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Research Statement

My research uses algebraic and geometric methods to prove theorems in extremal combinatorics. Going the other way, I also use combinatorial methods to prove algebraic results. Algebraic methods are deeply embedded in my work and nearly all of my success in graph theoretic research has come from attacking purely combinatorial problems through the lens of algebra, combinatorial number theory, or finite geometry. Conversely, my work in graph theory has allowed me to understand algebraic structure through combinatorial methods, yielding theorems in the same areas of algebra, combinatorial number theory, and finite geometry. As an overview, being able to work in the intersection of several different areas of mathematics has been fruitful and exciting. Interesting and open problems abound, and in this document I will describe a selection of problems that I have worked on and plan to work on in the future. Section 4 contains several specific problems that would make good projects for undergraduate or graduate students.

1 Introduction

The bulk of my work lies in three main areas: extremal graph theory, combinatorial number theory, and spectral graph theory. There are rich connections betweens these areas which are detailed below.

The basic question in *extremal graph theory* is to determine the maximum number of edges in a graph which does not contain a fixed forbidden subgraph. If the forbidden subgraph is F, the maximum number of edges in an F-free graph is denoted by $\mathrm{ex}(n, F)$ and is called the *Turán number* for F [67, 98, 147]. The interesting cases occur when the graph F is bipartite, and in this case very little is known. However, *all* of the best lower bounds come from either algebra or geometry. Further, there are some theorems and conjectures that say that in certain cases any extremal graph *must* come from an algebraic construction. That is, not only are algebraic and geometric constructions useful for the Turán problem, in some cases they are essential. Broadly speaking, my research in extremal graph theory is utilizing tools from algebra and geometry to answer combinatorial questions.

My research in *combinatorial number theory* does the opposite. For these problems, I use combinatorial arguments and graph theoretic techniques to prove theorems in number theory or algebra. Given an algebraic structure with addition and multiplication, theorems in this area give us information about the two operations or how they interact. For example, given a subset A, we may study its *sum set* $A + A = \{a + b : a, b \in A\}$ or its *product set* $A \cdot A = \{a \cdot b : a, b \in A\}$. We may be interested in how large these sets are or what structure they are required to have. In my work, I have been able to use combinatorial properties of these objects to prove algebraic results. Results like these have precedence, e.g. the Green-Tao Theorem [79] was proved using mostly "just" probabilistic combinatorics.

The third main area of my research is in *spectral graph theory*. The goal of this area is to associate a matrix with a graph and then to deduce properties of the graph from the eigenvalues or eigenvectors of the matrix. The strength of this area is that we are able to use linear algebra in surprising ways to prove graph theoretic results. My work is mostly in maximizing some function of the eigenvalues or eigenvectors of a matrix associated to a graph over a fixed family of graphs. By introducing linear algebra, theorems like these are often able to strengthen classical extremal graph theory theorems.

In the remainder of this document, I will describe in more detail the problems that I have worked on and where I think my research will go from here. I will give several specific goals which I hope to accomplish in the next few years, and I will note which of these projects are most suitable for collaboration with undergraduate or graduate students.

2 Extremal graph theory and combinatorial number theory

Problems in Turán theory ask how many edges may be in a graph that does not contain a fixed family of forbidden subgraphs. Many problems in extremal combinatorics can be phrased in this setting, and so Turán-type problems have become foundational in combinatorics. First examples of theorems in this area include Mantel's theorem [121] that any n-vertex graph with more than $n^2/4$ edges must contain a triangle or, earlier, the folklore result that any n-vertex graph with more than $3n-6$ edges cannot be planar (i.e. it must contain a K_5 minor or a $K_{3,3}$-minor). Turán's theorem [162] solved this problem when the forbidden subgraph is complete in 1941 and the celebrated Erdős-Stone theorem [49, 51] from 1946 gives an asymptotic solution when the forbidden subgraph has chromatic number at least 3.

However, much less is known when forbidding a bipartite graph, and in general this problem is notoriously difficult (for surveys, see [67, 147]). Writing $\mathrm{ex}(n,H)$ for the maximum number of edges an n vertex graph may have without containing H, the most well-studied bipartite Turán problems are determining $\mathrm{ex}(n,C_{2k})$ and $\mathrm{ex}(n,K_{s,t})$. For the first problem, we know the order of magnitude only for $\mathrm{ex}(n,C_4)$, $\mathrm{ex}(n,C_6)$, and $\mathrm{ex}(n,C_{10})$ [16, 23, 48], and determining the behavior for general even cycles is called the even-cycle problem [45, 23, 136]. For the latter, we understand the behavior of $\mathrm{ex}(n,K_{2,t})$ and $\mathrm{ex}(n,K_{s,t})$ for $t>(s-1)!$ [7, 26, 61, 100]. General upper bounds are known [63, 104, 167] for both problems, but we are lacking constructions.

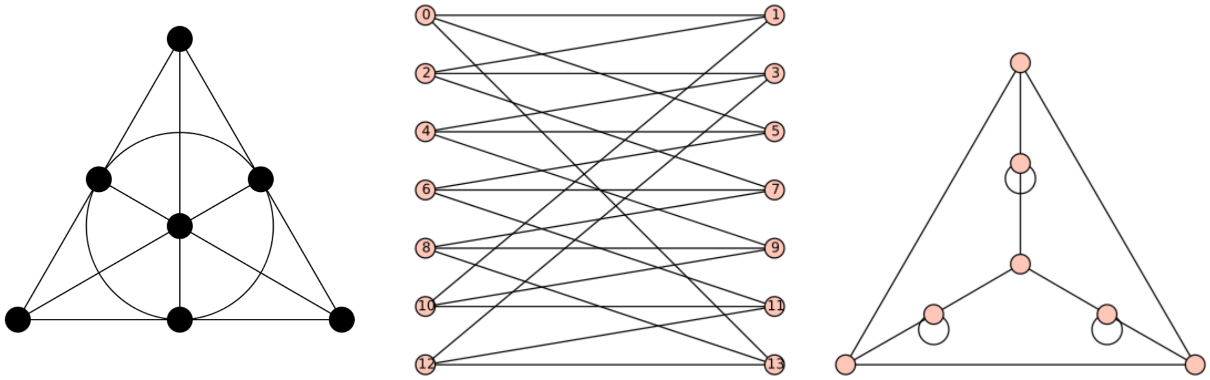
The constructions that we do have are all algebraic or geometric in nature. In all cases where the order of magnitude is known, a construction can be made by considering vertices as points in a vector space over a finite field and edges defined by points on a variety. Other constructions meeting the upper bounds are obtained by taking a subset of a group that has nice additive structure and creating a Cayley graph. It is not clear that these constructions are fundamentally different (see the discussion below Research Goal 3).

This part of my research plan has 3 fundamental goals, described below.

Research Goal 1. *Discover new algebraic constructions for extremal graphs.*

I have two primary avenues to attack this goal. The first is the recently developed "random polynomial method", invented by Blagojević, Bukh, and Karasev [19]. In this method, one defines a random graph where edges are formed if a certain system of polynomials over a finite field vanishes, where the polynomials are chosen in a suitable random way. One uses tools from algebraic geometry to analyze what subgraphs are likely to appear or not appear in this random graph. Bukh and Conlon [28] used this method to answer a longstanding open conjecture that for every rational $r\in(1,2)$ there is a family of graphs with Turán number $\Theta(n^r)$. Conlon [35], Bukh and I [29], and He and I [94] also used this method with success in attacking a generalization of the even-cycle problem, the study of the Turán number for theta graphs. Ma, Yuan, and Zhang [120] used the method to study Turán numbers for hypergraphs. I plan to find other situations in which this new method can be used.

The second approach is to find explicit equations that may define our extremal graphs. All of the best-performing constructions for extremal graphs can be rewritten in a way such that the edges are defined by points on a variety over a finite field. For example, the affine part of the celebrated Erdős-Rényi orthogonal polarity graph which is extremal for C_4 [48] has vertex set $\mathbb{F}_q^2$ and edge set $E=\{(x,y)\in\mathbb{F}_q^2\times\mathbb{F}_q^2 : x_1+y_1=x_2y_2\}$. Graphs defined in a similar manner are called *algebraically defined graphs* (see [107] for a survey). This family of graphs is far too large to search effectively for extremal graphs. One way to narrow the search is via a method of Woldar [166]. Lazebnik, Ustimenko, and Woldar [108, 109] currently have the densest construction of graphs with girth $2k+2$, and their construction was derived from Lie algebras. Woldar suggested studying the subset of algebraically defined graphs which are found in this way as a natural way to pare down the search space. Terlep and Williford [158] used this technique to find the densest known C_{14} free graphs. Not many people have the expertise to approach the problem in this way, and so the family remains mostly unsearched with many nice graphs waiting to be discovered [165].



A general theme of this line of research is that algebraic or geometric structure yields interesting combinatorial properties and interesting graphs

Related Work
Michael Tait and Craig Timmons. Sidon sets and graphs without 4-cycles. *Journal of Combinatorics*, Volume 5, Issue 2, 155 –165 (2014).

Michael Tait and Craig Timmons. Small dense subgraphs of polarity graphs and the extremal number for the 4-cycle. *The Australasian Journal of Combinatorics*, Volume 63 (1), 107–114, (2015).

Michael Tait. Degree Ramsey numbers of even cycles, *Discrete Mathematics*, 34(1), 104–108, (2017).

Boris Bukh and Michael Tait, On the Turán number for theta graphs, submitted.

Michael Tait and Craig Timmons, The Zarankiewicz problem in 3-partite graphs, submitted.

Research Goal 2. *Development of hypergraph Turán theory.*

In contrast to extremal graph theory, our knowledge of extremal hypergraph theory is severely lacking. For example, Turán proved his theorem in 1941, and yet we still do not know the asymptotics of how many hyperedges may be in a 3-uniform graph with no complete graph on 4 vertices [98, 140, 139]. These problems in general are out of our current reach. However, when one is able to put some graph structure on the problem then it becomes more tractable. Given a family of hypergraphs $\mathcal{F}$ we define $\mathrm{ex}_r(n, \mathcal{F})$ to be the maximum number of edges in an r-uniform graph which does not contain any hypergraph in $\mathcal{F}$.

One way that hypergraph cycles can resemble those in graphs is when considering them in the Berge sense. A *Berge cycle* is a sequence $v_1, \cdots, v_k$ of distinct vertices and a sequence $e_1, \cdots, e_k$ of distinct hyperedges where $v_i, v_{i+1} \in e_i$ for all i (with subscripts taken modulo k). When the hyperedges have size 2 this definition is equivalent to a cycle in a graph, but in general there are many non-isomorphic Berge cycles of length k. Since these cycles inherit some graph structure, the Turán problem for them is tractable and several authors [34, 41, 64, 66, 88, 89, 110] proved theorems resembling those for graphs. Gerbner and Palmer [75] generalized the notion of a Berge cycle to other graphs. A hypergraph H is said to be *Berge-F* if there is a bijection $\phi : E(F) \to E(H)$ such that $e \subset \phi(e)$ for all $e \in E(G)$. We abuse notation and write Berge-F for the family of hypergraphs which are a Berge-F. The Turán number for the family Berge-F for various graphs F has experienced an explosion of recent study [10, 12, 52, 73, 81, 86, 87, 92, 134, 161].

We also mention two other related problems. First, for a graph F, the *expansion* of F is a particular hypergraph in the family Berge-F. Turán numbers for expansions have also been heavily studied lately

[101, 103, 102, 124, 137]. Second given graphs G and F the generalized Turán number $\mathrm{ex}(n, G, F)$ is the maximum number of copies of G that an n-vertex F-free graph may have. Taking $G = K_2$ gives back the ordinary Turán number. This function was considered for particular combinations of G and F by many researchers [21, 44, 82, 90, 91, 93] before being systematically studied first by Alon and Shikhelmen [8]. Many papers have been written in the last 2 years since they proposed this investigation [5, 6, 9, 53, 65, 70, 71, 72, 74, 76, 83, 118, 119, 120, 133, 134, 143].

The abundance of papers on these topics in the last few years is evidence that the community is excited about new ways to make progress on hypergraph Turán problems. As part of my research, I will attempt to extend all of the algebraic techniques that have worked so well in graph theory to the hypergraph setting. I have already had partial success in [43, 94, 134]. Since this area is so new, this is the part of my research plan that has the highest chance of success. It is clear that many more papers on this subject will be written by the community in the next several years.

Related Work

Cory Palmer, Michael Tait, Craig Timmons, and Adam Wagner, Turán numbers for Berge-hypergraphs and related extremal problems, to appear in *Discrete Mathematics*.

Sean English, Dániel Gerbner, Abhishek Methuku, and Michael Tait, Linearity of Saturation for Berge Hypergraphs, submitted.

Sunny He[1] and Michael Tait, Hypergraphs with few Berge paths of fixed length between vertices, submitted.

Research Goal 3. *Form connections between finite geometry and combinatorial number theory.*

Both finite geometry and combinatorial number theory provide constructions in extremal graph theory. As a long term goal, I plan to investigate the connection between objects in finite geometry and combinatorial number theory. Some of these connections are already known. For example, graphs from projective planes can be used to prove sum-product estimates over finite fields [148, 163] or other settings [135, 159].

A *Sidon set* A in an abelian group has the property that if $a + b = c + d$ with $a, b, c, d \in A$, then $\{a, b\} = \{c, d\}$. It has long been known that both projective planes and Sidon sets are excellent objects to construct C_4-free graphs [1, 26, 48, 62, 152]. Are these constructions the same? Timmons and I showed [153] that the Cayley sum graph constructed using a Bose-Chowla Sidon set is isomorphic to a large induced subgraph of the Erdős-Rényi orthogonal polarity graph. One can show [160] that the same conclusion is true if instead of a Bose-Chowla Sidon set one uses Cilleruelo's Sidon set $\{(x, x^2) : x \in \mathbb{F}_q^2\}$ [32], or Ruzsa's multiplicative Sidon set $\{a^p + p : 1 \leq a \leq p - 1\} \subset \mathbb{Z}_{p^2}^*$ [142]. We note that these Sidon sets do not live in the same groups!

One of my major goals is to understand the connection between these algebraic objects. Here is a starting point: given a Sidon set A in an abelian group Γ one can create a point line incidence structure where every two points are on *at most* one line and every two lines intersect at *at most* one point. One makes this incidence structure by letting the point set be the elements of Γ and letting the line set be the translates of A. The Sidon property guarantees the incidence axioms, and if A is a perfect difference set then this defines a projective plane. Given a point-line incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ a *polarity* is a bijection on $\mathcal{P} \cup \mathcal{L}$ which is an involution and which preserves incidence. Given a polarity on a point-line incidence structure one can construct a *polarity graph* with vertex set $\mathcal{P}$ and $p_1 \sim p_2$ if and only if p_1 is on the line that p_2 is mapped to under the polarity. One can define a polarity on the Sidon set translate incidence structure by sending a point x to the line $A - x$ and the line $A - y$ to the point y. It would be interesting to know how dense a Sidon set must be to conclude that the

[1] Undergraduate coauthor

polarity graph formed in this way is isormorphic to a large induced subgraph of the Erdős-Rényi graph (the polarity graph defined by a classical finite projective plane where points and lines are mapped to their orthogonal complements). It would be even more interesting if one can construct a Sidon set that is asymptotically as dense as possible for which the conclusion does not hold. There are also natural stability questions that arise. For example, given a Sidon set A in a group of order $q^2 + q + 1$, how large does A need to be to guarantee that it is a subset of a perfect difference set?

These types of questions are not just a novelty: they are related to very deep questions in geometry and combinatorial number theory. First in geometry, there is an old question asked by Erdős and probably earlier [46] asking if any bipartite C_4 free graph is the subgraph of the incidence graph of a finite projective plane. In combinatorial number theory, Erdős [47] made a 1000 USD conjecture that given any Sidon set of integers, there is a prime p such that that set of integers is a subset of the Singer difference set in $\mathbb{Z}_{p^2+p+1}$. Additive stability theorems (a solution to Erdős's conjecture or others) could perhaps answer questions in finite geometry that ask when a partial linear space can be embedded in a projective plane [17, 18, 85, 106, 123]. Finally, results along these lines would help gain some understanding of an old and deep conjecture in geometry that every transitive projective plane is desarguesian [77, 97]. Even partial results on this conjecture are very difficult (e.g. using the Classification of Finite Simple Groups).

Another goal is to understand the connection between the constructions described additively and those described geometrically. Even if we can show that the graphs arising as Cayley graphs are the same as those from geometry, *why* is this true? Bukh [27] showed that one can recover the C_6 and C_{10} free graphs coming from generalized polygons by considering Cayley graphs on extra-special p-groups. Understanding the general connection between group theory and geometry might point towards a natural subset of Cayley graphs where one can search for extremal graphs.

This part of my research plan is the most speculative but is incredibly intriguing.

Related Work

Michael Tait and Craig Timmons, Orthogonal polarity graphs and Sidon sets, *Journal of Graph Theory* 82 (1), 103–116, (2016)

Thang Pham, Michael Tait, Craig Timmons, and Le Anh Vinh. A Szemerédi-Trotter type theorem, sum-product estimates in finite quasifields, and related results, *Journal of Combinatorial Theory Series A*, 147, 55–74, (2017).

Michael Tait, On a problem of Neumann, to appear in *Discrete Mathematics special issue on Algebraic and Extremal Graph Theory.*

3 Extremal Spectral Graph Theory

Spectral graph theory seeks to associate a matrix with a graph and then to deduce from the eigenvalues or eigenvectors of this matrix structural properties about the graph. In the intersection of spectral graph theory and extremal graph theory, one seeks to optimize some function of these eigenvalues or eigenvectors over a given family of graphs. Given a graph G, the *adjacency matrix* of G has its rows and columns indexed by $V(G)$, where the ij'th entry of the matrix is 1 if vertices i and j are adjacent and 0 otherwise. Since this is a real and symmetric matrix, it has a full set of real eigenvalues which we will denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Spectral extremal graph theory has a long history of study. Some classical examples of theorems in this area include Stanley's bound maximizing λ_1 over the class of graphs with m edges [150], The Alon-Boppana-Serre theorem [117, 125, 132] which minimizes λ_2 over the family of d-regular graphs, the Hoffman ratio bound [95], and Wilf's theorem relating λ_1 to the clique number of G [164]. Using the inequality that λ_1 is at least as large as the average degree of G, recent work has been done to strengthen classical theorems in extremal graph theory. In particular, Nikiforov [126] showed that if λ_1 is larger than the largest eigenvalue of the adjacency matrix of the Turán graph that avoids K_r, then G must have a clique on r vertices. This result implies Turán's theorem. Similarly, spectral strengthenings have been given for the Erdős-Stone-Bollobas theorem [128] and for the Kővari-Sós-Turán theorem [13, 129].

These types of theorems may be used not only to give alternate proofs of extremal graph theory results, but if one has more information can improve the results. One concrete example is the following. One can show [155] that if G is triangle free, then λ_1^2 is bounded above by the maxcut of G. This immediately implies Mantel's theorem, since the maxcut of G is bounded above by $n^2/4$ and λ_1 is bounded below by the average degree. If one knows that, for example, G is not bipartite (and so the inequality for maxcut cannot be tight) or that G is not regular (and so the inequality on the average degree cannot be tight), then one immediately improves the bound.

As a medium-term goal, I aim to resolve many open conjectures in this area. I have already had much recent success. In [155], Tobin and I characterized the unique graph maximizing λ_1 over the families of planar and outerplanar graphs, resolving conjectures of Boots and Royle from 1991 [24] and of Cvetkovic and Rowlinson from 1990 [40]. In [2], Aksoy, Chung, Tobin, and I gave an asymptotically tight lower bound to how small the second smallest eigenvalue of the normalized Laplacian may be over the family of n vertex connected graphs, resolving a conjecture about random walks on graphs of Aldous and Fill from at least 1994 [4]. Since eigenvalues control many parameters of random walks/discrete Markov chains [4, 112, 113] studying these extremal spectral problems fits into a large body of work on extremal problems for random walk parameters (for example, [25, 31, 38, 54, 56, 55, 105, 122]). Other recent successes are proofs of conjectures of Cioabă and Gregory [156] and of Aochiche et al [155] regarding describing which graph is the "most irregular", and results relating the spectral radius of a graph to minors that are contained in the graph and the Colin de Verdière parameter [151].

I have been able to achieve this recent success by combining stability techniques from extremal graph theory with linear algebra methods from spectral graph theory. A general blueprint to proving these theorems is emerging, and during the next few years I will continue this development and in the process continue to solve open problems.

Research Goal 4. *Use the techniques that the author has developed to prove several more conjectures in the field.*

Below I detail a selection of the conjectures that might be proven during the next several years.

- Counting independent sets in graphs is a fundamental problem which has seen recent breakthrough successes (c.f. [14, 36, 58, 144, 145]). One variation of this problem which has also been heavily worked on is finding the graph of maximum degree Δ which has the maximum number of cliques of some fixed size t (we are working in the complement and counting cliques instead of independent sets). This problem, studied in [39, 42, 68, 96, 69, 168], is still open in general. Since the problem is equivalent to maximizing the trace of A^3 where A is the adjacency matrix of a graph with maximum degree Δ, we may attack this problem spectrally. The solution to the optimization problem of maximizing the sum of λ_i^3 where $\lambda_i \leq \Delta$ (because of the maximum degree constraint) and $\sum \lambda_i = 0$ already gives you that the eigenvalues of the extremal graph must be very close to the conjectured extremal example, and so there is hope that one can apply stability techniques to solve the problem completely.

- In [2], we answered a conjecture of Aldous and Fill asking asymptotically how large the maximum relaxation time of a random walk on a connected graph can be. While we answered their question, we could not characterize the extremal graphs. In the spirit of Winkler [25], this would be interesting to do.

- Aldous and Fill also asked which connected *regular* graph has the maximum relaxation time of a random walk [4]. Guiduli [84] solved this if one assumes that the graph is 3-regular, but the solution of the actual conjecture remains open. In the near future I will attempt to extend ideas that we used in [2], where there was no regularity condition, to answer their question.
- Stanić [149] asked an adjacency matrix version of Aldous and Fill's problem about maximizing the relaxation time of a random walk on a connected graph, namely he made a conjecture about which connected graph minimizes $\lambda_1 - \lambda_2$ where these are the two largest adjacency eigenvalues. We [154] have some partial results that indicate the potential to combine the methods that we used in [2] and [156] to prove his conjecture.
- Bollobás, Lee, and Letzer [22] recently considered the question of maximizing the spectral radius of a subgraph on a fixed number of vertices of the hypercube. This question was posed by Fink (c.f. [22]) and by Friedman and Tillich [59] and is a variant of the classical isoperimetric problem in the cube, which has received a lot of attention including in [157]. In [22], several theorems were proved but many questions remain open.

Related Work

Michael Tait and Josh Tobin, Three conjectures in extremal spectral graph theory, *Journal of Combinatorial Theory Series B*, 126, 137–161, (2017).

Ghodratollah Aalipour, Aida Abiad, Zhanar Berikkyzy, Leslie Hogben, Franklin H. J. Kenter, Jephian C.-H. Lin, Michael Tait. Proof of a conjecture of Graham and Lovász concerning unimodality of coefficients of the distance characteristic polynomial of a tree. To appear in *The Electronic Journal of Linear Algebra*.

Vladimir Nikiforov, Michael Tait, and Craig Timmons, Degenerate Turán problems for hereditary properties, submitted.

Michael Tait, The Colin de Verdière parameter, excluded minors, and the spectral radius.

Research Goal 5. *Develop new techniques for spectral extremal graph theory problems where the extremal graph is dense.*

Our techniques that have had success so far have worked best when the extremal graph is sparse. There are also several open problems in this area where the conjectured extremal graph is dense. For example, a conjecture of Gregory, Hershkowitz, and Kirkland [80] says that for λ_1 and λ_n the largest and smallest eigenvalues of the adjacency matrix of a graph, the graph which maximizes $\lambda_1 - \lambda_n$ (called the *spread*) is the join of a clique on $2n/3$ vertices and an independent set on $n/3$ vertices. This problem has been considered in over 50 papers (c.f. [11]) but is far from being solved.

I believe that combining our previous techniques with analytical arguments could help to resolve these types of problems. The theory of graph limits [114, 116] has recently received much deserved attention. Understanding more about the spectrum of continuous operators could help to understand what the asymptotic structure of a dense extremal graph for a given conjecture is. Solving a discrete problem by using continuous methods is by no means a new idea. One example of this is the solution to the conjecture of Lovász and Simonovits [115] on the graph of a fixed order and size which contains the fewest number of cliques on r vertices. This conjecture was resolved, first for $r = 4$ by Nikiforov [130], and then in general by Reiher [141] by putting a continuous weighting on the problem and then applying analytic methods.

Research Goal 6. *Develop methods for hypergraph spectral theory*

While spectral graph theory as a field has many classical and seminal results, the theory of hypergraph spectra is much less well-developed. Several problems arise when trying to generalize the theory to hypergraphs, the first being that even the tensor to use as the generalization of an adjacency matrix is not clear (see the discussion in [37]). Furthermore, while computation of graph eigenvalues amounts to solving a system of linear equations, computation of hypergraph eigenvalues amounts to solving a system of polynomial equations and thus becomes a difficult algebraic geometry question.

Despite these difficulties, much recent progress has been made, an incomplete selection given by [30, 37, 60, 99, 111, 131, 138]. I propose to study spectral extremal questions on hypergraphs during the next several years. Some first feasible problems to try are to try to maximize the spectral radius of a uniform hypergraph while forbidding various families $\mathcal{F}$ of hypergraphs. This has been considered in [99] and depending on what $\mathcal{F}$ is will range greatly in difficulty.

This line of my research has the potential not only to produce publications, but to develop this exciting area that has connections to algebraic geometry and quasirandomness of set systems.

Research Goal 7. *Study of graph irregularity.*

All of my work in spectral extremal graph theory came about circuitously because I was studying a walk counting conjecture of Erdős and Simonovits [50]. Letting W_k be the average number of k-walks from a vertex in a graph, their conjecture says that for $k \geq j$, both odd natural numbers, $(W_k)^{1/k} \geq (W_j)^{1/j}$ for any graph. The case of their conjecture when $j = 1$ was answered affirmatively by Blakley and Roy in [20] and has been a useful tool in other graph theoretic problems. If a graph is d-regular then the number of k walks from any vertex is d^k, and so one has equality. I have worked on this problem on and off for several years, and despite some progress (for example, the conjecture is true for almost all graphs [154]) it remains open. This is perhaps the problem I would most like to solve.

The inequality seems to be close to equality only when the graph is "close" to being regular. This led us to consider what it means for a graph to be "very irregular". There are many different ways to measure graph irregularity (c.f. [3, 15, 33, 127]) and most of them are pairwise incomparable. Given such a measure, one may ask what is the most irregular graph with respect to that irregularity measure. I did this with Josh Tobin [156, 155] for two irregularity measures, and it would be interesting to pursue similar theorems for other measures.

Related Work

Michael Tait and Josh Tobin, Characterizing graphs of maximum principal ratio, *Electronic Journal of Linear Algebra*, 34.1, 61–70, (2018).

Sinan G. Aksoy, Fan Chung, Michael Tait, and Josh Tobin, The maximum relaxation time of a random walk, *Advances in Applied Mathematics*, Volume 101, 1–14, (2018).

4 Problems for students

I stated before that working on questions that border different areas of mathematics encourages collaboration with a large group of mathematicians. One group that I am most excited to work with is bright undergraduate students. I have already successfully mentored several undergraduate projects (detailed in my CV), and I am excited to continue working with students in the future. Below are problems which would be suitable for this purpose along with my opinion on the level of student for which each is optimal.

Student Problem 1. *Study the distance spectrum of various graphs.*

At the Graduate Research Workshop in Combinatorics in 2014, we worked on the distance spectrum of a graph. Given a graph, one can construct a matrix where the ij'th entry is the distance between vertex i and j in the graph. The eigenvalues of this matrix are its distance spectrum. This problem was studied in the 1970s at Bell Labs [78], but has largely remained dormant for the last 30 years or so. However, recently there is a renewed interest, and several papers have been written on this topic in the last few years. An undergraduate student who has excelled in a linear algebra course can get their hands dirty on this project right away, and it would be an excellent topic for an REU.

Student Problem 2. *Improve the lower bounds for* $\mathrm{ex}(n, C_4)$.

Learning about non-desarguesian projective planes requires knowledge of abstract and linear algebra at the level of a talented undergraduate. Just a couple of years ago, Jason Williford led an REU that determined $\mathrm{ex}(q^2 + q, C_4)$ for q a power of 2 [57]. This represented one of the rare exact results in Turán theory and was published in a top journal, *Journal of Combinatorial Theory, Series B*, showing that undergraduates have the ability to prove extremely strong results in this area.

Student Problem 3. *Choose a particular family of projective planes and examine its subplane structure.*

This is a fundamental but poorly understood question in the field. It would be a good project for an undergraduate who excels at linear algebra and basic enumeration techniques to consider a specific family of projective planes and determine whether or not they contain subplanes of order 2, 3, or 4. Additionally, computer aided results are very helpful in this field, and a student with programming experience could be very valuable. This would also be a good topic for a Masters thesis.

Student Problem 4. *Determine the minimum adjacency spectral gap that a graph may have.*

The work I did with Aksoy, Chung, and Tobin [2] solved this problem for the normalized Laplacian, and the solution of this problem would answer a conjecture of Stanic [149] Our proofs, while technical, did not require anything more than linear algebra and structural graph theory. So while the solution of this problem would make a good Masters thesis, it is not inconceivable that an undergraduate could solve it as well. It would also be a nice result for a PhD student to include as a part of a dissertation on spectral graph theory.

Student Problem 5. *Determine the order of magnitude of the independence and chromatic numbers of various polarity graphs.*

This topic is well-suited for a Masters thesis or a PhD dissertation. A very talented undergraduate who has a strong background in algebra, combinatorics, and linear algebra would be able to read papers in this area, though it would be a pleasant surprise for an undergraduate to prove any new results.

Student Problem 6. *Construct large k-fold Sidon sets.*

Constructing a k-fold Sidon set in $[n]$ of size $\Omega(\sqrt{n})$ would make an excellent PhD dissertation. An undergraduate or graduate student who is a strong coder would also be very helpful in this endeavor.

5 Future Work

Though I work on a variety of problems in several areas, there is a common theme to my research, and the problems above are highly interrelated. The goals described above represent the main line of questions that I hope to answer in the next few years. However, work on them will likely yield many tangential problems and directions that are also fascinating.

I am excited and passionate about the problems that I am working on. I hope that through my research I can foster collaboration between mathematicians with seemingly disjoint interests. I feel that the breadth of techniques I am able to employ to solve research problems is one of my greatest strengths. In the next several years, I hope to continue collaboration with undergraduates, graduate students, and senior faculty. Finally, graphs without small cycles are related to LDPC codes (c.f. [146]), and finite sets without solutions to the Sidon equation are equivalent to Golomb rulers. LDPC codes and Golomb rulers have several applications in error correction and data transmission, and thus it is possible that my work could yield practical and industrial applications.

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