Few T copies in H-saturated graphs

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Abstract

A graph is F-saturated if it is F-free but the addition of any edge creates a copy of F. In this paper we study the quantity $\text{sat}(n, H, F)$ which denotes the minimum number of copies of H that an F-saturated graph on n vertices may contain. This parameter is a natural saturation analogue of Alon and Shikhelmen's generalized Turán problem, and letting $H = K_2$ recovers the well-studied saturation function. We provide a first investigation into this general function focusing on the cases where the host graph is either K_s or C_k-saturated. Some representative interesting behavior is:

(a) For any natural number m, there are graphs H and F such that $\text{sat}(n, H, F) = \Theta(n^m)$.

(b) For many pairs k and l, we show $\text{sat}(n, C_l, C_k) = 0$. In particular, we prove that there exists a triangle-free C_k-saturated graphs on n vertices for any $k > 4$ and large enough n.

(c) $\text{sat}(n, K_3, K_4) = n - 2$, $\text{sat}(n, C_4, K_4) \sim \frac{n^2}{2}$, and $\text{sat}(n, C_6, K_5) \sim n^3$.

We discuss several intriguing problems which remain unsolved.

1 Introduction

Given graphs G and F, the graph G is *F-free* if G does not contain F as a subgraph. We write $\text{ex}(n, F)$ for the *Turán number* of F which is the maximum number of edges in an n-vertex F-free graph. This function is a fundamental object in combinatorics, c.f.

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[24] for a survey. An important generalization of the Turán number was introduced by Alon and Shikhelman [2]. For a graph H, write

$$\mathrm{ex}(n, H, F)$$

for the maximum number of copies of H in an F-free n-vertex graph. Taking $H = K_2$ gives back the ordinary Turán number $\mathrm{ex}(n, F)$. The function $\mathrm{ex}(n, H, F)$ has been studied by numerous researchers ([1, 7, 15, 16, 17, 19, 23] to name a few).

Next we discuss graph saturation. The graph G is *F-saturated* if G is F-free, but adding any nonedge to G creates at least one copy of F. The *saturation number* of F, written $\mathrm{sat}(n, F)$, is the minimum number of edges in an n-vertex F-saturated graph. Saturation in graphs has been studied extensively since the 1960s, c.f. [12] for a survey. Generalizing the function $\mathrm{sat}(n, F)$, we write

$$\mathrm{sat}(n, H, F)$$

for the minimum number of copies of H in an n-vertex F-saturated graph. Note that

$$\mathrm{sat}(n, K_2, F) = \mathrm{sat}(n, F).$$

If F is a subgraph of H, then $\mathrm{sat}(n, H, F) = 0$. This is because an n-vertex F-free graph with $\mathrm{ex}(n, F)$ edges is F-saturated, and has no copies of H.

One of the first results in this area is a theorem of Erdős, Hajnal, and Moon [8] which determines the saturation number of any complete graph.

Theorem 1.1 (Erdős, Hajnal, Moon) *If $s \geq 3$ is an integer, then*

$$\mathrm{sat}(n, K_s) = (s-2)(n-s+2) + \binom{s-2}{2}.$$

Furthermore, if G is an n-vertex K_s-saturated graph with $\mathrm{sat}(n, K_s)$ edges, then G is isomorphic to the join of a clique with $s-2$ vertices and an independent set with $n-s+2$ vertices.

While Theorem 1.1 solves the saturation problem for complete graphs, many other cases have since been studied, including cycles.

Our main results fall into two categories: counting graphs in K_s-saturated graphs or counting graphs in C_k-saturated graphs. We first discuss counting graphs in K_s-saturated graphs as this line of research is a natural generalization of Theorem 1.1 in the spirit of Alon and Shikhelman [2].

1.1 Clique saturated graphs

Theorem 1.2 *Let $s > r \geq 3$ be integers. There is a constant $n_{s,r}$ such that for all $n \geq n_{s,r}$,*

$$\begin{aligned} \max\left\{ \frac{\binom{s-2}{r-1}}{r-1}n - 2\binom{s-2}{r-1}, \left(\frac{\binom{s-2}{r-1} + \binom{s-3}{r-2}}{r} \right) n \right\} &\leq \mathrm{sat}(n, K_r, K_s) \\ &\leq (n-s+2)\binom{s-2}{r-1} + \binom{s-2}{r}. \end{aligned}$$

The join of a clique with $s-2$ vertices and an independent set with $n-s+2$ vertices gives the upper bound of Theorem 1.2. This is the same graph that is the unique extremal example for Theorem 1.1. For $r \geq \sqrt{s-1}+1$, the second entry in the maximum gives the better lower bound. When r is fixed and s tends to infinity, the lower bound is roughly $\frac{1}{r-1}\binom{s-2}{r-1}n$ which means there is a gap of a factor of $r-1$ between the lower and upper bounds.

Theorem 1.2 shows that $\mathrm{sat}(n, K_r, K_s) = \Theta(n)$ for $n \geq s > r \geq 3$, but it does not give an asymptotic formula. When $s = 4$ and $r = 3$, Theorem 1.2 implies

$$\frac{2n}{3} \leq \mathrm{sat}(n, K_3, K_4) \leq n-2.$$

In this special case we can determine $\mathrm{sat}(n, K_3, K_4)$ exactly.

Theorem 1.3 *For $n \geq 7$,*

$$\mathrm{sat}(n, K_3, K_4) = n-2.$$

Furthermore, the only n-vertex K_4-saturated graph with $n-2$ triangles is the join of an edge and an independent set with $n-2$ vertices.

Kászonyi and Tuza [18] proved that for any graph F, there is a constant C, depending only on F, for which

$$\mathrm{sat}(n, F) < Cn. \tag{1}$$

We can use the same construction that gives (1) to prove that $\mathrm{sat}(n, K_r, F)$ is also at most Cn.

Proposition 1.4 *Let $n \geq 1$ and $r \geq 2$ be integers. For any graph F, there is a constant $C = C(r, F)$ such that*

$$\mathrm{sat}(n, K_r, F) < Cn.$$

If one replaces K_r with an arbitrary graph H in Proposition 1.4, then it is not necessarily the case that $\mathrm{sat}(n, H, F) = O(n)$, as the following result shows.

Let H be a noncomplete graph with at most s vertices. Fix some nonedge h_1h_2 in H, and let u_1 and u_2 be a fixed pair of vertices in K_s. Let $f_{h_1,h_2}(H)$ be the number of copies of H in K_s where the vertices h_1, h_2 of H correspond to the vertices u_1, u_2, respectively, in the K_s.

Proposition 1.5 *Let $s \geq 3$ be an integer. If H contains at most s vertices and h_1h_2 is any nonedge of H, then*

$$\mathrm{sat}(n, H, K_s) \geq f_{h_1,h_2}(H)\left(\frac{n^2}{2(s-1)} - \frac{n}{2}\right).$$

Applying Proposition 1.5 with $H = C_4$, we can prove that $\mathrm{sat}(n, C_4, K_4) \sim \frac{n^2}{2}$. More precisely, we have:

Proposition 1.6 *Let $\delta > 0$ be a real number. There is an $n(\delta)$ such that for all $n \geq n(\delta)$,*

$$\binom{n}{2} - n^{5/3+\delta} \leq \mathrm{sat}(n, C_4, K_4) \leq \binom{n-2}{2}.$$

The graph obtained by taking a vertex of degree $n-1$ and putting $\lfloor \frac{n-1}{2} \rfloor$ disjoint edges in its neighborhood is $(K_4 - e)$-saturated and has no C_4, so

$$\mathrm{sat}(n, C_4, K_4 - e) = 0.$$

Thus, even though $K_4 - e$ differs from K_4 by only one edge, the functions $\mathrm{sat}(n, C_4, K_4 - e)$ and $\mathrm{sat}(n, C_4, K_4)$ have very different behavior.

Proposition 1.6 and Theorem 1.2 gives examples of graphs H and F where

$$\mathrm{sat}(n, H, F) = \Theta(n^m)$$

for $m = 1, 2$. In fact, for any integer $m \geq 3$, there are graphs H and F for which $\mathrm{sat}(n, H, F) = \Theta(n^m)$, as the following result shows.

Theorem 1.7 *For $s \geq 5$ and $r \leq 2s - 4$,*

$$\mathrm{sat}(n, C_r, K_s) = \Theta(n^{\lfloor \frac{r}{2} \rfloor}).$$

More precisely,

$$\begin{cases} \binom{\frac{(s-2)_k}{4 \cdot k}}{} \left(n^k - o(n^k)\right) \leq \mathrm{sat}(n, C_r, K_s) \leq \left(\frac{(s-2)_k}{2k}\right) \left(n^k + o(n^k)\right) & \text{if } 2 | r \\ \left(\frac{(s-2)_{k+1}(k-2)!}{r(r-3)(r)_k(s-1)}\right) \left(n^k - o(n^k)\right) \leq \mathrm{sat}(n, C_r, K_s) \leq \left(\frac{(s-2)_{k+1}}{2}\right) \left(n^k + o(n^k)\right) & \text{if } 2 \not| r \end{cases}$$

where $k = \lfloor \frac{r}{2} \rfloor$, and $(m)_k = m(m-1) \cdots (m-k+1)$.

In the special case of $\mathrm{sat}(n, C_6, K_5)$, Theorem 1.7 shows $\mathrm{sat}(n, C_6, K_5) = \Theta(n^3)$. With a more specialized argument, we can determine this function asymptotically.

Theorem 1.8 *We have*

$$(1 - o(1))n^3 \leq \mathrm{sat}(n, C_6, K_5) \leq 6\binom{n-3}{3}.$$

1.2 Cycle saturated graphs

Thus far, many of the results we have stated on $\mathrm{sat}(n, H, F)$ concern the cases when F is a complete graph, and when H is a cycle or a complete graph. The case when H and F are both cycles also is interesting. Some cases are fairly straightforward. A complete bipartite graph with large enough part sizes is C_{2k+1}-saturated and C_{2t+1}-free. Thus,

$$\mathrm{sat}(n, C_{2t+1}, C_{2k+1}) = 0$$

for all $t, k \geq 1$ and $n \geq k + 1$.

This shows that minimizing the number of triangles in a C_k-saturated graph when k is odd is trivial.

Our next theorem shows that there exist n-vertex triangle-free graphs that are C_k-saturated for any even $k \geq 5$ when n is large enough.

Theorem 1.9 *For any integer $k \geq 5$,*

$$\mathrm{sat}(n, K_3, C_k) = 0$$

for all $n \geq 2k + 2$.

The case $\mathrm{sat}(n, K_3, C_4)$ is not covered by Theorem 1.9 and appears difficult. This is discussed further in Section 4.1. We know that $\mathrm{sat}(n, K_3, C_4) = 0$ for $8 \leq n \leq 24$.

In the case of $\mathrm{sat}(n, C_4, C_k)$, we have the following result. The method of the proof used for Theorem 1.10 is very different from the proof of Theorem 1.9.

Theorem 1.10 *For all $n \geq 111$ and $k \in \{7, 8, 9, 10\}$,*

$$\mathrm{sat}(n, C_4, C_k) = 0.$$

The lower bound on n is not needed when $k \in \{7, 8\}$. For these two cases, we have $\mathrm{sat}(n, C_4, C_7) = 0$ for all $n \geq 8$, and $\mathrm{sat}(n, C_4, C_8) = 0$ for all $n \geq 9$. It is likely that the lower bound on n is a consequence of our proof technique and is not optimal for $k \in \{9, 10\}$.

Finally, simple bounds on $\mathrm{sat}(n, C_{2l}, C_{2k})$ can obtained by the following construction. Let C a clique on $2k - 2$ vertices and fix two vertices $x, z \in C$, and let $y_1, y_2, \ldots, y_{n-2k+2}$ be the vertices not in C. Let us add all the edges xy_i, y_iz for $1 \leq i \leq n - 2k + 2$. This shows that

$$\mathrm{sat}(n, C_{2l}, C_{2k}) \leq \begin{cases} 0 & \text{if } l \geq k, \\ O_{k,l}(n) & \text{if } l < k. \end{cases}$$

The table shown in Figure 1.2 gives a summary of our results with references.

The rest of this paper is organized as follows. In the next section, we introduce some of our notation and give the proofs of Propositions 1.4, 1.5, and 1.6. Many of the ideas used in the proofs of these propositions will be used at other places in the paper. Section 3 considers K_s-saturated graphs and contains the proofs of Theorems 1.2, 1.3, 1.7, and 1.8. Section 4 contains our results on $\mathrm{sat}(n, H, F)$ where H and F are both cycles. The proofs of Theorem 1.9 and 1.10 are given there along with more discussion. We end with some open problems in Section 5.

2 Notation and Proofs of Propositions 1.4, 1.5, and 1.6

Throughout the paper, we write $(n)_k$ to denote the falling factorial $(n)(n-1)\ldots(n-k+1)$.

For two graphs G and F, the *join* of G and F is written $G + F$. This is the graph obtained by taking the union of G and F, and joining every vertex of G to every vertex of F. For $r \geq 3$, $\overline{K_r}$ is the graph with r vertices and no edges. Write $N(v)$ for the neighborhood of v, and $N_2(v)$ for the vertices at distance 2 from v. A very useful fact

Result	Hypothesis	Reference
$\mathrm{sat}(n, K_r, F) = O(n)$	$n \geq 1, r \geq 2$	Proposition 1.4
$\mathrm{sat}(n, H, K_s) = \Omega(n^2)$	$H \neq K_s, \|V(H)\| = s \geq 3$	Proposition 1.5
$\mathrm{sat}(n, K_r, K_s) = \Theta(n)$	$s > r \geq 3$	Theorem 1.2
$\mathrm{sat}(n, K_3, K_4) = n - 2$	$n \geq 7$	Theorem 1.3
$\mathrm{sat}(n, C_4, K_4) \sim \frac{n^2}{2}$		Proposition 1.6
$\mathrm{sat}(n, C_r, K_s) = \Theta(n^{\lfloor \frac{r}{2} \rfloor})$	$s \geq 5, r \leq 2s - 4$	Theorem 1.7
$\mathrm{sat}(n, C_6, K_5) \sim n^3$		Theorem 1.8
$\mathrm{sat}(n, K_3, C_k) = 0$	$k \geq 5, n \geq 2k + 2$	Theorem 1.9
$\mathrm{sat}(n, C_4, C_k) = 0$	$n \geq 111, k \in \{7, 8, 9, 10\}$	Theorem 1.10
$\mathrm{sat}(n, C_{2l}, C_{2k}) = 0$	$l \geq k$	
$\mathrm{sat}(n, C_{2l}, C_{2k}) = O_{k,l}(n)$	$l < k$	
$\mathrm{sat}(n, H, C_{2k+1}) = 0$	$n \geq 2k + 2 \geq 4$, H is not bipartite	Proposition 4.1
$\mathrm{sat}(n, C_t, C_k) = 0$	$n \geq t \geq k \geq 3$	Proposition 4.2
$\mathrm{sat}(m(r-1) + 1, C_t, C_k) = 0$	$t \geq r + 1, 2r - 2 \geq k \geq r + 1$	Proposition 4.2
$\mathrm{sat}(n, K_3, C_4) \leq \lfloor \frac{n-1}{2} \rfloor$	$n \geq 4$	
$\mathrm{sat}(10t + 1, C_4, C_6) \leq 2t$	$t \geq 1$	Theorem 4.8

Figure 1.2: Summary of our results

about a K_s-saturated graph is that its diameter is 2 provided $s \geq 3$. Thus, if v is any vertex in a K_s-saturated graph G, then

$$V(G) = \{v\} \cup N(v) \cup N_2(v).$$

We will slightly abuse notation and use $N(v)$ and $N_2(v)$ to represent the subgraph of G induced by $N(v)$ and $N_2(v)$, respectively, when it is convenient. One observation that we will use frequently is that in a K_s-saturated graph, if u and v are two vertices that are not adjacent, then $N(u) \cap N(v)$ must contain a copy of K_{s-2}. This also implies that the minimum degree in a K_s-saturated graph is at least $s - 2$.

Let us finish this section by giving the proofs of Propositions 1.4, 1.5, and 1.6.

Proof of Proposition 1.4. Let $r \geq 2$ be an integer and F be a graph. We use the construction of Kászonyi and Tuza [18] to build an n-vertex graph G that is F-saturated. Start with a clique C with $|V(F)| - \alpha(F) - 1$ vertices, and join all vertices of C to an independent set J with $n - (|V(F)| - \alpha(F) - 1)$ vertices. The graph constructed so far is F-free. This is because all of the edges in this graph can be covered with $|V(F)| - \alpha(F) - 1 = \beta(F) - 1$ vertices (here $\beta(F)$ is the minimum number of vertices in F needed to touch all edges of F). We now add edges to the independent set J one by one until we obtain an F-saturated graph G. We will never add more than $\alpha(F)$ edges incident to a single vertex in J because this would create a copy of F. Indeed, if $v \in J$ and v has $\alpha(F)$ other neighbors in J, then the clique C together with v forms a clique of size $|V(F)| - \alpha(F)$, and all of these vertices are joined to $\alpha(F)$ neighbors of v in J. This subgraph contains a copy of F. Thus, at the end of the process the subgraph induced by J has maximum degree less than $\alpha(F)$, and all vertices in J have degree at most $|V(F)| - \alpha(F) - 1 + \alpha(F) - 1 = |V(F)| - 2$.

The last step is to estimate the number of K_r's in G. There are $\binom{|V(F)|-\alpha(F)-1}{r}$ copies of K_r that do not contain a vertex in J. The number of K_r's that contain at least one vertex in J is at most

$$n\binom{|V(F)|-2}{r-1}$$

since the degree of a vertex in J is no more than $|V(F)|-2$. We conclude that there are at most

$$n\binom{|V(F)|-2}{r-1}+\binom{|V(F)|-\alpha(F)-1}{r}$$

copies of K_r in the F-saturated graph G. ■

Proof of Proposition 1.5. Let G be an n-vertex K_s-saturated graph. So if xy is a nonedge of G, then there is an $(s-2)$-clique, say on $\{z_1,\dots,z_{s-2}\}$, where each z_i is adjacent to both x and y. There are $f_{h_1,h_2}(H)$ copies of H in G which are contained in the vertex set $\{x,y,z_1,\dots,z_{s-2}\}$, such that the vertices h_1, h_2 of each copy of H correspond to x, y respectively. Let $H_1,\dots,H_f$ be these copies of H where $f=f_{h_1,h_2}(H)$. We claim that each time we choose a nonedge and obtain the corresponding f many copies of H, we will never see the same copy of H twice. To see this, suppose xy and $x'y'$ are distinct nonedges of G. Let $H_1,\dots,H_f$ and $H_1',\dots,H_f'$ be copies of H obtained from xy and $x'y'$, respectively. The vertex set of each H_i is $\{x,y,z_1,\dots,z_{s-2}\}$ where $\{z_1,\dots,z_{s-2}\}$ is an $(s-2)$-clique, and x and y are joined to every z_i. If some H_j' has the same vertex set as H_i, then $x',y'\in\{x,y,z_1,\dots,z_{s-2}\}$. This is a contradiction since $x'y'$ is a nonedge and the only missing edge from $\{x,y,z_1,\dots,z_{s-2}\}$ is xy. This shows that the number of copies of H in G is at least

$$f_{h_1,h_2}(H)\left(\binom{n}{2}-e(G)\right)\geq f_{h_1,h_2}(H)\left(\binom{n}{2}-\mathrm{ex}(n,K_s)\right).$$

The proposition follows from the bound $\mathrm{ex}(n,K_s)\leq\left(1-\frac{1}{s-1}\right)\frac{n^2}{2}$. Observe that since H has a nonedge, $f_{h_1,h_2}(H)\geq 1$. ■

Proof of Proposition 1.6. For the upper bound, notice that $K_2+\overline{K}_{n-2}$ is K_4-saturated with $\binom{n-2}{2}$ copies of C_4. To prove the lower bound, let G be an n-vertex graph that is K_4-saturated. If $e(G)>n^{5/3+\delta}$, then the number of C_4's in G is at least

$$\frac{2e(G)^4}{n^4}-\frac{3}{4}e(G)n=\left(2n^{3\delta}-\frac{3}{4}\right)n^{8/3+\delta}>\frac{n^2}{2}$$

for large enough n in terms of δ (see Lemma 2.5 of [14]). Now assume $e(G)\leq n^{5/3+\delta}$. The argument of Proposition 1.5 shows that G contains at least

$$\binom{n}{2}-e(G)\geq\binom{n}{2}-n^{5/3+\delta}$$

copies of C_4. ■

3 Counting subgraphs of clique-saturated graphs

In this section we focus on graphs which are K_s-saturated. In Section 3.1, we prove Theorem 1.2. We improve the results for the case $s = 4$ and $r = 3$ in Theorem 1.3 in Section 3.2. In Section 3.3 we count cycles to prove Theorems 1.7 and 1.8.

3.1 Proof of Theorem 1.2

We begin this section with a lemma that is certainly known, but a proof is included for completeness. A similar result was proved by Amin, Faudree, and Gould [3] in the case that $s = 4$.

Lemma 3.1 *Let $n > s \geq 3$ be integers. If G is a K_s-saturated graph with $\delta(G) \leq s - 2$, then G is isomorphic to $K_{s-2} + \overline{K}_{n-s+2}$.*

Proof. Suppose G is a K_s-saturated graph with n vertices. Note that $\delta(G) \leq s - 2$ implies $\delta(G) = s - 2$ since G is K_s-saturated. Choose a vertex v with $d(v) = s - 2$. If $u \in N_2(v)$, then $N(v) \cap N(u)$ must contain a K_{s-2}, but since $|N(v)| = s - 2$, $N(v)$ must then be a clique. If u_1 and u_2 are distinct vertices in $N_2(v)$, then u_1 cannot be adjacent to u_2, otherwise $\{u_1, u_2\} \cup N(v)$ is a K_s in G. This shows that G contains a copy $K_{s-2} + \overline{K}_{n-s+2}$ where $N_2(v) \cup \{v\}$ is the independent set of size $n - s + 2$. The graph $K_{s-2} + \overline{K}_{n-s+2}$ is K_s-saturated and has n vertices, so G must be this graph. ■

The graph $K_{s-2} + \overline{K}_{n-s+2}$ has the property that $n - s + 2$ vertices have exactly one K_{s-2} in their neighborhood. The next lemma shows that this cannot occur when there are no vertices of degree $s - 2$.

Lemma 3.2 *Let $n \geq 2s - 2$ and $s \geq 3$ be integers. If G is a K_s-saturated graph on n vertices with $\delta(G) \geq s - 1$, then no vertex has just one copy of K_{s-2} in its neighborhood.*

Proof. Suppose G is a K_s-saturated graph with n vertices and $\delta(G) \geq s - 1$. Aiming for a contradiction, assume v is a vertex with exactly one copy of K_{s-2} in $N(v)$. Let $S \subseteq N(v)$ be the vertices that induce the unique K_{s-2} in $N(v)$.

Case 1: $d(v) \leq n - 2$

For any vertex $u \in N_2(v)$, $N(u) \cap N(v)$ contains a K_{s-2}. By uniqueness, this K_{s-2} must be S. This implies $S \subseteq N(u) \cap N(v)$ and in particular, u is adjacent to all vertices in S. As every vertex in $N_2(v)$ is joined to S, the set $N_2(v)$ must be an independent set, otherwise G contains a K_s. By assumption, $d(v) \geq s - 1$ and so there is a vertex $v' \in N(v)$ with $v' \notin S$. As there is only one K_{s-2} in $N(v)$, vertex v' cannot be adjacent to all vertices in S. Say v' is not adjacent to $v_1 \in S$. The set $N(v_1) \cap N(v')$ must contain a K_{s-2}. Let S' be the vertices of such an $(s-2)$-clique. Note $v_1 \notin S'$ and $v' \notin S'$ since $S' \subseteq N(v_1) \cap N(v')$. If $|S' \cap N(v)| \geq s - 3$, then there is more than one K_{s-2} in $N(v)$. Indeed, $(S' \cap N(v)) \cup \{v'\}$ would contain a K_{s-2} in $N(v)$ different from S. This also shows $v \notin S'$ otherwise, $S' \subseteq \{v\} \cup N(v)$. Since the lemma is trivially true for $s = 3$, assume that $s \geq 4$. As $|S' \cap N(v)| \leq s-4$ and $v \notin S'$, S' contains at least two vertices in $N_2(v)$. This contradicts the fact that $N_2(v)$ is an independent set.

Case 2: $d(v) = n-1$

Let W be the neighbors of v that are not in S. First suppose there is a pair of nonadjacent vertices, say w_1 and w_2, in W. Then $N(w_1) \cap N(w_2)$ must contain a K_{s-2}, say S' are the vertices of such a $(s-2)$-clique. If $v \notin S'$, then $S' = S$, but then we can remove a vertex from S and replace it with w_1 to get a K_{s-2} in $N(v)$ different from S. Therefore, v must be in S' and $|S' \backslash \{v\}| = s-3$. But then $w_1 \cup S'$ is an $(s-2)$-clique in $N(v)$ that is different from S. This shows that W is a clique and so $|W| < s-1$ as G is K_s-free. This contradicts the assumption that $n \geq 2s-2$.

■

Lemma 3.2 shows that the neighborhood of any vertex in a K_s-saturated graph G with $\delta(G) \geq s-1$ must have at least two copies of K_{s-2} in its neighborhood. We now use this lemma to characterize K_s-saturated graphs with $\delta(G) = s-1$.

For integers $n > s \geq 3$, let $(K_{s-1}-e)+\overline{K}_{n-s+1}$ be the graph obtained by taking a K_{s-1} and removing an edge e, and then joining all vertices of this graph to an independent set of size $n-s+1$. This graph is the same as the complete $(s-1)$-partite graph with part sizes $1, 1, \ldots, 1$ ($s-3$ times), 2, and $n-s+1$.

Let W be the 6-vertex graph obtained by taking a 5-cycle $a_1a_2a_3a_4a_5a_1$ and joining a new vertex b to each vertex on the 5-cycle. We call b the *central vertex*. For $s \geq 3$ and positive integers m_1, m_3, m_4 with $m_1 + m_3 + m_4 = n-s+1$, let $W_s(m_1, 1, m_3, m_4, 1)$ be the graph obtained from W by replacing a_i with an independent set I_i with $|I_i| = m_i$ ($i = 1, 3, 4$), and replacing the central vertex b with a clique of size $s-3$. If x and y are vertices that replaced a_i and a_j, respectively, then x and y are adjacent if and only if a_i and a_j are adjacent in W. Vertices in the $(s-3)$-clique that replaced the central vertex b are adjacent to all vertices in the graph and so have degree $n-1$.

Amin, Faudree, and Gould [3] showed that if G is an n-vertex K_4-saturated graph that is 3-connected, then G is isomorphic to $(K_3-e)+\overline{K}_{n-3}$, or to $W_4(m_1, 1, m_3, m_4, 1)$ for some $m_1 + m_3 + m_4 = n-3$. We prove a similar result for K_s-saturated graphs that have minimum degree $s-1$.

Lemma 3.3 *If G is a K_s-saturated n-vertex graph with $\delta(G) = s-1$, then G is isomorphic to $(K_{s-1}-e)+\overline{K}_{n-s+1}$, or to $W_s(m_1, 1, m_3, m_4, 1)$ for some $m_1+m_3+m_4 = n-s+1$.*

Proof. Suppose v is a vertex in a K_s-saturated n-vertex graph G where $\delta(G) = s-1$ and $d(v) = s-1$. By Lemma 3.2, there must be at least two $(s-2)$-cliques in $N(v)$. If there are more than two $(s-2)$-cliques in $N(v)$, then $N(v)$ is complete, which gives a K_s in G. Thus, $N(v)$ contains exactly two $(s-2)$-cliques. Let $S_1 = \{v_1, v_2, \ldots, v_{s-3}, v_{s-2}\}$ be the first K_{s-2}, and $S_2 = \{v_1, v_2, \ldots, v_{s-3}, v_{s-1}\}$ be the second (so the only edge missing from $N(v)$ is $v_{s-2}v_{s-1}$).

Let T_1 be all vertices in $N_2(v)$ that are adjacent to every vertex in S_1, but not adjacent to v_{s-1}. Similarly, let T_2 be all vertices in $N_2(v)$ that are adjacent to all vertices in S_2, but not adjacent to v_{s-2}. Lastly, let T_3 be all vertices in $N_2(v)$ that are adjacent to all vertices in $S_1 \cup S_2 = N(v)$. Since $N(v) \cap N(t)$ must contain a K_{s-2} for any $t \in N_2(v)$, the sets T_1, T_2, and T_3 form a partition of $N_2(v)$. Also, both $T_1 \cup T_3$ and $T_2 \cup T_3$ are independent sets since G is K_s-free.

If $T_1 = T_2 = \emptyset$, then T_3 is an independent set on $n-s+1$ vertices that are all joined to each vertex in $\{v_1, v_2, \ldots, v_{s-2}, v_{s-1}\}$. Vertex v is also joined to these vertices, but is not joined to any vertex in T_3. This shows that G contains a subgraph isomorphic to $(K_{s-1} - e) + \overline{K}_{n-s+1}$. This last graph is K_s-saturated and has n vertices so G must be this graph.

Now suppose $T_1 \neq \emptyset$ and let $t \in T_1$. Since t is not adjacent to v_{s-1}, $N(t) \cap N(v_{s-1})$ must contain a K_{s-2}. The intersection $N(t) \cap N(v_{s-1})$ contains the $(s-3)$-clique $\{v_1, v_2, \ldots, v_{s-3}\}$ so there must be another vertex x for which x is adjacent to both t and v_{s-1}. If $x \in T_1 \cup T_3$, then we contradict the fact that $T_1 \cup T_3$ is an independent set. Therefore, $x \in T_2$ and so $T_2 \neq \emptyset$. This argument shows that $T_1 \neq \emptyset$ if and only if $T_2 \neq \emptyset$. Next, let $y \in T_1$ and $z \in T_2$ be arbitrary vertices. We will show that y and z are adjacent. If they are not, then $N(y) \cap N(z)$ must contain a K_{s-2}. Now $N(y) \cap N(z) \cap N(v) = \{v_1, v_2, \ldots, v_{s-3}\}$, and so there must be a vertex in $N_2(v)$ that is adjacent to both y and z. This is impossible though since $y \in T_1 \cup T_3$, $z \in T_2 \cup T_3$, $T_1 \cup T_3$ and $T_2 \cup T_3$ are independent sets, and $N_2(v) = T_1 \cup T_2 \cup T_3$. Thus, every vertex in T_1 is joined to every vertex in T_2. At this point, we have a K_s-saturated subgraph that is isomorphic to

$$W_s(|T_3| + 1, 1, |T_1|, |T_2|, 1).$$

Indeed, $\{v_1, v_2, \ldots, v_{s-3}\}$ is a $(s-3)$-clique and every vertex in this set has degree $n-1$. If this clique replaces the central vertex b in the graph W defined before Lemma 3.3, and we replace a_1 with $T_3 \cup \{v\}$, a_2 with v_{s-2}, a_3 with T_1, a_4 with T_2, and a_5 with v_{s-1}, we obtain a $W_s(m_1, 1, m_3, m_4, 1)$. This last graph is K_s-saturated and has n vertices, so G must be this graph. ■

Let us summarize what we have shown so far. Let G be an n-vertex K_s-saturated graph.

1. If $\delta(G) \leq s-2$, then G is isomorphic to $K_{s-2} + \overline{K}_{n-s+2}$.

2. If $\delta(G) = s-1$, then G is isomorphic to $(K_{s-1} - e) + \overline{K}_{n-s+1}$, or some

   $$W_s(m_1, 1, m_3, m_4, 1)$$

   with $m_1 + m_3 + m_4 = n - s + 1$.

We now use Lemmas 3.1, 3.2, and 3.3 to prove Theorem 1.2.

Proof of Theorem 1.2. Let G be a K_s-saturated graph with n vertices. We first show that there are at least

$$\frac{1}{r}\left(\binom{s-2}{r-1} + \binom{s-3}{r-2}\right) n$$

copies of K_r in G.

If $\delta(G) = s-2$, then G is isomorphic to $K_{s-2} + \overline{K}_{n-s+2}$ by Lemma 3.1. This graph has $\binom{s-2}{r} + (n-s+2)\binom{s-2}{r-1}$ copies of K_r. For large enough n, this is at least $\frac{1}{r}\left(\binom{s-2}{r-1} + \binom{s-3}{r-1}\right) n$.

If $\delta(G) = s-1$, then by Lemma 3.3, G is isomorphic to $(K_{s-1} - e) + \overline{K}_{n-s+1}$ or $W_s(m_1, 1, m_3, m_4, 1)$ for some $m_1 + m_3 + m_4 = n - s + 1$. The first graph has

$$\binom{s-2}{r} + \binom{s-3}{r-1} + (n-s+1)\left(\binom{s-2}{r-1} + \binom{s-3}{r-2}\right)$$

copies of K_r. A member of $W_s(m_1, 1, m_3, m_4, 1)$ that minimizes the number of K_r's is obtained when two of the m_i's are 1, and the other is $n-s-1$. The number of K_r's in this graph is

$$(n-s-1)\left(\binom{s-2}{r-1} + \binom{s-3}{r-1}\right) + \binom{s-3}{r} + 4\binom{s-3}{r-1} + 3\binom{s-3}{r-2}.$$

In both cases, we have at least $\frac{1}{r}\left(\binom{s-2}{r-1} + \binom{s-3}{r-2}\right)n$ copies of K_r for large enough n.

Assume $\delta(G) \geq s$. By Lemma 3.2, every vertex has at least two distinct copies of K_{s-2} in its neighborhood. Thus, for all $v \in V(G)$, the number of K_{r-1}'s in $N(v)$ is at least

$$\binom{s-2}{r-1} + \binom{s-3}{r-2}$$

as the two $(s-2)$-cliques in $N(v)$ cannot form a K_{s-1} (this would create a K_s, using v, in G). The number of K_r's in G is at least

$$\frac{1}{r}\sum_{v \in V(G)} (\text{number of } K_{r-1}\text{'s in } N(v)) \geq \frac{1}{r}\left(\binom{s-2}{r-1} + \binom{s-3}{r-2}\right)n.$$

Next we show that there are also at least

$$\frac{1}{r-1}\binom{s-2}{r-1}n - \binom{s-2}{r-1} - o_n(1)$$

copies of K_r in G. By a result of Erdős [11] for $r = 3$ and Mubayi [22] for $r \geq 4$, there is a positive constant $\alpha_{r,s}$, depending only on r and s, such that if G has at least $\text{ex}(n, K_r) + \alpha_{r,s}$ edges, then G has at least $\binom{s-2}{r-1}n$ copies of K_r, in which case we are done. Now assume that

$$e(G) \leq \text{ex}(n, K_r) + \alpha_{r,s} \leq \left(1 - \frac{1}{r-1}\right)\frac{n^2}{2} + \alpha_{r,s}.$$

Consider a pair of nonadjacent vertices x and y. Their common neighborhood contains at least one copy of K_{s-2} since G is K_s-saturated. This gives $2\binom{s-2}{r-1}$ copies of K_r that contain the vertex x or contain the vertex y. Now x has at least $s-2$ neighbors, and so each copy of K_r containing x obtained in this way (by choosing a nonedge containing x and looking at the common neighborhood) is counted at most $n-s+2$ times. Thus, the number of copies of K_r in G is at least

$$\frac{2\binom{s-2}{r-1}e(\overline{G})}{n-s+2} \geq \frac{2\binom{s-2}{r-1}}{n}\left(\binom{n}{2} - e(G)\right) \geq \frac{2\binom{s-2}{r-1}}{n}\left(\frac{n^2}{2(r-1)} - \frac{n}{2} - \alpha_{r,s}\right).$$

For large enough n, this is at least

$$\frac{1}{r-1}\binom{s-2}{r-1}n - 2\binom{s-2}{r-1}.$$

■

3.2 Proof of Theorem 1.3

Let G be a K_4-saturated graph on n vertices. We must show that G has at least $n-2$ triangles, and if G has $n-2$ triangles, then G is isomorphic to $K_2 + \overline{K}_{n-2}$. A *triangle block* is a maximal subgraph of G constructed by starting with a triangle and repeatedly adding triangles to it such that each new triangle shares at least one edge with a previous triangle. One can easily see that if a triangle block contains x vertices, then it contains at least $x-2$ triangles. In fact, $K_2 + \overline{K}_{n-2}$ is a triangle block on n vertices. Also notice that if two triangle blocks have at least two vertices in common, and their union contains x vertices, then it contains at least $x-2$ triangles.

A *triangle cluster* is a maximal union of triangle blocks $B_1, B_2, \ldots, B_k$ such that each block B_i (for $2 \leq i \leq k$) shares at least two vertices with the union of blocks $B_1, B_2, \ldots, B_{i-1}$. A triangle cluster also has the property that if it has x vertices, then it has at least $x-2$ triangles. More importantly, note that any two triangle clusters share at most one vertex in common. Indeed, otherwise their union is contained in a triangle cluster contradicting the maximality.

Claim 3.4 *If a triangle cluster C has three triangles of the form abc, bcd, cde, then G has more than $n-2$ triangles.*

Proof. Consider any vertex v not in C. Then v is not adjacent to at least three of the vertices $x, y, z \in \{a, b, c, d, e\}$. Now notice that any two non-adjacent vertices p and q belong to the same triangle cluster. Indeed, adding pq to G must create a K_4, so there exist vertices r, s such that prs and qrs are triangles in G, so p and q belong to the same triangle block, and so they belong to the same triangle cluster as well. Suppose v and x belong to a triangle cluster C_1, v and y belong to C_2, and v and z belong to C_3. Now C_1, C_2, C_3 are distinct triangle clusters because if, say $C_1 = C_2$, then C_1 and C would share two vertices (x and y), a contradiction. This implies that every vertex v not in C belongs to at least three different triangles. Suppose C has m vertices and let $t(u)$ denote the number of triangles containing a vertex u. Then since C contains at least $m-2$ triangles, we have $\sum_u t(u) \geq 3(n-m) + 3(m-2) + 1 > 3(n-2)$. On the other hand, the sum $\sum_u t(u)$ counts each triangle 3 times exactly, proving the claim. ∎

By Claim 3.4, we can assume that every triangle cluster C consists of triangles of the form $abx_1, abx_2, \ldots, abx_r$ for integer $r \geq 1$. If $r \geq 2$, let us call ab the base of a triangle abx_i (for any $i \in \{1, 2, \ldots, r\}$) and x_i as its tip. For a vertex u, let us define $p(u)$ as the number of triangles whose tip is the vertex u. If there is a triangle cluster with n vertices, then G is isomorphic to $K_2 + \overline{K}_{n-2}$ and we are done. Assume this is not the case.

Claim 3.5 *For any vertex v, there is a triangle cluster that does not contain v. Moreover, $p(v) \geq 2$.*

Proof. Consider any triangle cluster C and a vertex u not in C. If we take any triangle abc in C, then u is not adjacent to at least two of the vertices a, b, c, otherwise u would have to be in C. Suppose without loss of generality that u is not adjacent to a or b.

Therefore, the vertices u and a belong to a triangle cluster C', and the vertices u and b belong to a triangle cluster C''. Then, by the linearity of triangle clusters, C, C', C'' are distinct and there is no vertex contained in all three of them, proving the first part of the claim. Thus, for any vertex v, there is a triangle cluster D not containing it; moreover v is not adjacent to some two vertices a, b in D. The second part of the claim simply follows by using the fact that adding the pairs va or vb must create a K_4 in G. ■

By Claim 3.5, $\sum_u p(u) \geq 2n$. Moreover, the sum $\sum_u p(u)$ counts each triangle at most once (notice that the triangles that do not share an edge with another triangle are not counted by this sum). So the number of triangles in G is at least $2n$. Since if G has $2n \geq n-2$ triangles when $G \neq K_2 + \overline{K}_{n-2}$, we conclude that $\mathrm{sat}(n, K_3, K_4) = n-2$ and the extremal example is uniquely achieved by $K_2 + \overline{K}_{n-2}$.

3.3 Proof of Theorems 1.7 and 1.8

Proof of Theorem 1.7. For an upper bound on $\mathrm{sat}(n, C_r, K_s)$ consider the graph $G = K_{s-2} + \overline{K}_{n-s+2}$. Let A be an independent set of $k = \lfloor \frac{r}{2} \rfloor$ vertices in $\overline{K}_{n-s+2}$. There are $\binom{n-s+2}{k}$ ways to pick an independent set of size k. If r is even, then there are $\frac{(s-2)_k (k-1)!}{2}$ C_r subgraphs containing A and each C_r is counted once. If r is odd, then there are $\frac{(s-2)_{k+1} k!}{2}$ C_r subgraphs containing A and each C_r is counted once. Furthermore, there does not exist a C_r subgraph with more than $\lfloor \frac{r}{2} \rfloor$ vertices in $\overline{K}_{n-s+2}$ since $\alpha(C_r) = \lfloor \frac{r}{2} \rfloor$. Thus,

$$\mathrm{sat}(n, C_r, K_s) \leq \begin{cases} \left(\frac{(s-2)_k}{2k}\right)\left(n^k + o(n^k)\right) & \text{if } 2|r \\ \left(\frac{(s-2)_{k+1}}{2}\right)\left(n^k + o(n^k)\right) & \text{if } 2 \not| r \end{cases}.$$

Let G be a graph that witnesses $\mathrm{sat}(n, C_r, K_s)$ for $s \geq 5$, $2s-4 \geq r \geq s+1$. Notice that if $xy \notin E(G)$, then there exists a K_{s-2} subgraph in the common neighborhood of x and y. Furthermore, if $xy \notin E(G)$, then there exists $s-2$ internally disjoint x, y-paths of length 2.

Case 1: r is even.

Let $A \subseteq V(G)$ be an independent set of size k. Fix the index for $a_1 \in A$. There are $(k-1)!$ ways to index the remaining elements. Notice that for each $1 \leq i \leq k-1$, the vertices a_i, a_{i+1} have at least $s-2$ common neighbors (since there is a copy of K_{s-2} in their common neighborhood), and similarly a_1, a_k have at least $s-2$ common neighbors.

If r is even, then we will select distinct $b_i \in N(a_i) \cap N(a_{i+1})$ for $1 \leq i \leq k-1$ and $b_k \in N(a_1) \cap N(a_k)$. So we pick k different elements to form a set $B = \{b_i : 1 \leq i \leq k\}$. Since $r \leq 2s-4$, we have that $k \leq s-2$; so we can always pick B in at least $(s-2)_k$ ways. Since

$$a_1 b_2 a_2 \ldots a_k b_k$$

is a cycle of length r, the total number of cycles of length r we see is at least $\frac{(s-2)_k (k-1)!}{2}$ times the number of independent sets of size k in G.

By Theorem 1** in [9], there exists $c, c' > 0$ such that for any graph G with

$$|E(G)| \geq cn^{2-2/r},$$

there exists

$$c'n^r\left(\frac{|E(G)|}{n^2}\right)^{(r/2)^2}$$

copies of $K_{r/2,r/2}$. Each copy of $K_{r/2,r/2}$ contains many copies of C_r. Therefore, if $|E(G)|=\epsilon n^2$ and n is sufficiently large, there are $\Theta(n^r)$ copies of C_r. Thus, we can assume that $|E(G)|=o(n^2)$ and that G has $n^2/2-o(n^2)$ non-edges. Using the Moon-Moser Theorem, we know that G has at least $\binom{n}{k}-o(n^k)$ independent sets of size k.

Each C_r in G is counted at most 2 times (C_r induces at most two independent sets of size k). Putting our estimates together gives,

$$\operatorname{sat}(n,C_r,K_s)\geq\left(\frac{(s-2)_k(k-1)!}{4}\right)\left(\binom{n}{k}-o(n^k)\right)=\left(\frac{(s-2)_k}{4\cdot k}\right)\left(n^k-o(n^k)\right).$$

Case 2: r is odd.

Let $xy\notin E(G)$, $Z=\{z_1,\ldots,z_{s-2}\}\subseteq N(x)\cap N(y)$ induce a clique, and $A=\{a_2,\ldots,a_{k-1}\}\subseteq V(G)\setminus(Z\cup\{x,y\})$. These elements could be indexed in $(k-2)!$ ways.

Construct an x,y-path P as follows. Let $a_1=x$ and $a_k=y$. For each $1\leq i\leq k$ with $a_ia_{i+1}\notin E(G)$ we can choose $b_i\in N(a_i)\cap N(a_{i+1})$ that has not been used to create an a_1,a_k-path that traverses the vertices a_i in the order of their index. Let B be the set of the b_i's chosen, $B=\{b_i\in N(a_i)\cap N(a_{i+1}):a_ia_{i+1}\notin E(G)\}$. Notice that $|B|\leq k-1$ since we choose at most one b_i per pair a_ia_{i+1}. It follows that $|V(P)|=k+|B|<r$ and $|Z\setminus V(P)|=s-2-|B|>0$. The set B can be chosen in at least $(s-2)_{|B|}$ ways.

Since $|Z\setminus V(P)|=s-2-|B|$ and $|V(P)|=k+|B|$, we can extend P to a C_r subgraph if $r-k-|B|\leq s-2-|B|$. This condition is met by our assumption on r. Extending P with Z can be done in $(s-2-|B|)_{k+1-|B|}$ ways. Therefore, for any fixed choice of $xy\notin E(G)$, Z, and A, we see at least $(k-2)!(s-2)_{k+1}$ copies of C_s in G.

Notice that there are at least $\left(\binom{n}{2}-\operatorname{ex}(n,K_s)\right)\geq\left(\frac{n^2}{2(s-1)}-\frac{n}{2}\right)$ ways to choose $xy\notin E(G)$. Then there are $\binom{n-s}{k-1}$ ways to pick A. If we count copies of C_r in this way, each copy of C_r in G is counted at most $\left(\frac{r(r-3)}{2}\right)\cdot\binom{r-2}{k-2}$ times (choose a non-edge in the cycle, and then choose the set A). Thus,

$$\begin{aligned}\operatorname{sat}(n,C_r,K_s)&\geq\frac{(k-2)!(s-2)_{k+1}}{\left(\frac{r(r-3)}{2}\right)\cdot\binom{r-2}{k-2}}\left(\frac{n^2}{2(s-1)}-\frac{n}{2}\right)\binom{n-s}{k-2}\\&\geq\left(\frac{(s-2)_{k+1}(k-2)!}{r(r-3)(r)_k(s-1)}\right)\left(n^k-o(n^k)\right).\end{aligned}$$

■

Next we prove Theorem 1.8. First we need a lemma which relates the number of copies of a C_6 in a K_5-saturated graph to the number of independent sets of size 3.

Lemma 3.6 *If G is a K_5-saturated graph, then the number of copies of C_6 in G is at least*

$$6i_3(G),$$

where $i_3(G)$ is the number of independent sets of size 3 in G.

Proof. Let G be a K_5-saturated graph. Let $\mathcal{I}_3(G)$ be the set of all independent sets in G having three vertices. Partition $\mathcal{I}_3(G)$ into two sets $\mathcal{I}_3'(G)$ and $\mathcal{I}_3''(G)$ where $\{x, y, z\} \in \mathcal{I}_3'(G)$ if and only if there is a triangle in the common neighborhood of $\{x, y, z\}$. We will count the number of copies of C_6 of the form $x\alpha y\beta z\gamma x$ where at least one of $\{x, y, z\}$ or $\{\alpha, \beta, \gamma\}$ is an independent set in G. Then the only C_6's that will be counted more than once are those for which both $\{x, y, z\}$ and $\{\alpha, \beta, \gamma\}$ are independent sets. (Note that this situation does not occur until Case 3 given below.) These C_6's will be counted twice.

For vertices x and y, let $N(x, y)$ be the vertices adjacent to both x and y. When x and y are not adjacent, $N(x, y)$ contains at least one triangle, since G is K_5-saturated.

Let $\{x, y, z\} \in \mathcal{I}_3'(G)$. Choose a triangle $abca$ with $a, b, c \in N(x, y) \cap N(x, z) \cap N(y, z)$. Then we have six copies of C_6 using x, y, z, a, b, and c:

$$xaybzcx,\ xayczbx,\ xbyazcx,\ xbyczax,\ xcyazbx,\ xcybzax. \tag{2}$$

If we choose another $\{x_1, y_1, z_1\} \in \mathcal{I}_3'(G)$ distinct from $\{x, y, z\}$, and a triangle $a_1b_1c_1a_1$ that lies in the common neighborhood of $\{x_1, y_1, z_1\}$, then none of the 6-cycles

$$\begin{array}{ccc} x_1a_1y_1b_1z_1c_1x_1, & x_1a_1y_1c_1z_1b_1x_1, & x_1b_1y_1a_1z_1c_1x_1, \\ x_1b_1y_1c_1z_1a_1x_1, & x_1c_1y_1a_1z_1b_1x_1, & x_1c_1y_1b_1z_1a_1x_1 \end{array}$$

will coincide with a 6-cycle listed in (2). This is because the only three independent vertices in $\{x, y, z, a, b, c\}$ are x, y, and z, and the only three independent vertices in $\{x_1, y_1, z_1, a_1, b_1, c_1\}$ are x_1, y_1, and z_1. Thus, we have that the number of copies of C_6 counted so far is

$$6|\mathcal{I}_3'(G)|. \tag{3}$$

Furthermore, the vertex set of each C_6 counted by (3) induces a subgraph that is isomorphic to $K_6 - K_3$ (the graph obtained by removing a triangle from K_6).

Now let $\{x, y, z\} \in \mathcal{I}_3''(G)$. We are going to count copies of C_6 of the form $x\alpha y\beta z\gamma x$. The first observation to make is that we will not count copies of C_6 that are counted by (3). The reason for this is that if $x\alpha y\beta z\gamma x$ is any 6-cycle with $\{x, y, z\} \in \mathcal{I}_3''(G)$, then $\{x, y, z, \alpha, \beta, \gamma\}$ does not induce $K_6 - K_3$ since x, y, and z have no triangle in their common neighborhood. Let us proceed now with the counting. Let $abca$ be a triangle in $N(x, y)$. Then z is not adjacent to all 3 of of a, b, and c and so we consider cases depending on the number of adjacencies between z and $\{a, b, c\}$.

Case 1: z is adjacent to a and b, but not c

Let $t \in N(x, z) \backslash \{a, b\}$. Such a t exists since x and z have at least 3 common neighbors. We are not claiming that a, b, and t form a triangle. We now consider two subcases.

First suppose that y is also adjacent to t. Then we have the following list of 12 C_6's:

$$\begin{array}{cccccc} xaybztx, & xaytzbx, & xbyaztx, & xbytzax, & xcyazbx, & xcyaztx \\ xcybzax, & xcytzax, & xcytzax, & xcytzbx, & xtybzax, & xtyazbx. \end{array}$$

Each one of these C_6's contains at least two of a, b, and c. Thus, each is of the form $x\alpha y\beta z\gamma x$ where $\{\alpha, \beta, \gamma\}$ is not an independent set of size 3.

Now suppose that other than a and b, there is no vertex adjacent to each of x, y, and z. Let $s \in N(y,z)$. Such a vertex exists since $N(y,z)$ must contain a triangle. In this case, we have the following list of 13 C_6's:

$$\begin{array}{ccccc} xaybztx, & xayszbx, & xbyaztx, & xbyszax, & xcyazbx, \\ xcyaztx, & xcybzax, & xcybztx, & xcyszax, & xcyszbx, \\ & xaysztx, & xbysztx, & xcysztx. & \end{array}$$

The first ten in the list are of the form $x\alpha y\beta z\gamma x$ where $\{\alpha,\beta,\gamma\}$ is not an independent set of size 3.

The conclusion is that in both of these subcases, we have at least 10 copies of C_6 of the form $x\alpha y\beta z\gamma x$ where $\{x,y,z\}$ is an independent set, $\{\alpha,\beta,\gamma\}$ is not an independent set, and $\{x,y,z,\alpha,\beta,\gamma\}$ does not induce a graph isomorphic to $K_6 - K_3$.

Case 2: z is adjacent to a, but not adjacent to b or c

Since $N(x,z)$ must contain a triangle, there must be a pair of adjacent vertices s and t with $\{s,t\} \cap \{a,b,c\} = \emptyset$ and $s,t \in N(x,z)$. Our goal is to find at least 6 copies of C_6 of the form $x\alpha y\beta z\gamma x$ where $\{\alpha,\beta,\gamma\}$ is not an independent set. Four such C_6's are

$$xbyaztx, \quad xbyazsx, \quad xcyaztx, \quad xcyastx.$$

If y is adjacent to t, then two more are $xaytzsx$ and $xaysztx$ and we are done. Assume that y is not adjacent to t or s. Since $N(y,z)$ contains a triangle, there is a new vertex u with u adjacent to both y and z. Then $xbyuzax$ and $xcyuzax$ are two more C_6's with the property that we need.

The conclusion is that in Case 2, we have at least 6 copies of C_6 of the form $x\alpha y\beta z\gamma x$ where $\{x,y,z\}$ is an independent set, $\{\alpha,\beta,\gamma\}$ is not an independent set, and $\{x,y,z,\alpha,\beta,\gamma\}$ does not induce a graph isomorphic to $K_6 - K_3$.

Case 3: z is not adjacent to a, b, or c

Let $uvwu$ be a triangle in $N(x,z)$ where the vertices x,y,z,a,b,c,u,v,w are all distinct.

First suppose that y is adjacent to w. Then we have the following 6 copies of C_6:

$$xaywzvx, \quad xaywzux, \quad xbywzvx, \quad xbywzux, \quad xcywzvx, \quad xcywzux.$$

Each of these C_6's is of the form $x\alpha y\beta z\gamma x$ where $\{x,y,z\}$ is an independent set, $\{\alpha,\beta,\gamma\}$ is not (they all contain at least two of $\{u,v,w\}$), and $\{x,y,z,\alpha,\beta,\gamma\}$ does not induce a $K_6 - K_3$.

Now suppose that y is not adjacent to any of u, v, or w. Let $rstr$ be a triangle in $N(y,z)$ where all of the vertices x,y,z,a,b,c,u,v,w,r,s,t are distinct. Using these vertices we find 27 copies of C_6 that are all of the form $x\alpha y\beta z\gamma x$ because α can be any one of $\{a,b,c\}$, β can be any one of $\{r,s,t\}$, and γ can be any one of $\{u,v,w\}$. While we know that $\{x,y,z\}$ is independent and $\{x,y,z,\alpha,\beta,\gamma\}$ does not induce a $K_6 - K_3$, we do not know if $\{\alpha,\beta,\gamma\}$ is independent. In the case that $\{\alpha,\beta,\gamma\}$ is independent, we obtain the 6-cycle $x\alpha y\beta z\gamma x$ in two ways; once when we choose $\{x,y,z\} \in \mathcal{I}_3''(G)$ and again when we choose $\{\alpha,\beta,\gamma\} \in \mathcal{I}_3''(G)$. Dividing by two takes care of this over counting and so we obtain at least 27 copies of C_6 provided we divide this count by 2.

Combining Cases 1 through 3, we get at least

$$6|\mathcal{I}_3''(G)|$$

copies of C_6. Therefore, the number of C_6's in G is at least

$$6|\mathcal{I}_3'(G)| + 6|\mathcal{I}_3''(G)| = 6|\mathcal{I}_3(G)|.$$

■

Proof of Theorem 1.8. Let G be a graph that witnesses $\mathrm{sat}(n, C_6, K_5)$ with n vertices. First we claim that we may assume G has $O(n^{3/2})$ edges. Suppose that G contains at least $Cn^{3/2}$ edges for some large enough constant $C > 0$. A supersaturation result of Simonovits (see [10]) implies that there is a constant $c > 0$ such that G contains at least $c \cdot C^6 n^3$ many copies of C_6. Thus we can choose C large enough to obtain the desired lower bound on the number of C_6's. From now on, we assume G has $O(n^{3/2})$ edges, so the number of pairs $xy \notin E(G)$ is $\binom{n}{2} - C'n^{3/2}$ for some constant $C' > 0$.

Now using the Goodman bound on the number of triangles, we know that G has $\binom{n}{3}$ independent sets of size 3, asymptotically. If m is the number of non-edges in G, then

$$\begin{aligned}
\#(\{x,y,z\} : xy, zy, zx \notin E(G)) &\geq \frac{m(4m - n^2)}{3n} \\
&= \frac{(\binom{n}{2} - C'n^{3/2})(4\binom{n}{2} - 4C'n^{3/2} - n^2)}{3n} \\
&= \frac{n^4 - o(n^4)}{6n} \\
&= \frac{n^3}{6} - o(n^3).
\end{aligned}$$

By Lemma 3.6, the number of copies of C_6 in G is at least

$$6\left(\frac{n^3}{6} - o(n^3)\right) = (1 - o(1))n^3.$$

■

4 Cycles in C_k-saturated graphs

The focus of this section is cycles in C_k-saturated graphs. We begin with a few easy propositions.

Proposition 4.1 *Let $n \geq 2k + 2 \geq 4$ be integers. For any nonbipartite graph H,*

$$\mathrm{sat}(n, H, C_{2k+1}) = 0.$$

Proof. Let $n \geq 2k + 2$ and consider a complete bipartite graph where both parts have at least $k + 1$ vertices. This graph is H-free and C_{2k+1}-saturated. ■

Proposition 4.2 *(i) For any $n \geq t \geq k \geq 3$,*

$$\mathrm{sat}(n, C_t, C_k) = 0.$$

(ii) Let $r \geq 3$ be an integer. For any integer $m \geq 1$,

$$\mathrm{sat}(m(r-1)+1, C_t, C_k) = 0$$

whenever $t \geq r+1$ and $r+1 \leq k \leq 2r-1$.

Proof. First we prove (i). Given a positive integer $n \geq t$, write $n = 1 + q(k-2) + r$ where q and r are nonnegative integers with $r \in \{0, 1, \ldots, k-3\}$. Take q copies of K_{k-2} and one copy of K_r, and join every vertex in these complete graphs to a new vertex v. This graph is C_k-saturated, and has no cycle of length greater than $k-1$.

Next we prove (ii). Let $t \geq r+1$ and $r+1 \leq k \leq 2r-1$ where $r \geq 3$. Consider m copies of K_r identified on a single vertex v (the case when $r = 3$ is the Friendship Graph on $2m+1$ vertices and $3m$ edges). The longest cycle in this graph has length r. When an edge is added in this graph, we obtain a cycle of length k for each $k \in \{r+1, r+2, \ldots, 2r-1\}$. ■

In light of Propositions 4.1 and 4.2, we focus our attention on

$$\mathrm{sat}(n, C_t, C_k)$$

where $t < k$, and at least one of t or k is even. Our arguments used to prove upper bounds on $\mathrm{sat}(n, C_t, C_k)$ depend on t, and so we divide this section into some further subsections.

4.1 Triangles in C_4-saturated graphs

By Proposition 4.1,

$$\mathrm{sat}(n, K_3, C_{2k+1}) = 0$$

for all $k \geq 1$ and $n \geq 2k+2$.

The first nontrivial case that we consider is $\mathrm{sat}(n, K_3, C_4)$. Note that a C_4-saturated graph has diameter at most 3, otherwise adding an edge between a pair of vertices at distance more than 3 does not create a C_4. Both the 5-cycle and the Petersen graph have girth 5 and so are K_3-free and C_4-free. One can also check that these two graphs are C_4-saturated. Both are examples of Moore graphs, and the next proposition makes this connection between triangle-free C_4-saturated graphs and Moore graphs precise.

Proposition 4.3 *Let G be a triangle-free C_4-saturated graph. Then either G has diameter 3, or G is a Moore graph.*

Proof. Suppose that G has diameter 2 and let x and y be non-adjacent vertices and $v \in N(x) \cap N(y)$. Since G is C_4-saturated, there exists an x, y-path P of length 3. Since G is triangle free, the path P does not contain v. Therefore, Pv is a C_5 subgraph of G and the girth of G is 5. Since G has diameter 2 and girth 5, it is a Moore graph [25]. ■

The 5-cycle, the Petersen graph, and the Hoffman-Singleton graph are all examples of triangle-free C_4-saturated graphs with diameter 2. For several small values of n, there are n-vertex triangle-free C_4-saturated graphs with diameter 3. These were found by a computer search and show that

$$\text{sat}(n, K_3, C_4) = 0 \quad \text{for} \quad n \in \{8, 9, \dots, 24\}.$$

Miller and Codish [5] investigated extremal graphs of girth at least 5 and at most 32 vertices. They determined all graphs with n vertices, girth 5, and the maximum number of edges for n in the range $\{20, 21, \dots, 32\}$. We checked several of their extremal graphs. Some were C_4-saturated while others were not. For example, the unique extremal graph of girth 5 having 20 vertices is C_4-saturated. On the other hand, of the three extremal graphs of girth 5 with 21 vertices, none are C_4-saturated. All three extremal graphs on 22 vertices are C_4-saturated, and the largest girth 5 extremal graph found in [5], which has 32 vertices, is also C_4-saturated. These claims were verified using Mathematica [26].

We do not know if $\text{sat}(n, K_3, C_4) = 0$ for infinitely many n, and we were unable to show that $\text{sat}(n, K_3, C_4) > 0$ for infinitely many n. By taking a vertex of degree $n-1$ and putting a matching of size $\lfloor \frac{n-1}{2} \rfloor$ in its neighborhood, we obtain the upper bound

$$\text{sat}(n, K_3, C_4) \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Determining the behavior of this function is an intriguing open problem.

4.2 Triangles in C_k-saturated graphs ($k \geq 5$) and the proof of Theorem 1.9

The method we use here is to find a small K_3-free C_{2k}-saturated graph with a special set of vertices. The presence of this special set of vertices, which will be made precise in a moment, will allow us to clone vertices, yet maintain both the K_3-free property and the C_{2k}-saturated property.

Lemma 4.4 *Let $k \geq 5$, $d \geq 2$, and G be a C_k-saturated graph. Suppose that G contains d vertices $v_1, \dots, v_d$ such that v_i and v_j have the same neighborhood for all $1 \leq i, j \leq d$, and this neighborhood has size at most d. If G_u is the graph obtained by adding a new vertex u to G and making $N(u)$ the same as $N(v_1)$, then G_u is C_k-saturated.*

Proof. Suppose G is C_k-saturated and has d vertices $v_1, \dots, v_d$ with the same neighborhood $N(v_1)$, and $|N(v_1)| \leq d$. Let G_u be as in the statement of the lemma. If G_u contains a k-cycle C, then C must contain u. Let xu and uy be the unique edges of C that contain u. If a vertex v_i is on C, then two vertices in $N(v_i)$ are also on C. Since u is on C, at most $d-1$ of the vertices $v_1, \dots, v_d$ can be on C because $|N(v_1)| \leq d$. Without loss of generality, assume that v_d is not on C. Then we can replace xu and uy on C with xv_d and v_dy to get a k-cycle that is in G. This is a contradiction, so G_u must be C_k-free. To finish the proof, we must show that if w is a vertex that is not adjacent to u, then there is a path of length $k-1$ from w to u. If w is not adjacent to u, then w is not adjacent to v_1 and so there is a path P of length $k-1$ from w to v_1. We then remove v_1 from P and replace it with u to get a path of length $k-1$ from w to u. ■

Lemma 4.4 is very useful in proving upper bounds on $\text{sat}(n, K_3, C_{2k})$ because adding a vertex, in the way that is described in Lemma 4.4, will not create a triangle. Once we find a K_3-free C_{2k}-free graph on m vertices with a subset of vertices having the same neighborhood as in Lemma 4.4, we get $\text{sat}(n, K_3, C_{2k}) = 0$ for all $n \geq m$. The next step then is to construct a small K_3-free C_{2k}-saturated graph.

Let $k \geq 2$ be an integer. Let $G(4k)$ be the graph with $4k+2$ vertices whose vertex set is the disjoint union of five sets

$$\{v, u_1, u_2\} \cup X \cup Y \cup A \cup B$$

where $X = \{x_1, \ldots, x_k\}$, $Y = \{y_1, \ldots, y_k\}$, $A = \{a_1, \ldots, a_k\}$, $B = \{b_1, \ldots, b_{k-1}\}$, and

- v is a degree 2 vertex with neighbors u_1 and u_2,
- u_1 is adjacent to v and all vertices in X, and u_2 is adjacent to v and all vertices in A,
- every vertex in X is joined to every vertex in $\{u_1\} \cup A \cup Y$, and
- every vertex in A is joined to every vertex in $\{u_2\} \cup X \cup B$.

This completes the description of the first graph that is needed. Now we introduce the second graph which is similar.

Let $k \geq 1$ be an an integer. Let $G(4k+2)$ be the graph with $4k+4$ vertices whose vertex set, like $G(4k)$, is the disjoint union of five sets $\{v, u_1, u_2\} \cup X \cup Y \cup A \cup B$, except now $A = \{a_1, \ldots, a_k, a_{k+1}\}$ and $B = \{b_1, \ldots, b_{k-1}, b_k\}$. The adjacencies are defined in the same way as they were defined for $G(4k)$ so the edge sets are the same, except in $G(4k+2)$ we also join b_k to every vertex in A, and join a_{k+1} to every vertex in $\{u_2\} \cup X \cup B$.

Lemma 4.5 *(i) If $k \geq 2$ is an integer, then $G(4k)$ is K_3-free and C_{4k}-saturated.*

(ii) If $k \geq 1$ is an integer, then $G(4k+2)$ is K_3-free and C_{4k+2}-saturated.

Proof. (i) First we show that for each nonedge e of $G(4k)$, when e is added to $G(4k)$ there is a $4k$-cycle that contains e. There are several cases to consider and it will be extremely useful to introduce some notation to make the argument concise. Suppose that

1. $\alpha_1\alpha_2$ is a nonedge of $G(4k)$,
2. $C = \alpha_1\alpha_2\alpha_3\alpha_4 \ldots \alpha_{4k}\alpha_1$ is a $4k$-cycle in the graph obtained by adding $\alpha_1\alpha_2$ to $G(4k)$,
3. β_1 and β_2 are the unique pair of vertices in $G(4k)$ not on C.

In this case, we will write

$$\alpha_1\alpha_2 : \alpha_1\alpha_2\alpha_3\alpha_4\ldots\alpha_{4k}\alpha_1, \qquad \{\beta_1,\beta_2\}$$

Now we use this notation and list $4k$-cycles we get when adding nonedges to $G(4k)$.

Starting with nonedges on v,

$va_1 :\ va_1b_1a_2b_2a_3\ldots b_{k-1}a_kx_ky_{k-1}x_{k-1}y_{k-2}\ldots y_1x_1u_1v, \quad \{y_k,u_2\}$

$vb_1 : vb_1a_1x_1y_1x_2y_2\ldots y_{k-1}x_ka_kb_{k-1}a_{k-1}b_{k-2}\ldots b_2a_2u_2v, \quad \{y_k,u_1\}$

$vx_1 : vx_1y_1x_2y_2x_3\ldots x_{k-1}y_{k-1}x_ka_kb_{k-1}a_{k-1}b_{k-2}\ldots a_2b_1a_1u_2v, \quad \{y_k,u_1\}$

$vy_1 : vy_1x_2y_2x_3\ldots x_{k-1}y_{k-1}x_ka_kb_{k-1}a_{k-1}b_{k-2}\ldots b_2a_2b_1a_1x_1u_1v, \quad \{y_k,u_2\}$

This covers all possible missing edges on v and we no longer check missing edges that contain v. Moving on to those containing u_1, the possibilities we must consider are adding the nonedges u_1z where $z\in\{u_2\}\cup A\cup B\cup Y$.

$u_1u_2 :\ u_1u_2a_1b_1a_2b_2a_3\ldots b_{k-1}a_kx_ky_{k-1}x_{k-1}y_{k-2}\ldots y_1x_1u_1, \quad \{y_k,v\}$

$u_1a_1 :\ u_1a_1b_1a_2b_2\ldots a_{k-2}b_{k-2}a_{k-1}x_1y_1x_2y_2\ldots x_{k-1}y_{k-1}x_ka_ku_2vu_1, \quad \{y_k,b_{k-1}\}$

$u_1b_1 :\ u_1b_1a_1u_2a_2b_2a_3b_3\ldots b_{k-1}a_kx_ky_{k-1}x_{k-1}y_{k-2}\ldots y_1x_1u_1, \quad \{y_k,v\}$

$u_1y_1 :\ u_1y_1x_2y_2x_3y_3\ldots y_{k-1}x_ka_kb_{k-2}a_{k-1}b_{k-3}\ldots a_3b_1a_2x_1a_1u_2vu_1, \quad \{y_k,b_{k-1}\}$

This covers all possible missing edges on u_1, and we no longer check missing edges that contain u_1 or v. Concerning missing edges on u_2, we must check nonedges of the form u_2z with $z\in X\cup Y\cup B$.

$u_2x_1 :\ u_2x_1y_1x_2y_2x_3y_3\ldots y_{k-2}x_{k-1}a_1b_1a_2b_2\ldots a_{k-1}b_{k-1}a_kx_ku_1vu_2, \quad \{y_{k-1},y_k\}$

$u_2y_1 :\ u_2y_1x_ky_{k-1}x_{k-1}y_{k-2}\ldots y_2x_2u_1x_1a_kb_{k-1}a_{k-1}b_{k-2}\ldots b_2a_2b_1a_1u_2, \quad \{y_k,v\},$

$u_2b_1 :\ u_2b_1a_1x_1a_2b_2a_3b_3\ldots a_{k-1}b_{k-1}a_kx_2y_1x_3y_2x_4\ldots y_{k-2}x_ku_1vu_2, \quad \{y_{k-1},y_k\}$

The remaining nonedges all have their endpoints in $A\cup B\cup X\cup Y$. A careful check shows that the list below covers the remaining cases.

$x_1b_1 :\ x_1b_1a_2b_2a_3b_3\ldots b_{k-1}a_kx_ky_{k-2}x_{k-1}y_{k-3}x_{k-2}\ldots y_2x_3y_1x_2a_1u_2vu_1x_1, \quad \{y_{k-1},y_k\}$

$x_1x_2 :\ x_1x_2y_3x_3y_4x_4\ldots y_kx_ku_1vu_2a_1b_1a_2b_2\ldots b_{k-1}a_kx_1, \quad \{y_1,y_2\}$

$a_1a_2 :\ a_1a_2b_1a_3b_2a_4\ldots b_{k-3}a_{k-1}b_{k-2}a_kx_ky_{k-1}x_{k-1}\ldots y_2x_2y_1x_1u_1vu_2a_1, \quad \{y_k,b_{k-1}\}$

$a_1y_1 :\ a_1y_1x_2y_2x_3y_3\ldots x_{k-1}y_{k-1}x_ka_ku_2vu_1x_1a_{k-1}b_{k-2}a_{k-2}b_{k-3}\ldots b_1a_1, \quad \{y_k,b_{k-1}\}$

$b_1b_2 :\ b_1b_2a_2b_3a_3b_4\ldots b_{k-1}a_{k-1}x_ka_ku_2vu_1x_{k-1}y_{k-2}x_{k-2}y_{k-3}\ldots y_2x_2y_1x_1a_1b_1, \{y_{k-1},y_k\}$

$y_1y_2 :\ y_1y_2x_3y_3x_4\ldots x_{k-1}y_{k-1}x_ka_kb_{k-2}a_{k-1}b_{k-3}\ldots b_2a_3b_1a_2x_1u_1vu_2a_1x_2y_1, \{y_k,b_{k-1}\}$

$b_1y_1 :\ b_1y_1x_1y_2x_2\ldots y_{k-1}x_{k-1}u_1x_ka_kb_{k-1}a_{k-1}b_{k-2}\ldots a_3b_2a_2u_2a_1b_1, \quad \{v,y_k\}$

We finish the proof of (i) by showing that $G(4k)$ is C_{4k}-free. Suppose, for contradiction, that C is a $4k$-cycle in $G(4k)$. Observe that any cycle of length more than $2k$ cannot contain all vertices in Y, because the only cycles in $G(4k)$ that contain all vertices in Y are cycles of length $2k$ having k vertices in X and k vertices in Y. Without loss of generality, assume y_k is not on C, and let

$$Y' = Y\backslash\{y_k\}.$$

If u_1 is not on C, then v cannot be on C, but then C has less than $4k$ vertices. Thus, u_1 is on C and similarly, u_2 is also on C. We consider two cases depending on whether or not v is on C.

If v is not on C, then C must contain all vertices in $\{u_1, u_2\} \cup A \cup B \cup X \cup Y'$. To contain all vertices in Y', C must have a subpath of length $2k-1$ that starts and ends in X, and alternates between vertices in X and Y'. By relabeling vertices if necessary, we may assume that

$$P = x_1 y_1 x_2 y_2 \dots x_{k-1} y_{k-1} x_k$$

is this subpath. Now u_1 is on C but v is not on C, and the only neighbors of u_1 in the union $\{u_1, u_2\} \cup A \cup B \cup X \cup Y'$ are vertices in X. Therefore, x_1 and x_k must be joined to u_1 on C. This is a contradiction as adding u_1 to the endpoints of P closes the cycle C before it touches vertices in A. We conclude that v is on C, and so u_1 and u_2 must also be on C. The rest of the vertices on C are all but one vertex in $A \cup B \cup X \cup Y'$. Since $|A| = |X|$ and $|B| = |Y'|$, the sets X, Y' and A, B are symmetric. We may assume, without loss of generality that C contains all vertices in Y', otherwise if C misses a vertex in Y' and B, then C has at most $4k-1$ vertices. As C contains all vertices in Y', C contains the subpath

$$P_1 = a_1 u_2 v u_1 x_1 y_1 x_2 y_2 \dots x_{k-1} y_{k-1} x_k a_2$$

where we relabel vertices within the sets A, X, and Y as needed. The path P_1 contains all vertices in X and the edges touching u_1 and u_2, so all of the other edges of C that are not in P_1 must have one endpoint in A and the other in B. If P_2 is the subpath of C from a_2 to a_1 that is different from P_1, then P_2 must contain an even number of edges as it starts and ends in A, and alternates between vertices in A and B. This is a contradiction however since P_1 has $2k+3$ edges.

We have shown that no $4k$-cycle exists in $G(4k)$ so $G(4k)$ is C_{4k}-free.

(ii) Now we show that $G(4k+2)$ is C_{4k+2}-saturated. Recall that $G(4k+2)$ is almost the same as $G(4k)$, except A and B contain one more vertex each so $A = \{a_1, \dots, a_k, a_{k+1}\}$, and $B = \{b_1, \dots, b_{k-1}, b_k\}$. We use the same notation as used in (i), and we give our list of nonedges and $(4k+2)$-cycles containing the nonedges here:

$va_1 :\ va_1 u_2 a_2 b_1 a_3 b_2 \dots a_{k-1} b_{k-2} a_k b_{k-1} a_{k+1} x_k y_{k-1} x_{k-1} y_{k-2} \dots x_2 y_1 x_1 u_1 v, \quad \{b_k, y_k\}$

$vb_1 : vb_1 a_1 b_2 a_2 \dots b_{k-1} a_{k-1} b_k a_k x_k y_{k-1} x_{k-1} y_{k-2} \dots x_2 y_1 x_1 a_{k+1} u_2 v, \quad \{u_1, y_k\}$

$vx_1 : vx_1 y_1 x_2 y_2 \dots y_{k-1} x_k a_{k+1} b_k a_k b_{k-1} \dots a_2 b_1 a_1 u_2 v, \quad \{u_1, y_k\}$

$vy_1 : vy_1 x_1 a_1 b_1 a_2 b_2 \dots a_{k-1} b_{k-1} a_k b_k a_{k+1} x_2 y_2 x_3 y_3 \dots x_{k-1} y_{k-1} x_k u_1 v, \quad \{u_2, y_k\}$

$u_1 u_2 :\ u_1 u_2 a_1 b_1 a_2 b_2 \dots a_{k-1} b_{k-1} a_k b_k a_{k+1} x_1 y_1 x_2 y_2 \dots x_{k-1} y_{k-1} x_k u_1, \quad \{v, y_k\}$

$u_1 a_1 :\ u_1 a_1 x_1 y_1 x_2 y_2 \dots x_{k-1} y_{k-1} x_k a_{k+1} b_k a_k b_{k-1} a_{k-1} \dots a_3 b_2 a_2 u_2 v u_1, \quad \{y_k, b_1\}$

$u_1 b_1 :\ u_1 b_1 a_1 b_2 a_2 b_3 \dots a_{k-2} b_{k-1} a_{k-1} b_k a_k u_2 a_{k+1} x_1 y_1 x_2 y_2 \dots y_{k-1} x_k u_1, \quad \{y_k, v\}$

$u_1 y_1 :\ u_1 y_1 x_1 y_2 x_2 y_3 \dots x_{k-1} y_k x_k a_1 b_1 a_2 b_2 \dots a_{k-1} b_{k-1} a_k u_2 v u_1, \quad \{a_{k+1}, b_k\}$

$u_2 x_1 :\ u_2 x_1 a_1 b_1 a_2 b_2 \dots a_k b_k a_{k+1} x_2 y_1 x_3 y_2 \dots x_{k-1} y_{k-2} x_k u_1 v u_2, \quad \{y_{k-1}, y_k\}$

$u_2y_1:\quad u_2y_1x_1u_1x_2y_2x_3y_3\dots x_{k-1}y_{k-1}x_ka_1b_1a_2b_2\dots a_kb_ka_{k+1}u_2,\quad \{y_k,v\}$

$u_2b_1:\quad u_2b_1a_1b_2a_2\dots b_ka_kx_1y_1x_2y_2\dots x_{k-1}y_{k-1}x_ku_1vu_2,\quad \{a_{k+1},y_k\}$

$x_1b_1:\quad x_1b_1a_1b_2a_2b_3a_3\dots b_ka_ku_2vu_1x_2y_1x_3y_2\dots x_{k-1}y_{k-2}x_ky_{k-1}x_1,\quad \{y_k,a_{k+1}\}$

$x_1x_2:\quad x_1x_2y_1x_3y_2x_4y_3\dots y_{k-2}x_ka_1b_1a_2b_2\dots a_kb_ka_{k+1}u_2vu_1x_1,\quad \{y_k,y_{k-1}\}$

$a_1a_2:\quad a_1a_2b_1a_3b_2a_4\dots a_kb_{k-1}a_{k+1}x_1y_1x_2y_2\dots x_{k-1}y_{k-1}x_ku_1vu_2a_1,\quad \{b_k,y_k\}$

$a_1y_1:\quad a_1y_1x_1y_2x_2\dots y_kx_ku_1vu_2a_2b_1a_3b_2\dots a_kb_{k-1}a_1,\quad \{a_{k+1},b_k\}$

$b_1b_2:\quad b_1b_2a_1b_3a_2b_4\dots a_{k-2}b_ka_{k-1}x_1y_1x_2y_2\dots x_{k-1}y_{k-1}x_ku_1vu_2a_kb_1,\quad \{a_{k+1},y_k\}$

$y_1y_2:\quad y_1y_2x_1y_3x_2\dots y_kx_{k-1}u_1vu_2a_1b_1a_2b_2\dots a_{k-1}b_{k-1}a_kx_ky_1,\quad \{b_k,a_{k+1}\}$

$b_1y_1:\quad b_1y_1x_1y_2x_2y_3\dots y_{k-1}x_{k-1}u_1x_ka_{k+1}u_2a_kb_ka_{k-1}b_{k-1}\dots a_1b_1,\quad \{v,y_k\}$

To show $G(4k+2)$ is C_{4k+2}-free, we again use proof by contradiction. Suppose C is a $(4k+2)$-cycle in $G(4k+2)$. If C contains a subpath of the form aba' where $a,a'\in A$ and $b\in B$, then by replacing aba' with a, we can obtain a cycle of length $4k$ in $G(4k)$ which we have already shown is impossible. This means that C cannot contain any vertices in B, and we also know that C must miss at least one vertex in Y (the same argument used to show this for $G(4k)$ applies here as well since $|X|=|Y|$ in $G(4k+2)$). Thus, $k=|B|\le 1$. It is then easy to check that $G(6)$ is C_6-free. ■

We now have all of the lemmas needed to prove Theorem 1.9.

Proof of Theorem 1.9. By the comments preceding the statement of Theorem 1.9, we only need to consider cycles of even length. That is, we must show that for all $n\ge 2k+2\ge 6$, there is a K_3-free C_{2k}-saturated graph on n vertices. By Lemma 4.4, it is enough to find a K_3-free C_{2k}-saturated graph with a set of $d\ge 2$ vertices having the same neighborhood whose size is at most d. When k is even, say $k=2r$, then the graph $G(4r)$ has $4r+2$ vertices and is K_3-free and C_{2k}-saturated by Lemma 4.5. The vertices in Y form a set of k vertices that all have the same neighborhood X which has size k. By Lemma 4.4, we may duplicate vertices in y as many times as needed to obtain a K_3-free C_{2k}-saturated graph on $n\ge 2k+2$ vertices. When k is odd, say $k=2r+1$, the same argument applies except we use the graph $G(4r+2)$. ■

4.3 4-cycles in C_k-saturated graphs, $k>6$

In this subsection, we consider how many C_4's must be in a C_k-saturated graph. We will assume that $k\ge 5$ throughout. Our approach does not use Lemma 4.4 because when a vertex is duplicated, we will create new C_4's. Instead, we use the idea of C_k-builders introduced by Barefoot et. al. [4]. A graph G is a C_k-*builder* if G is C_k-saturated, and there is a distinguished vertex v in G such that if v in one copy of G is identified with v in the other copy of G, then the resulting graph is C_k-saturated. Barefoot et. al. [4] use C_k-builders to obtain upper bounds on $\mathrm{sat}(n,C_k)$ for different values of k. They were also used by Zhang, Luo, and Shigno [27] in the special case $k=6$.

If G is a C_k-builder with distinguished vertex v and G is C_4-free, then the graph obtained by taking two copies of G and identifying v is C_k-saturated and C_4-free. This observation, like in the case of $\mathrm{sat}(n, K_3, C_{2k})$, allows us to reduce the problem to finding a small C_4-free C_k-builder. If G is a C_k-builder with distinguished vertex v, then for any ordered pair of vertices (u_1, u_2) where $u_1 \neq v$ and $u_2 \neq v$, there must be positive integers k_1, k_2 with $k_1 + k_2 = k - 1$ and u_i is joined to v by a path of length k_i $(i = 1, 2)$. We generalize this observation to two different builders in the next lemma.

Lemma 4.6 *Let m_1 and m_2 be positive integers. Let G_1 and G_2 be C_k-builders with distinguished vertices v_1 and v_2, respectively. Suppose for every ordered pair of vertices $(u, w) \in (G_1 \backslash v_1) \times (G_2 \backslash v_2)$ there is a path of length k_1 from u to v_1 in G_1, and a path of length k_2 from w to v_2 in G_2 with $k_1 + k_2 = k - 1$. If G is the graph obtained by taking m_1 copies of G_1 and m_2 copies of G_2 and identifying each of the m_1 copies of v_1 and the m_2 copies of v_2 all into a single vertex v, then G is C_k-saturated and has*

$$m_1(|V(G_1)| - 1) + m_2(|V(G_2)| - 1) + 1$$

vertices. Furthermore, if each of G_1 and G_2 are H-free where H is a graph with no cut vertex, then G is also H-free.

Proof. Let G be the graph described in the lemma. It is clear that G has

$$m_1(|V(G_1)| - 1) + m_2(|V(G_2)| - 1) + 1$$

vertices. Consider now a pair of nonadjacent vertices x and y in G. If this pair belongs to the same copy of some G_i, $i = 1$ or $i = 2$, then they are joined by a path of length $k - 1$ since G_i is C_k-saturated. If x and y are in different copies of G_i, then since G_i is a C_k-builder, there is a path of length k_1 from x to v in the copy of G_i that contains x, and a path of length k_2 from y to v in the copy of G_i that contains y, where $k_1 + k_2 = k - 1$. Finally, assume that x is in a copy of G_1 and y is in a copy of G_2. Then, by hypothesis, there is a path of length $k - 1$ from x to y that uses the vertex v.

Lastly, since v is a cut-vertex, any copy of H in G must be contained in some copy of G_1 or G_2. ∎

Let us call a pair of C_k-builders satisfying the conditions of Lemma 4.6 *compatible*.

Lemma 4.7 *Let H be a graph with no cut vertex and $k \geq 3$ be an integer. If G_1 and G_2 are H-free C_k-builders that are compatible and $|V(G_1)| - 1$ is relatively prime to $|V(G_2)| - 1$, then*

$$\mathrm{sat}(n, H, C_k) = 0$$

for all $n \geq n_0$ where n_0 depends only on $|V(G_1)|$ and $|V(G_2)|$.

Proof. By Lemma 4.6, the graph obtained by identifying the distinguished vertices in m_1 copies of G_1 and m_2 copies of G_2 into a single vertex is C_k-saturated. This graph is also H-free since each of the builders G_1 and G_2 are H-free, so no copy of H is contained in a single copy of a builder. If we find a copy of H whose vertices are in more than one builder, then H contains a cut vertex which is not possible. Therefore, we have an H-free C_k-saturated graph on

$$m_1(|V(G_1)| - 1) + m_2(|V(G_2)| - 1) + 1 \tag{4}$$

vertices. Since $|V(G_1)| - 1$ and $|V(G_2)| - 1$ are relatively prime, all sufficiently large positive integers can be written in the form (4) for some nonnegative integers m_1 and m_2. ■

Proof of Theorem 1.10. By Lemma 4.7, it is enough to find compatible C_4-free C_k-builders such that the respective number of vertices minus one are coprime. The adjacency matrices of C_4-free compatible C_k-builders for $k \in \{7, 8, 9, 10\}$ are given in the appendix. The computations establishing that the corresponding graphs have the needed properties was done using Mathematica [26]. ■

Remark: The lower bound on n in Theorem 1.10 comes from the number of vertices in the compatible C_k-builders. The worst case is $k = 10$ where our builders have 12 and 13 vertices. A short computation shows that every integer $n \geq 111$ can be written in the from $1 + 11m_1 + 12m_2$ for some nonnegative integers m_1 and m_2. As mentioned in the introduction, we have verified computationally that $\text{sat}(n, K_3, C_7) = 0$ and $\text{sat}(n, K_3, C_8) = 0$ for the cases not covered by Theorem 1.10 [26].

4.4 4-cycles in C_6-saturated graphs

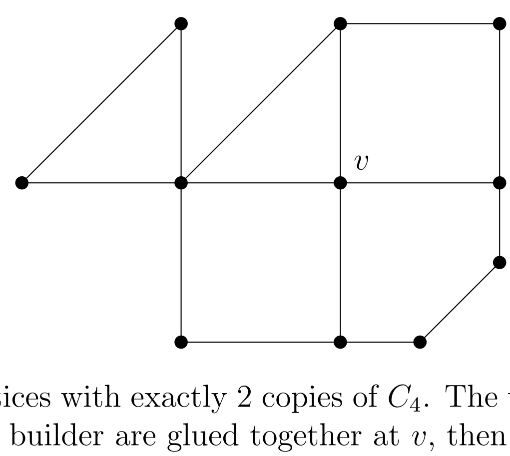
In this subsection we discuss 4-cycles in C_6-saturated graphs. Like $\text{sat}(n, K_3, C_4)$, we were unable to show that $\text{sat}(n, C_4, C_6) > 0$ for infinitely many n. Using a computer search, we were able to find C_4-free C_6-saturated graphs for $n \in \{14, 15, 18, 20, 22, 26\}$. The graphs on 26 vertices that are C_4-free and C_6-saturated are two of the three 3-regular graphs of girth 7 [20]. The Coxeter graph on 28 vertices is also has girth 7 and is C_6-saturated.

Using a C_6-builder with 11 vertices and exactly two copies of C_4, we can prove the following upper bound on $\text{sat}(n, C_4, C_6)$.

Theorem 4.8 *If $t \geq 1$ is an integer, then*

$$\text{sat}(10t + 1, C_4, C_6) \leq 2t.$$

Proof. The graph



is a C_6-builder on 11 vertices with exactly 2 copies of C_4. The vertex v is a distinguished vertex. If t copies of this builder are glued together at v, then we obtain a C_6-saturated graph on $10t + 1$ vertices with $2t$ triangles. The computations that show this graph is a C_6-builder (and has 2 copies of C_4) may be found in [26]. ■

5 Open Problems

We end with some open problems. First, when H and F were both cliques we were not able to determine the function $\text{sat}(n, H, F)$ except when counting triangles in a K_4-saturated graph. We believe that the natural construction giving the upper bound in Theorem 1.2 is correct.

Problem 5.1 *Let $s > r \geq 3$ be integers. Determine the exact value of* $\text{sat}(n, K_r, K_s)$.

One of the most intriguing questions for us was counting triangles in C_4-saturated graphs. In Section 4.1 we showed that $\limsup_{n\to\infty} \text{sat}(n, K_3, C_4) \leq \frac{n-1}{2}$, but we could not show that $\liminf_{n\to\infty} \text{sat}(n, K_3, C_4)$ is positive.

Problem 5.2 *Determine if* $\text{sat}(n, K_3, C_4)$ *is positive for infinitely many n.*

We ask the same question when counting copies of C_4 in a C_6-saturated graph.

Problem 5.3 *Determine if* $\text{sat}(n, C_4, C_6)$ *is positive for infinitely many n.*

Finally, we focused on graphs which are either C_k-saturated or K_s-saturated. It would be interesting to consider other nontrivial combinations of graphs H and F, for example when one of them is a tree.

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