of solutions to a class of nonelliptic variational problems. In *Symposia Mathematica, Vol. XIV (Convegno di Teoria Geometrica dell'Integrazione e Varietà Minimali, INDAM, Roma, Maggio 1973)*, pages 499–508. Academic Press, London, 1974.

[Tay75] J. E. Taylor. Unique structure of solutions to a class of nonelliptic variational problems. In *Differential geometry (Proc. Sympos. Pure. Math., Vol. XXVII, Stanford Univ., Stanford, Calif., 1973), Part 1*, pages 419–427. Amer. Math. Soc., Providence, R.I., 1975.

[Tay78] J. E. Taylor. Crystalline variational problems. *Bull. Am. Math. Soc.*, 84(4):568–588, 1978.

[WX11] G. Wang and C. Xia. A characterization of the Wulff shape by an overdetermined anisotropic PDE. *Arch. Ration. Mech. Anal.*, 199(1):99–115, 2011.

[XZ] C. Xia and X. Zhang. ABP estimate and geometric inequalities. *to appear in Comm. Anal. Geom.*

(Matias Delgadino)
Abdus Salam International Center for Theoretical Physics,
Strada Costiera 11, I-34151, Trieste, Italy.
E-mail address: mdelgadi@ictp.it

(Francesco Maggi)
Abdus Salam International Center for Theoretical Physics,
Strada Costiera 11, I-34151, Trieste, Italy.
On leave from the University of Texas at Austin
E-mail address: fmaggi@ictp.it

(Cornelia Mihaila)
Department of Mathematics, The University of Texas at Austin,
2515 Speedway Stop C1200, Austin, TX 78712, USA
E-mail address: cmihaila@math.utexas.edu

(Robin Neumayer)
Department of Mathematics, The University of Texas at Austin,
2515 Speedway Stop C1200, Austin, TX 78712, USA
E-mail address: rneumayer@math.utexas.edu