Spectral bounds for the k-independence number of a graph

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Abstract

In this paper, we obtain two spectral upper bounds for the k-independence number of a graph which is is the maximum size of a set of vertices at pairwise distance greater than k. We construct graphs that attain equality for our first bound and show that our second bound compares favorably to previous bounds on the k-independence number.

Keywords: k-independence number; graph powers; eigenvalues; Expander-Mixing lemma.

1 Introduction

The independence number of a graph G, denoted by $\alpha(G)$, is the size of the largest independent set of vertices in G. A natural generalization of the independence number is the *k-independence number* of G, denoted by $\alpha_k(G)$ with $k \geq 0$, which is the maximum number of vertices that are mutually at distance greater than k. Note that $\alpha_0(G)$ equals the number of vertices of G and $\alpha_1(G)$ is the independence number of G.

The k-independence number of a graph is related to its injective chromatic number [16], packing chromatic number [13], strong chromatic index [21] and has also connections to coding theory, where codes and anticodes are k-independent sets in appropriate associated graphs. This parameter has been studied in various other contexts by many researchers [1, 6, 9, 10, 11, 12, 22, 18]. It is known that determining α_k is NP-Hard in general [19].

In this article, we prove two spectral upper bounds for α_k that generalize two well-known bounds for the independence number: Cvetković's inertia bound [3] and the Hoffman ratio bound (see [2, Theorem 3.5.2] for example). Note that α_k is the independence number of G^k, the k-th power of G. The graph G^k has the same vertex set as G and two distinct vertices are adjacent in G^k if their distance in G is k or less. In general, even the simplest spectral or combinatorial parameters of G^k cannot be deduced easily from the similar parameters of G (see [4, 5, 17] for example). Our bounds depend only on the spectrum of the adjacency matrix of G and do not require the spectrum of G^k. We prove our main results in Section 3 and Section 4. We end with a comparison of our bounds to previous work and some directions for future work.

2 Preliminaries

Throughout this paper $G = (V, E)$ will be a graph (undirected, simple and loopless) on vertex set V with n vertices, edge set E and adjacency matrix A with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. The following result was proved by Haemers in his Ph.D. Thesis (see [15] for example).

Lemma 2.1 (Eigenvalue Interlacing, [15]). *Let A be a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. For some integer $m < n$, let S be a real $n \times m$ matrix such that $S^\top S = I$ (its columns are orthonormal), and consider the $m \times m$ matrix $B = S^\top AS$, with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$. Then, the eigenvalues of B interlace the eigenvalues of A, that is, $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$, for $1 \leq i \leq m$.*

If we take $S = [\, I \;\; O \,]$, then B is just a principal submatrix of A and we have:

Corollary 2.2. *If B is a principal submatrix of a symmetric matrix A, then the eigenvalues of B interlace the eigenvalues of A.*

3 Generalized inertia bound

Cvetković [3] (see also [2, p.39] or [14, p.205]) obtained the following upper bound for the independence number.

Theorem 3.1 (Cvetković's inertia bound, [3]). *If G is a graph, then*

$$\alpha(G) \leq \min\{|i : \lambda_i \geq 0|, |i : \lambda_i \leq 0|\}. \tag{1}$$

Let $w_k(G) = \min_i (A^k)_{ii}$ be the minimum number of closed walks of length k where the minimum is taken over all the vertices of G. Similarly, let $W_k(G) = \max_i (A^k)_{ii}$ be the maximum number of closed walks of length k where the maximum is taken over all the vertices of G. Our first main theorem generalizes Cvetković's inertia bound which can be recovered when $k = 1$.

Theorem 3.2. *Let G be a graph on n vertices. Then,*

$$\alpha_k(G) \le |\{i : \lambda_i^k \ge w_k(G)\}| \quad \textit{and} \quad \alpha_k(G) \le |\{i : \lambda_i^k \le W_k(G)\}|. \tag{2}$$

Proof. Because G has a k-independent set U of size α_k, the matrix A^k has a principal submatrix (with rows and columns corresponding to the vertices of U) whose off-diagonal entries are 0 and whose diagonal entries equal the number of closed walks of length k starting at vertices of U. Corollary 2.2 leads to

$$\alpha_k(G) \le |\{i : \lambda_i^k \ge w_k(G)\}| \quad \text{and} \quad \alpha_k(G) \le |\{i : \lambda_i^k \le W_k(G)\}|.$$

□

3.1 Construction attaining equality

In this section, we describe a set of graphs for which Theorem 3.2 is tight. For $k, m \ge 1$ we will construct a graph G with $\alpha_{2k+2}(G) = \alpha_{2k+3}(G) = m$.

Le H be the graph obtained from the complete graph K_n by removing one edge. The eigenvalues of H are $\frac{n-3\pm\sqrt{(n+1)^2-8}}{2}$, 0 (each with multiplicity 1), and -1 with multiplicity $n-3$. This implies $|\lambda_i(H)| < 2$ for $i > 1$.

Let $H_1, ..., H_m$ be vertex disjoint copies of H with $u_i, v_i \in V(H_i)$ and $u_i \not\sim v_i$ for $1 \le i \le m$. Let x be a new vertex. For each $1 \le i \le m$, create a path of length k with x as one endpoint and u_i as the other. Let G be the resulting graph which has $nm + (k-2)m + 1$ vertices with $m\left(\binom{n}{2} - 1\right) + mk$ edges.

Because the distance between any distinct v_is is $2k + 4$, we get that

$$\alpha_{2k+2}(G) \ge \alpha_{2k+3}(G) \ge m. \tag{3}$$

We will use Theorem 3.2 to show that equality occurs in (3) for sufficiently large n.

Starting from any vertex of G, one can find a closed walk of length $2k+2$ or $2k+3$ that contains an edge of some H_i. Therefore, $w_{2k+2}(G) \ge n-2$ and $w_{2k+3}(G) \ge n-2$. Choose n so that $n - 2 > (\sqrt{m} + 4)^{2k+3}$. If we can show that

$$|\lambda_i(G)| \le \sqrt{m} + 4 \tag{4}$$

for all $i > m$, then Theorem 3.2 will imply that $\alpha_{2k+3}(G) \le \alpha_{2k+2}(G) \le m$ and we are done. To show (4), note that the edge-set of G is the union of m edge disjoint copies of H, the star $K_{1,m}$, and m vertex disjoint copies of P_{k-1}. Since the star $K_{1,m}$ has spectral radius $\sqrt{m}$ and a disjoint union of paths has spectral radius less than 2, applying the Courant-Weyl inequalities again along with the triangle inequality yields that $|\lambda_i(G)| < \sqrt{m} + 4$ for all $i > m$ and finishes our proof.

4 Generalized Hoffman bound

The following bound on the independence number is an unpublished result of Hoffman known as the Hoffman's ratio bound (see [2, p.39] or [14, p.204]).

Theorem 4.1 (Hoffman bound)**.** *If G is regular then $\alpha(G) \leq n\frac{-\lambda_n}{\lambda_1-\lambda_n}$ and if a coclique C meets this bound then every vertex not in C is adjacent to precisely $-\lambda_n$ vertices of C.*

Let G be a d-regular graph on n vertices (undirected, simple, and loopless) having an adjacency matrix A with eigenvalues $d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -d$. Let $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$. We use Alon's notation and say G is an (n, d, λ)-graph (see also [20, p.19]). Let $\tilde{W}_k = \max_i \sum_{j=1}^{k} (A^j)_{ii}$ be the maximum over all vertices of the number of closed walks of length at most k. Our second theorem is an extension of the Hoffman bound to k-independent sets.

Theorem 4.2. *Let G be an (n, d, λ)-graph and k a natural number. Then*

$$\alpha_k(G) \leq n\frac{\tilde{W}_k + \sum_{j=1}^{k} \lambda^j}{\sum_{j=1}^{k} d^j + \sum_{j=1}^{k} \lambda^j}. \tag{5}$$

The proof of Theorem 4.2 will be given as a corollary to a type of Expander-Mixing Lemma. For k a natural number, denote

$$\lambda^{(k)} = \lambda + \lambda^2 + \cdots + \lambda^k,$$

and

$$d^{(k)} = d + d^2 + \cdots + d^k.$$

Theorem 4.3 (k-Expander Mixing Lemma)**.** *Let G be an (n, d, λ)-graph. For $S, T \subseteq G$ let $W_k(S, T)$ be the number of walks of length at most k with one endpoint in S and one endpoint in T. Then for any $S, T \subseteq V$, we have*

$$\left| W_k(S,T) - \frac{d^{(k)}|S||T|}{n} \right| \leq \lambda^{(k)} \sqrt{|S||T| \left(1 - \frac{|S|}{n}\right)\left(1 - \frac{|T|}{n}\right)} < \lambda^{(k)} \sqrt{|S||T|}.$$

Proof. Let $S, T \subset V(G)$ and let $\mathbf{1}_S$ and $\mathbf{1}_T$ be the characteristic vectors for S and T respectively. Then

$$W_k(S,T) = \mathbf{1}_S^t \left(\sum_{j=1}^{k} A^j \right) \mathbf{1}_T.$$

Let $x_1, ..., x_n$ be an orthonormal basis of eigenvectors for A. Then $\mathbf{1}_S = \sum_{i=1}^{n} \alpha_i x_i$ and $\mathbf{1}_T = \sum_{i=1}^{n} \beta_i x_i$, where $\alpha_i = \langle \mathbf{1}_S, x_i \rangle$ and $\beta_i = \langle \mathbf{1}_T, x_i \rangle$. Note that $\sum \alpha_i^2 = \langle \mathbf{1}_S, \mathbf{1}_S \rangle = |S|$ and similarly, $\sum \beta_i^2 = |T|$. Because G is d-regular, we get that $x_1 = \frac{1}{\sqrt{n}}\mathbf{1}$ and so $\alpha_1 = \frac{|S|}{n}$ and $\beta_1 = \frac{|T|}{n}$. Now, since $i \neq j$ implies $\langle x_i, x_j \rangle = 0$, we have

$$\begin{aligned}W_k(S,T) &= \left(\sum_{i=1}^{n}\alpha_i x_i\right)^t \left(\sum_{j=1}^{k}A^j\right)\left(\sum_{i=1}^{n}\beta_i x_i\right)\\ &= \sum_{i,j}(\alpha_i x_i)((\beta_j(\lambda_j+\lambda_j^2+\cdots+\lambda_j^k)x_j)\\ &= \sum_{i=1}^{n}(\lambda_i+\lambda_i^2+\cdots+\lambda_i^k)\alpha_i\beta_i\\ &= \frac{d_k}{n}|S||T| + \sum_{i=2}^{n}(\lambda_i+\lambda_i^2+\cdots+\lambda_i^k)\alpha_i\beta_i\end{aligned}$$

Therefore, we have

$$\begin{aligned}\left|W_k(S,T) - \frac{d_k}{n}|S||T|\right| &= \left|\sum_{i=2}^{n}(\lambda_i+\lambda_i^2+\cdots+\lambda_i^k)\alpha_i\beta_i\right|\\ &\le \lambda^{(k)}\sum_{i=2}^{n}|\alpha_i\beta_i|\\ &\le \lambda^{(k)}\left(\sum_{i=2}^{n}\alpha_i^2\right)^{1/2}\left(\sum_{i=2}^{n}\beta_i^2\right)^{1/2},\end{aligned}$$

where the last inequality is by Cauchy-Schwarz. Now since

$$\sum_{i=2}^{n}\alpha_i^2 = |S| - \frac{|S|^2}{n^2}$$

and

$$\sum_{i=2}^{n}\beta_i^2 = |T| - \frac{|T|^2}{n^2},$$

we have the result. □

Now we are ready to prove the bound from Theorem 4.2.

Proof. [Proof of Theorem 4.2] Let S be a k-independent set in G with $|S| = \alpha_k(G)$, and let $W_k(S,S)$ be equal to the number of closed walks of length at most k starting in S. Theorem 4.3 gives

$$\frac{d^{(k)}|S|^2}{n} - W_k(S,S) \le \lambda^{(k)}|S|\left(1-\frac{|S|}{n}\right).$$

Recalling that $\tilde{W}_k = \max_i \sum_{j=1}^{k}(A^j)_{ii}$, we have $W_k(S,S) \le |S|\tilde{W}_k$. This yields

$$\frac{d^{(k)}|S|}{n} - \tilde{W}_k \le \lambda^{(k)}\left(1-\frac{|S|}{n}\right).$$

Solving for $|S|$ and substituting $|S| = \alpha_k$ gives

$$\alpha_k \leq n\frac{\tilde{W}_k + \lambda^{(k)}}{d^{(k)} + \lambda^{(k)}}.$$

□

Note that the bound from Theorem 4.2 behaves nicely if $\tilde{W}_k$ and λ_k are small with respect of d_k. It is easy to see that $\tilde{W}_k \leq \frac{d^k-1}{d-1}$ (we expand d in each step but the last step we do not have any freedom since we assume that we are counting closed walks). Since G is d-regular and we know that $\tilde{W}_k \leq d^{k-1}$, the above bound performs well for graphs with a good spectral gap.

5 Concluding Remarks

In this section, we note how our theorems compare with previous upper bounds on α_k. Our generalized Hoffman bound for α_k is best compared with Firby and Haviland [12], who proved that if G is a connected graph of order $n \geq 2$ then

$$\alpha_k(G) \leq \frac{2(n-\epsilon)}{k+2-\epsilon} \tag{6}$$

where $\epsilon \equiv k \pmod 2$. If d is large compared to k and $\lambda = o(d)$, then Theorem 4.2 is much better than (6). We note that almost all d-regular graphs have $\lambda = o(d)$ as $d \to \infty$.

In [8], Fiol (improving work from [9]) obtained the bound

$$\alpha_k(G) \leq \frac{2n}{P_k(\lambda_1)}, \tag{7}$$

when G is a regular graph (later generalized to nonregular graphs in [7]), and P_k is the k-alternating polynomial of G. The polynomial P_k is defined by the solution of a linear programming problem which depends on the spectrum of the graph G. It is nontrivial to compute P_k, and it is unclear how (7) compares with our theorems. However, we note that (7) cannot be strictly better in general than our Theorem 3.2, as there are cases where Theorem 3.2 is sharp, shown in subsection 3.1. Furthermore, our theorems require less information to apply than (7).

If p is a polynomial of degree at most k, and U is a k-independent set in G, then $p(A)$ has a principal submatrix defined by U that is diagonal, with diagonal entries defined by a linear combination of various closed walk. Theorems 3.2 and 4.2 are obtained by taking $p(A) = A^k$, but hold also for in general for other polynomials of degree at most k. It is not clear how to choose a polynomial $p(A)$ to optimize our bounds and we leave as an open problem. Finally, we were able to construct graphs attaining equality in Theorem3.2 but not in Theorem 4.2. We leave open whether the bound in Theorem 4.2 is attained for some graphs or can be improved in general.

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