Using random polynomials in extremal graph theory

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AMS Fall Central Sectional
University of Michigan

October 20, 2018

Turán numbers

The *Turán number* of a graph F is the maximum number of edges that an n vertex graph may have under the condition that it does not contain F as a subgraph, denoted

$$\mathrm{ex}(n, F).$$

Theorem (Erdős-Stone 1946)

Let $\chi(F) \geq 2$ be the chromatic number of F. Then

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1}\right)\binom{n}{2} + o(n^2).$$

Theorem (Kővári-Sós-Turán 1954)

For integers $2 \leq s \leq t$,

$$\text{ex}(n, K_{s,t}) \leq \frac{1}{2}(t-1)^{1/s} n^{2-1/s} + \frac{1}{2}(s-1)n.$$

$\text{ex}(n, F) < n^{2-\epsilon}$ for bipartite F.

The even-cycle problem

Determine $\mathrm{ex}(n, C_{2k})$.

Order of magnitude only known when $k \in \{2, 3, 5\}$!

$\mathrm{ex}(n, C_4) \sim \frac{1}{2} n^{3/2}$	KST 1954, ERS and Brown 1966
$.5338 n^{4/3} \leq \mathrm{ex}(n, C_6) \leq .6272 n^{4/3}$	Füredi-Naor-Verstraëte 2006
$\mathrm{ex}(n, C_{10}) = \Theta(n^{6/5})$	Benson 1966, Wenger 1991

Connections to finite geometry, design theory, additive combinatorics, LDPC codes.

$\mathrm{ex}(n, C_{2k}) \leq O(n^{1+1/k})$	Erdős unpublished
$20kn^{1+1/k}$	Bondy-Simonovits 1974
$8(k-1)n^{1+1/k}$	Verstreäte 2000
$(k-1+o(1))n^{1+1/k}$	Pikhurko 2012
$(80+o(1))\sqrt{k}\log kn^{1+1/k}$	Bukh-Jiang 2017

Theorem (Lazebnik-Ustimenko-Woldar 1995)

$$\mathrm{ex}(n, C_{2k}) = \Omega\left(n^{1+\frac{2}{3k-3+\eta}}\right)$$

The even-cycle problem is hard! Let's make it easier. C_{2k} is 2 internally disjoint paths of length k between a pair of vertices.

Easier question

How many edges may be in a graph such that no pair of vertices contains t internally disjoint paths of length k between vertices? ie determine

$$\mathrm{ex}(n, \Theta_{k,t}).$$

$\Theta_{k,2} = C_{2k}$.

Theorem (Faudree-Simonovits 1983)

$$\mathrm{ex}(n, \Theta_{k,t}) \le c_k t^{k^2} n^{1+1/k}.$$

Theorem (Conlon 2014)

For any k there exists a C_k such that

$$\mathrm{ex}(n, \Theta_{k,C_k}) = \Omega(n^{1+1/k}).$$

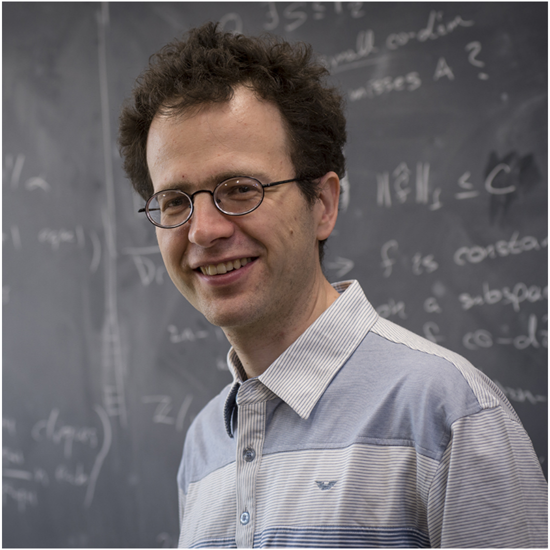
What is the dependence on t?

Theorem (Bukh-Tait)

$$\mathrm{ex}(n, \Theta_{k,t}) \le c_k t^{1-1/k} n^{1+1/k}.$$

For odd k

$$\mathrm{ex}(n, \Theta_{k,t}) \ge c_k' t^{1-1/k} n^{1+1/k}.$$



Upper bounds: a depth first search “looks like” a tree, each level grows a lot and so one sees all of the vertices after k steps.

Lower bound: use "random polynomial graph" to construct a bipartite graph with:

- $N = \frac{n}{t/C}$ vertices
- No pair of vertices has more than C paths of length at most k between them (not necessarily disjoint)
- $\epsilon N^{1+1/k}$ edges

Blow up each vertex by t/C vertices

- n vertices
- $\epsilon \left(\frac{n}{t/C}\right)^{1+1/k} \left(\frac{t}{C}\right)^2 = \Omega\left(t^{1-1/k} n^{1+1/k}\right)$ edges.
- How many k paths between vertices?

G random polynomial graph, G' blown up graph. X in G is a "supervertex" of x in G' if x was one of the t/C vertices that X blew up to.

- Let x and y have m internally disjoint paths of length k between them
- Each path $(x, u_1, \cdots, u_{k-1}, y)$ maps to a sequence of supervertices $(X, U_1, \cdots, U_{k-1}, Y)$.
- These sequences are not necessarily disjoint or distinct. However, since the paths are internally disjoint, each U_i can appear at most t/C times.
- There are at least $m/(t/C)$ distinct sequences of supervertices
- Since k is odd, X and Y are distinct.
- Each distinct sequence of supervertices corresponds to a walk from X to Y in G.
- There are $m/(t/C) \leq C$ distinct paths of length at most k from X to Y in G. $m \leq t$.

Let $\mathcal{P}_d^s$ be the set of polynomials over $\mathbb{F}_q$ of total degree at most d in s variables. Linear combinations of $X_1^{d_1} \cdots X_s^{d_s}$ with $\sum d_i \leq d$.

Definition

We use the term *random polynomial* to refer to a polynomial chosen uniformly from $\mathcal{P}_d^s$.

Choosing a random polynomial is equivalent to choosing the coefficient of each monomial $X_1^{d_1} \cdots X_s^{d_s}$ independently and uniformly from $\mathbb{F}_q$. Given a fixed $\vec{x} \in \mathbb{F}_q^s$,

$$\mathbb{P}(f(\vec{x}) = 0) = \frac{1}{q}.$$

If d is large enough,

$$\mathbb{P}(f(\vec{x_1}) = f(\vec{x_2}) = \cdots = f(\vec{x_m}) = 0) = \left(\frac{1}{q}\right)^m.$$

Definition: Random polynomial graph

Define G a bipartite graph with partite sets $U = V = \mathbb{F}_q^k$.Let $f_1, \cdots, f_{k-1}$ be random polynomials chosen independently from $\mathcal{P}_{2k^2}^{2k}$. $\vec{u} \in U$ is adjacent to $\vec{v} \in V$ if and only if

$$f_1(\vec{u},\vec{v}) = f_2(\vec{u},\vec{v}) = \cdots = f_{k-1}(\vec{u},\vec{v}) = 0.$$

$\mathbb{P}(\vec{u} \sim \vec{v}) = \left(\frac{1}{q}\right)^{k-1}$. $\mathbb{E}(\text{edges}) = q^{2k}\frac{1}{q^{k-1}} = q^{k+1} = \Omega\left(N^{1+1/k}\right)$.

G	$G_{N,p}$
q^{k+1} edges	q^{k+1} edges
$\mathbb{P}(\text{fixed set of } m \text{ edges}) = \left(\frac{1}{q^{k-1}}\right)^m$	$\mathbb{P}(m \text{ edges}) = \left(\frac{1}{q^{k-1}}\right)^m$
$\mathbb{P}(\text{fixed } k \text{ path from } x \text{ to } y) = \left(\frac{1}{q^{k-1}}\right)^k$	$\mathbb{P}(\text{fixed } k \text{ path}) = \left(\frac{1}{q^{k-1}}\right)^k$
$S = \#k$ paths from x to y	$T = \#k$ paths from x to y
$\mathbb{E}(S) = N^{k-1}\left(\frac{1}{q^{k-1}}\right)^k = 1$	$\mathbb{E}(T) = 1$
S is the number of points on a variety	$T \sim$ poisson

Lang-Weil: Either $S \leq C$ or $S \geq q$.

$$\mathbb{P}(S > C) = \mathbb{P}(S > q) \leq \frac{\mathbb{E}(S)}{q} = \frac{1}{q}.$$

Sunny

Let $\mathrm{ex}_r(n, \Theta_{k,t})$ be the maximum number of edges in an r uniform hypergraph where no pair of vertices has t internally disjoint (Berge) paths of length k between them. Several researchers have studied $\mathrm{ex}_r(n, \Theta_{k,2})$.

Theorem (He-Tait)

For each k, there is a constant C_k such that

$$\mathrm{ex}_r(n, \Theta_{k,t}) = O_{k,t,r}\left(n^{1+1/k}\right),$$

and

$$\mathrm{ex}_r(n, \Theta_{k,C_k}) = \Omega_{k,r}\left(n^{1+1/k}\right).$$

Open Problems

- For k even $\epsilon t^{1/k} n^{1+1/k} \leq \mathrm{ex}(n, \Theta_{k,t}) \leq c t^{1-1/k} n^{1+1/k}$.
- Lower the dependence on k in the constant C_k.
- Determine the dependence on t in the hypergraph question.