Induced Turán numbers

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Abstract

The classical Kővári-Sós-Turán theorem states that if G is an n-vertex graph with no copy of $K_{s,t}$ as a subgraph, then the number of edges in G is at most $O(n^{2-1/s})$. We prove that if one forbids $K_{s,t}$ as an *induced* subgraph, and also forbids *any* fixed graph H as a (not necessarily induced) subgraph, the same asymptotic upper bound still holds, with different constant factors. This introduces a nontrivial angle from which to generalize Turán theory to induced forbidden subgraphs, which this paper explores. Along the way, we derive a nontrivial upper bound on the number of cliques of fixed order in a K_r-free graph with no induced copy of $K_{s,t}$. This result is an induced analog of a recent theorem of Alon and Shikhelman and is of independent interest.

1 Introduction

Turán-type problems represent some of the oldest investigations in Extremal Combinatorics, with many intriguing questions still notoriously open. They share a common theme of asking for the maximum number of edges in a graph (or similar combinatorial structure) with a given number of vertices, subject to the condition of forbidding certain substructures. In this paper, we open the systematic study of a natural yet new direction in this area, focusing on induced substructures, and demonstrate connections between existing areas of research and the new results and problems.

The most basic Turán question concerns ordinary graphs and asks to determine $\mathrm{ex}(n, H)$, defined as the maximum number of edges in an n-vertex graph with no subgraph isomorphic to H. Turán's original theorem [47] solves this completely when H is a complete graph. For non-complete H, the condition obviously does not require the forbidden subgraph to be induced, or else the answer would trivially be $\binom{n}{2}$. The classical Erdős-Stone-Simonovits theorem [14, 20, 22] shows that the asymptotic behavior is determined by the chromatic number $\chi(H)$, namely

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right)\binom{n}{2} + o(n^2). \tag{1}$$

This determines $\mathrm{ex}(n, H)$ asymptotically for non-bipartite H. For bipartite H, it is often quite difficult to obtain good estimates on the Turán number. The classical theorem of Kővári, Sós, and Turán [32] states that $\mathrm{ex}(n, K_{s,t}) < c_{s,t} n^{2 - \frac{1}{s}}$ but this is overwhelmed by the $o(n^2)$ error term in (1). Many interesting and longstanding open problems remain unsolved in this case, often called the *degenerate case*, as surveyed by Füredi and Simonovits [26] and Sidorenko [42].

Many other generalizations have been considered, such as to hypergraphs (surveyed in [31]) where even the most basic questions remain unanswered, or to non-complete host graphs (e.g., replacing K_n with the hypercube as in [1, 17]), or other combinatorial objects such as partially ordered sets [27, 33, 45]. In all contexts, analogous questions with multiple simultaneously forbidden sub-configurations have been studied.

1.1 Induced substructures

Although the opening section dismissed as trivial the situation of induced subgraphs in the ordinary graph Turán problem, it turns out that this first impression is wrong, and there are natural and interesting questions. Induced Turán-type problems have previously surfaced in many of the above contexts. On the topic of one of the central open problems in hypergraph Turán theory, Razborov [41] established the conjectured upper bound for $K_4^{(3)}$-free hypergraphs under the additional condition of forbidding induced sub-hypergraphs with four vertices and exactly one edge. In the context of partially ordered sets, the induced Turán problem is nontrivial because not all sets are comparable, and this has been studied as well [5, 35].

It has been less clear what induced question to study in the original graph context. In the late 1980's, F. Chung, Gyárfás, Trotter, and Tuza [9] studied a version which was posed by Bermond et al. [4] and also by Nešetřil and Erdős, in which the maximum degree was specified instead of the number of vertices. Specifically, they determined the maximum number of edges in a connected graph with maximum degree Δ and no induced subgraph isomorphic to the 4-vertex graph formed by two vertex-disjoint edges. Connectivity is obviously required, or else one can generate arbitrarily large graphs by taking disjoint unions. Several other authors continued this line of investigation with different forbidden induced subgraphs [10, 11, 12]. However, this quantity is usually infinite unless the forbidden induced subgraph has a very simple structure (generally disjoint unions of paths).

Around that time, while studying hereditary properties, Prömel and Steger [38, 39, 40] introduced another extremal induced subgraph problem: determine the maximum number of edges a graph $G = (V_n, E)$ can have such that there exists a graph $G_0 = (V_n, E_0)$ on the same vertex set with $E_0 \cap E = \emptyset$ such that $(V_n, E_0 \cup X)$ does not contain an induced H-subgraph for all $X \subset E$. This was natural in the context of their investigation of counting the number of graphs in a hereditary family, and generalized the Erdős, Frankl, and Rödl [18] estimate on the number of H-free, graphs being $2^{(1+o(1)) \mathrm{ex}(n,H)}$, to induced-$H$-free graphs. Rates of growth of hereditary properties were further studied by several researchers (e.g., Bollobás and Thomason [7], and Balogh, Bollobás, and Weinreich [3]).

1.2 New general problem

When a single non-complete graph F is forbidden as an induced subgraph, the maximum number of edges is trivially $\binom{n}{2}$. We introduce the question of simultaneously forbidding both an induced copy of F and a (not necessarily induced) copy of H, defining

$$\mathrm{ex}(n, \{H, F\text{-ind}\})$$

to be the maximum number of edges over all such graphs with n vertices. The answer is no longer trivially $\binom{n}{2}$ because H is not necessarily induced, and this general question is related to two areas of Extremal Combinatorics which have received much attention: Ramsey-Turán Theory and the Erdős-Hajnal Conjecture.

Introduced by Sós [44], the Ramsey-Turán number $\mathbf{RT}(n, H, m)$ is the maximum number of edges that an n-vertex graph with independence number less than m may have without containing H as a (not necessarily induced) subgraph. When $m = o(n)$, one may not use a Turán graph as a construction, and a variety of interesting constructions and methods were developed as a result. Ramsey-Turán theory has been heavily studied in the last half-century, including the first application of Szemerédi's Regularity Method [46]; see, e.g., the nice survey by Simonovits and Sós [43]. Our new general problem is precisely the Ramsey-Turán problem in the case where F is an independent set of order m.

Another question which has received much study is the Erdős-Hajnal Problem, which seeks to prove that if a graph F is forbidden as an induced subgraph, then there is always a large clique or a large independent set. The Erdős-Hajnal Conjecture [19] states that for any fixed F, there is a constant $c > 0$ such that every F-induced-free graph on n vertices contains a clique or independent set of order at least n^c, which is much larger than what is guaranteed without the F-induced-free condition. This problem has been the focus of extensive research (see, e.g., the survey of Chudnovsky [8]). The relationship to our new problem is that an upper bound on $\mathrm{ex}(n, \{K_t, F\text{-ind}\})$ of the form $nd/2$ implies an average degree of at most d. Turán's theorem then guarantees an independent set of order at least $\frac{n}{d+1}$. This shows that a graph with no induced copy of F contains either a clique of size t or an independent set of size $\frac{n}{d+1}$. We will discuss this further in the concluding remarks.

1.3 New results

Throughout this paper, we consider only non-complete graphs F. Our new function $\mathrm{ex}(n, \{H, F\text{-ind}\})$ sometimes reduces to the ordinary Turán number $\mathrm{ex}(n, \{H, F\})$ where both H and F are forbidden as (not necessarily induced) subgraphs. Indeed, if $H = C_3$ and $F = C_4$, every graph which is both C_3-free and C_4-induced-free is also C_4 free, and every graph which is C_4-free is obviously C_4-induced-free.

As mentioned early in the introduction, if F is non-bipartite, Turán's theorem establishes that $\mathrm{ex}(n, F\text{-ind})$ and $\mathrm{ex}(n, F)$ are both quadratic in n. However, for bipartite F, $\mathrm{ex}(n, F\text{-ind}) = \binom{n}{2}$, while the Kővári-Sós-Turán theorem trivially establishes a sub-quadratic upper bound $n^{2-\frac{1}{s}}$ for some s for which $F \subset K_{s,t}$. The two functions therefore deviate asymptotically for all bipartite F. Our first main result shows that in fact, when $F = K_{s,t}$, we can recover the same asymptotic upper bound as Kővári-Sós-Turán by forbidding *any* other fixed graph H.

Theorem 1.1 *If G is an n-vertex graph with no copy of K_r as a subgraph and no copy of $K_{s,t}$ as an induced subgraph, then*

$$e(G) \leq n^{2-1/s} 4^s \left((r+t)^{t/s} + (r+s) + 2(r+t)^{t(s+1)/s^2}(r+s) \right) + 2 \cdot 4^s n.$$

As a corollary, this shows that for any positive integers s and t and any fixed graph H,

$$\text{ex}(n, \{H, K_{s,t}\text{-ind}\}) = O\left(n^{2-\frac{1}{s}} \right)$$

where the implied constant depends on H, s, and t. Note that if the forbidden induced subgraph F is bipartite but not complete bipartite, then the complete bipartite graph $K_{n/2,n/2}$ provides a construction which shows that $\text{ex}(n, \{K_r, F\text{-ind}\})$ is quadratic in n for all $r > 2$.

Our proof method draws another connection between this problem and a recent Turán-type problem of Alon and Shikhelman [2]. For graphs T and H, denote by $\text{ex}(n, T, H)$ the maximum number of copies of T in an H-free graph with n vertices. When $T = K_2$, this is the classical Turán number. Several authors have studied this problem before (cf. [6, 16, 30]), and [2] is the first systematic study of the parameter. A key ingredient in the proof of our main theorem gives an upper bound on the number of complete subgraphs in a graph that does not contain H or an induced copy of $K_{s,t}$. In particular, in [2], the quantity $\text{ex}(n, K_m, K_{s,t})$ is studied. The following theorem is a natural extension of $\text{ex}(n, K_m, K_{s,t})$ to graphs with no *induced* copy of $K_{s,t}$. We used it as a tool for proving Theorem 1.1, but due to the connection with Alon and Shikhelman's problem, it may be of independent interest.

Theorem 1.2 *Let G be an n-vertex, K_r-free graph with no copy of $K_{s,t}$ as an induced subgraph. If $t_m(G)$ is the number of cliques of order m in G, then*

$$m \cdot t_m(G) \leq 2(t+r)^{tm/s}(r+s)^s n^{m-\frac{m-1}{s}} + (r+s)^s n^{m-1}.$$

The bound of Theorem 1.2 holds for all parameters and is enough to prove Theorem 1.1. When n tends to infinity and r, s, and t are fixed, then one can improve Theorem 1.2 to $t_m(G) = o(n^{m-\frac{m-1}{s}} + o^{m-1})$ using an additional tool from extremal graph theory. This is discussed further in the final section of the paper.

1.4 Sharper results for special families

In this section, we dive deeper into the constant factors, opening the study with specific families of graphs for F and H in $\text{ex}(n, \{H, F\text{-ind}\})$. As was historically studied by others in graphs, we start with complete bipartite graphs and cycles. Theorem 1.1 gives that if G is a graph with no induced copy of $K_{s,t}$ and G has significantly more than $n^{2-1/s}$ edges, then G must contain a large complete subgraph. This leads us to the work of Gyárfás, Hubenko, and Solymosi on cliques in graphs with no induced $K_{2,2}$. In [28], answering a question of Erdős, they show that any n-vertex graph with no induced $K_{2,2}$ must have a clique of order at least $\frac{d^2}{10n}$, where d is the average degree. We extend this result to graphs with no induced $K_{2,t}$. In this special case, we obtain a much better bound than what is implied by Theorem 1.1. Here and in the remainder, the clique number $\omega(G)$ denotes the maximum order of a clique contained in G.

Theorem 1.3 *Let $t \geq 2$ be an integer. If G is a graph with n vertices, minimum degree d, and no induced $K_{2,t+1}$, then*

$$\omega(G) \geq (1 - o(1)) \left(\frac{d^2}{2nt} \right)^{1/t}.$$

Corollary 1.4 *Let H be a graph with v_H vertices. For any integer $t \geq 2$,*

$$\mathrm{ex}(n, \{H, K_{2,t+1}\text{-ind}\}) < (\sqrt{2} + o(1))t^{1/2}v_H^{t/2}n^{3/2}.$$

When $\chi(H) \geq 3$, we can obtain a lower bound of the same order of magnitude by considering a max cut in a $K_{2,t+1}$-free graph with n vertices and $\frac{1}{2}\sqrt{t}n^{3/2} - o(n^{3/2})$ edges. Such graphs were constructed by Füredi in [25]. A max cut in a $K_{2,t+1}$-free graph will clearly not contain a copy of H and will not contain an induced copy of $K_{2,t+1}$. This gives a lower bound of

$$\frac{1}{4}\sqrt{t}n^{3/2} - o(n^{3/2}) \leq \mathrm{ex}(n, \{H, K_{2,t+1}\text{-ind}\}) \tag{2}$$

for any $t \geq 2$ and non-bipartite H.

Theorem 1.1 shows that when one forbids induced copies of $K_{s,t}$ and any other subgraph, the number of edges is bounded above by something that is the same order of magnitude than what is given by the Kővári-Sós-Turán theorem, leaving the question of the multiplicative constant. We have also remarked that there are instances where the problem reduces to the ordinary Turán number, for example $\mathrm{ex}(n, \{C_3, C_4\text{-ind}\}) = \mathrm{ex}(n, \{C_3, C_4\})$. A nice construction based on the incidence graph of a projective plane was used by Bollobás and Győri [6] to show that there are C_5-free n-vertex graphs with many triangles. It turns out that this same construction shows that for any q that is a power of a prime,

$$\mathrm{ex}(3(q^2 + q + 1), \{C_5, C_4\text{-ind}\}) \geq 2(q + 2)(q^2 + q + 1).$$

A standard densities of primes argument then gives

$$\mathrm{ex}(n, \{C_5, C_4\text{-ind}\}) \geq \frac{2}{3\sqrt{3}}n^{3/2} - o(n^{3/2}),$$

while Erdős and Simonovits [21] proved that

$$\mathrm{ex}(n, \{C_4, C_5\}) \leq \frac{1}{2\sqrt{2}}n^{3/2} + 4\left(\frac{n}{2}\right)^{1/2}.$$

This shows that there are cases when the numbers $\mathrm{ex}(n, \{H, F\text{-ind}\})$ and $\mathrm{ex}(n, \{H, F\})$ may have different multiplicative constants.

Finally, we note that while Theorem 1.1 gives an upper bound matching the Kővári-Sós-Turán theorem in order of magnitude, the multiplicative constant is dependent on certain Ramsey numbers in r, s, and t, and so is likely not tight. Our final results display how one may lower the multiplicative constant when one knows more about the forbidden (not necessarily induced) subgraph H. We give the following theorems for H an odd cycle, but note that the proof technique could be applied to a wide family of graphs.

Theorem 1.5 *For any integers $k \geq 2$ and $t \geq 2$ there is a constant β_k, depending only on k, such that*

$$\mathrm{ex}(n, \{C_{2k+1}, K_{2,t}\text{-ind}\}) \leq (\alpha(k,t)^{1/2} + 1)^{1/2} \frac{n^{3/2}}{2} + \beta_k n^{1+1/2k}$$

where $\alpha(k,t) = (2k-2)(t-1)((2k-2)(t-1) - 1)$.

Observe that (2) gives a lower bound on $\mathrm{ex}(n, \{C_{2k+1}, K_{2,t}\text{-ind}\})$ since C_{2k+1} is not bipartite. Therefore, Theorem 1.5 is correct in both order of magnitude and its dependence on t. We leave open the question of whether Theorem 1.5 gives the correct growth rate as a function of k.

1.5 Notation and organization

Let the Ramsey number $R(s,t)$ denote the smallest n such that in any red and blue coloring of the edges of K_n, there is either a red K_s or a blue K_t. We write $t_m(G)$ for the number of complete subgraphs of G that have exactly m vertices. An independent set of order s is called an s-independent set. We define

$$\mathcal{I}_s(G) = \{\{x_1, \ldots, x_s\} \subset V(G) : x_1, \ldots, x_s \text{ are distinct and non-adjacent in } G\}.$$

Similarly, a clique of order m is called an m-clique, and $\mathcal{K}_m(G)$ denotes the set of all m-cliques in G. Given a set of vertices $\{x_1, \ldots, x_s\} \subset V(G)$, we write $N(x_1, \ldots, x_s)$ to denote the vertices in G that are adjacent to every vertex in the set $\{x_1, \ldots, x_s\}$, and we let

$$d(x_1, \ldots, x_s) = |N(x_1, \ldots, x_s)|.$$

We write $\Gamma(x_1, \ldots, x_s)$ for the subgraph of G induced by the vertices in $N(x_1, \ldots, x_s)$. Lastly, $\overline{H}$ denotes the complement of the graph H.

This paper is organized as follows. We prove our two main results, Theorems 1.1 and 1.2, in Sections 2 and 3. Theorem 1.3 is proved in Section 1.3. We prove Theorem 1.5 in Subsection 4.1. The final section contains some concluding remarks and open problems.

2 The number of edges in H-free graphs with no induced $K_{s,t}$

In this section, let r, s, and t be fixed positive integers. Let G be an n-vertex graph with no copy of K_r as a subgraph and no copy of $K_{s,t}$ as an induced subgraph. We will prove Theorem 1.1, showing that $e(G) = O\left(n^{2-1/s}\right)$, where the implied constant depends on r, s, and t. The proof will rely on an upper bound on the number of cliques of a fixed order in G, for which we will apply Theorem 1.2. We will delay the proof of Theorem 1.2 to Section 3. We will need the following claim which also counts cliques. A much stronger version is given by Conlon in [13], but we only need a weaker version that can be proved using an elementary counting argument of Erdős [16].

Lemma 2.1 *If F is a graph on $n > 2 \cdot 4^s$ vertices, then*

$$t_s(F) + t_s(\overline{F}) \geq \frac{n^s}{2^s 4^{s^2}}.$$

Proof. Since it is well known that $R(s,s) < 4^s$, any set of 4^s vertices in $V(F)$ must contain a clique of order s in either F or $\overline{F}$. Each set of s vertices is contained in $\binom{n-s}{4^s-s}$ sets of order 4^s. Therefore,

$$t_s(F) + t_s(\overline{F}) \geq \frac{\binom{n}{4^s}}{\binom{n-s}{4^s-s}} > \frac{(n-4^s)^s}{(4^s)^s} > \frac{n^s}{2^s 4^{s^2}}$$

where in the last inequality we have used the assumption that $n > 2 \cdot 4^s$. ∎

Proof of Theorem 1.1. Let G be an n-vertex graph that is K_r-free and has no induced copy of $K_{s,t}$. We must show that $e(G) < cn^{2-1/s}$ where c is a constant depending only on r, s, and t. First we show that we can assume that the minimum degree of G is at least $2 \cdot 4^s$. Indeed, if the minimum degree of G is less than $2 \cdot 4^s$, then we remove a vertex of minimum degree and we observe that the number of edges removed is at most $2 \cdot 4^s$. If the resulting subgraph has minimum degree at least $2 \cdot 4^s$, then we are done. Otherwise, we continue removing vertices of minimum degree. The maximum number of edges removed during this process is $2 \cdot 4^s n$ and we are either left with an empty graph, or a graph with at most n vertices and minimum degree at least $2 \cdot 4^s$. In particular, when the minimum degree is at least $2 \cdot 4^s$ we can apply Lemma 2.1 to the graph $\Gamma(v)$ for any $v \in V(G)$. We now proceed with the main part of the proof of Theorem 1.1.

Since G does not contain an induced copy of $K_{s,t}$, an s-independent set cannot contain a t-independent set in its common neighborhood. Also, no set of vertices can contain a clique of order r in its neighborhood since G is K_r-free. We conclude that for any $\{x_1, \ldots, x_s\} \in \mathcal{I}_s(G)$,

$$d(x_1, \ldots, x_s) \leq R(r,t).$$

Therefore, using the Erdős-Szekeres [23] bound $R(r,t) \leq \binom{r+t-2}{t-1}$,

$$\sum_{\{x_1,\ldots,x_s\}\in\mathcal{I}_s(G)} d(x_1,\ldots,x_s) \leq \binom{n}{s} R(r,t) < (r+t)^t n^s. \qquad (3)$$

On the other hand, we may double count to see that

$$\sum_{\{x_1,\ldots,x_s\}\in\mathcal{I}_s(G)} d(x_1,\ldots x_s) = \sum_{v\in V(G)} t_s\left(\overline{\Gamma(v)}\right).$$

Using Lemma 2.1 and then convexity, we get

$$\sum_{\{x_1,\ldots,x_s\}\in\mathcal{I}_s(G)} d(x_1,\ldots,x_s) \geq \sum_{v\in V(G)} \left(\frac{d(v)^s}{2^s 4^{s^2}} - t_s(\Gamma(v))\right)$$

$$\geq \frac{n}{2^s 4^{s^2}} \left(\frac{1}{n}\sum_{v\in V(G)} d(v)\right)^s - \sum_{v\in V(G)} t_s(\Gamma(v))$$

$$= \frac{n}{2^s 4^{s^2}} \left(\frac{2e(G)}{n}\right)^s - (s+1)t_{s+1}(G)$$

$$= \frac{(e(G))^s}{n^{s-1} 4^{s^2}} - (s+1)t_{s+1}(G).$$

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This inequality, together with (3), gives

$$(r + t)^t n^s \geq \frac{(e(G))^s}{n^{s-1}4^{s^2}} - (s + 1)t_{s+1}(G). \tag{4}$$

By Theorem 1.2,

$$(s + 1) \cdot t_{s+1}(G) \leq 2(t + r)^{t(s+1)/s}(r + s)^s n^s + (r + s)^s n^s$$

and this estimate, together with the previous inequality, gives the result. ∎

3 Clique counting with forbidden induced subgraphs

As in the previous section, r, s, and t are positive integers. In this section we prove our upper bound on the number of m-cliques in any n-vertex, K_r-free graph with no induced copy of $K_{s,t}$.

Proof of Theorem 1.2. Let G be such a graph. We will write $\mathcal{I}_s$ for $\mathcal{I}_s(G)$ and $\mathcal{K}_{m-1}$ for $\mathcal{K}_{m-1}(G)$. Consider the set of pairs

$$S := \{(\{x_1, \ldots, x_s\}, v) : \{x_1, \ldots, x_s\} \in \mathcal{I}_s, v \in \Gamma(x_1, \ldots, x_s)\}.$$

As observed in the proof of Theorem 1.1, the common neighborhood of an s-independent set has no t-independent set or an r-clique. Therefore,

$$|S| = \sum_{\{x_1,\ldots,x_s\} \in \mathcal{I}_s} d(x_1, \ldots, x_s) \leq \binom{n}{s} R(t, r) < (t + r)^t n^s. \tag{5}$$

To give a lower bound on $|S|$, we count from the perspective of $(m - 1)$-cliques with s-independent sets in their neighborhood.

Claim 3.1 *If $\{x_1, \ldots, x_{m-1}\} \in \mathcal{K}_{m-1}$ and $d(x_1, \ldots, x_{m-1}) > R(r - m + 1, s)$, then the number of s-independent sets in $\Gamma(x_1, \ldots, x_{m-1})$ is at least*

$$\left(\frac{d(x_1, \ldots, x_{m-1})}{2(r + s)^s}\right)^s$$

Proof of Claim. If $\{x_1, \ldots, x_{m-1}\}$ forms a clique in G, then its neighborhood can have no clique of order $r - m + 1$. Thus, every set of order $R(r - m + 1, s)$ in its neighborhood must contain an s-independent set. Now, any set of order s in $\Gamma(x_1, \ldots, x_{m-1})$ is contained in at most $\binom{d(x_1,\ldots,x_{m-1})-s}{R(r-m+1,s)-s}$ sets of order $R(r - m + 1, s)$. Therefore,

$$|\mathcal{I}_s(\Gamma(x_1, \ldots, x_{m-1}))| \geq \frac{\binom{d(x_1,\ldots,x_{m-1})}{R(r-m+1,s)}}{\binom{d(x_1,\ldots,x_{m-1})-s}{R(r-m+1,s)-s}} \geq \frac{(d(x_1, \ldots, x_{m-1}) - s)^s}{R(r - m + 1, s)^s}.$$

Using the estimates $d(x_1, \ldots, x_{m-1}) - s > d(x_1, \ldots, x_{m-1})/2$ and $R(r - m + 1, s) < (s + r)^s$ proves the claim. ∎

Returning to the proof of Theorem 1.2, since the vertices in an s-independent set have at most $R(t, r)$ common neighbors, each s-independent set may be contained in the common neighborhood of at most

$$\binom{R(t, r)}{m - 1} < (r + t)^{t(m-1)}$$

$(m-1)$-cliques. Let $B \subset \mathcal{K}_{m-1}$ be the $(m-1)$-cliques in G where the vertices of each $(m-1)$-clique in B have more than $R(r - m + 1, s)$ common neighbors. We have

$$|S| \geq \frac{1}{(r + t)^{t(m-1)}} \sum_{\{x_1, \ldots, x_{m-1}\} \in B} |\mathcal{I}_s(\Gamma(x_1, \ldots, x_{m-1}))|$$

$$\geq \frac{1}{(r + t)^{t(m-1)}} \sum_{\{x_1, \ldots, x_{m-1}\} \in B} \left(\frac{d(x_1, \ldots, x_{m-1})}{2(r + s)^s} \right)^s$$

$$\geq \frac{1}{(r + t)^{t(m-1)}} \frac{\left(\sum_B d(x_1, \ldots, x_{m-1}) \right)^s}{2^s (r + s)^{s^2} |B|^{s-1}}$$

where the second inequality is by Claim 3.1 and the third is by convexity. Since

$$mt_m(G) = \sum_{\{x_1, \ldots, x_{m-1}\} \in \mathcal{K}_{m-1}} d(x_1, \ldots, x_{m-1}),$$

we have

$$\sum_{\{x_1, \ldots, x_{m-1}\} \in B} d(x_1, \ldots, x_{m-1})^s \geq \left(mt_m(G) - \binom{n}{m - 1} R(r - m + 1, s) \right)^s$$

$$\geq \left(mt_m(G) - (r + s)^s n^{m-1} \right)^s.$$

Combining this inequality with our lower bound on $|S|$ and the trivial inequality $|B| < n^{m-1}$ gives

$$|S| \geq \frac{\left(mt_m(G) - (r + s)^s n^{m-1} \right)^s}{(r + t)^{t(m-1)} 2^s (r + s)^{s^2} n^{(s-1)(m-1)}}. \tag{6}$$

Combining (5) and (6) finishes the proof of Theorem 1.2. ∎

4   Sharper results for $K_{2,t+1}$

In this section we prove Theorem 1.3 and Corollary 1.4. We must show that a graph with n vertices, minimum degree d, and no induced $K_{2,t+1}$ must have a clique of order at least $(1 - o(1)) \left(\frac{d^2}{2nt} \right)^{1/t}$. Our argument extends the methods of Gyárfás, Hubenko, and Solymosi [28].

Lemma 4.1 *Let G be a graph with n vertices, minimum degree d, and no induced copy of $K_{2,t+1}$. For any $b \in \{1, 2, \ldots, \alpha(G)\}$,*

$$\omega(G) \geq \left(\frac{bd - n}{t \binom{b}{2}} \right)^{1/t}.$$

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Proof. Let $\{x_1, \ldots, x_b\}$ be a b-independent set. If we let $m = \max_{i \neq j} |N(x_i, x_j)|$, then

$$bd - \binom{b}{2}m \leq \sum_{i=1}^{b} |N(x_i)| - \sum_{1 \leq i < j \leq b} |N(x_i, x_j)| \leq \left| \bigcup_{i=1}^{b} N(x_i) \right| \leq n$$

which implies

$$m \geq \frac{bd - n}{\binom{b}{2}}.$$

Fix a pair $1 \leq i < j \leq b$ with $|N(x_i, x_j)| = m$. If $N(x_i, x_j)$ contains an independent set of order $t+1$, then we get an induced $K_{2,t+1}$. Therefore, by the Erdős-Szekeres bound on Ramsey numbers, $N(x_i, x_j)$ must contain a clique of order at least

$$\left(\frac{|N(x_i, x_j)|}{t} \right)^{1/t}.$$

We conclude that

$$\omega(G) \geq \left(\frac{m}{t} \right)^{1/t} \geq \left(\frac{bd - n}{t\binom{b}{2}} \right)^{1/t}.$$

∎

Proof of Theorem 1.3. Let G be a graph with n vertices, minimum degree d, and no induced copy of $K_{2,t+1}$. Let $\alpha = \alpha(G)$ and let S be an independent set of order α, say $S = \{x_1, \ldots, x_\alpha\}$. Let B_i be the vertices in G whose only neighbor in S is x_i, and let $B_{i,j}$ be the vertices in G that are adjacent to both x_i and x_j (and possibly other vertices of S). Since S is an independent set with the maximum number of vertices, each B_i is a clique and so $\{x_i\} \cup B_i$ is a clique. Also,

$$V(G) = \left(\bigcup_{i=1}^{\alpha} (\{x_i\} \cup B_i) \right) \bigcup \left(\bigcup_{1 \leq i < j \leq \alpha} B_{i,j} \right) \tag{7}$$

otherwise we could create a larger independent set by adding a vertex to S.

Claim 4.2 *Either* $|\{x_i\} \cup B_i| \geq \left(\frac{d^2}{2nt} \right)^{1/t}$ *for some* $i \in \{1, 2, \ldots, \alpha\}$, *or*

$$\omega(G) \geq \left(\frac{n - \alpha \left(\frac{d^2}{2nt} \right)^{1/t}}{t\binom{\alpha}{2}} \right)^{1/t}. \tag{8}$$

Proof of Claim. Assume that $|\{x_i\} \cup B_i| \leq \left(\frac{d^2}{2nt} \right)^{1/t}$ for $1 \leq i \leq \alpha$. By (7),

$$n \leq \alpha \left(\frac{d^2}{2nt} \right)^{1/t} + \sum_{1 \leq i < j \leq \alpha} |B_{i,j}|.$$

By averaging, there is a pair $1 \leq i < j \leq \alpha$ such that

$$|B_{i,j}| \geq \frac{n - \alpha \left(\frac{d^2}{2nt}\right)^{1/t}}{\binom{\alpha}{2}}.$$

The set $B_{i,j}$ cannot contain a $(t+1)$-independent set otherwise we have an induced $K_{2,t+1}$ using the vertices x_i and x_j. We conclude that

$$\omega(G) \geq \left(\frac{n - \alpha \left(\frac{d^2}{2nt}\right)^{1/t}}{t\binom{\alpha}{2}}\right)^{1/t}.$$

$\blacksquare$

Returning to the proof of Theorem 1.3, we now use Lemma 4.1 and Claim 4.2 to show that

$$\omega(G) \geq (1 - o(1)) \left(\frac{d^2}{2nt}\right)^{1/t}.$$

If $\alpha(G) \geq \frac{2n}{d}$, then we take $b = \frac{2n}{d}$ and apply Lemma 4.1 to get,

$$\omega(G) \geq \left(\frac{(2n/d)d - n}{t(1/2)(2n/d)^2}\right)^{1/t} = \left(\frac{d^2}{2nt}\right)^{1/t}.$$

Assume that $\alpha(G) \leq \frac{2n}{d}$. We then apply Claim 4.2 to get that either $|\{x_i\} \cup B_i| \geq (d^2/2nt)^{1/t}$, in which case $\omega(G) \geq (d^2/2nt)^{1/t}$, or (8) holds. Using $\alpha(G) \leq \frac{2n}{d}$ together with (8) gives

$$\omega(G) \geq \left(\frac{2(n - (2n/d)(d^2/2n)^{1/t})}{4tn^2/d^2}\right)^{1/t} \geq \left(\frac{d^2}{2nt}\right)^{1/t} \left(1 - \frac{2}{d}\left(\frac{d^2}{2n}\right)^{1/t}\right)^{1/t}.$$

Since $t \geq 2$, we have $\frac{2}{d}\left(\frac{d^2}{2n}\right)^{1/t} \leq \frac{2}{n^{1/t}} = o(1)$ so

$$\omega(G) \geq \left(\frac{d^2}{2nt}\right)^{1/t} (1 - o(1))^{1/t}.$$

$\blacksquare$

Proof of Corollary 1.4. Let $t \geq 2$ be an integer and let H be a graph with v_H vertices. Suppose G is an n-vertex H-free graph with no induced $K_{2,t+1}$. Let d be the average degree of d. Let G' be an H-free subgraph of G with minimum degree $d/2$ and no induced $K_{2,t+1}$ (see Proposition 1.2.2 of Diestel's book [14]). By Theorem 1.3, G' has a clique of order at least

$$(1 - o(1)) \left(\frac{d^2}{8nt}\right)^{1/t}.$$

Since G' is H-free, G' cannot have a clique of order v_H so

$$(1 - o(1)) \left(\frac{d^2}{8nt}\right)^{1/t} < v_H.$$

Since $d = e(G)/n$, we can solve this inequality for $e(G)$ to get that

$$e(G) < (\sqrt{2} + o(1))t^{1/2}v_H^{t/2}n^{3/2}.$$

$\blacksquare$

4.1   Forbidding an odd cycle

In this section we prove Theorem 1.5. We must show that for integers $k \geq 2$ and $t \geq 2$, any n-vertex C_{2k+1}-free graph with no induced $K_{2,t}$ has at most

$$\left(\alpha(k,t)^{1/2} + 1\right)^{1/2} \frac{n^{3/2}}{2} + \beta_k n^{1+1/2k}$$

edges where $\alpha(k,t) = (2k-2)(t-1)((2k-2)(t-1)-1)$.

Proof of Theorem 1.5.   Suppose G is a C_{2k+1}-free graph with n vertices and no induced copy of $K_{2,t}$. For any pair of distinct non-adjacent vertices x and y, the common neighborhood $N(x,y)$ cannot contain a path of length $2k-1$ or an independent set of order t. A classical result of Erdős and Gallai is that any graph with at least $(a-1)(b-1)+1$ vertices must contain a path of length a or an independent set of order b (see Parsons [36]). Therefore,

$$d(x,y) \leq (2k-2)(t-1). \tag{9}$$

Let $\overline{e} = \binom{n}{2} - e(G)$. By convexity and (9),

$$\frac{\alpha(k,t)}{2}\binom{n}{2} \geq \sum_{\{x,y\}\notin E(G)} \binom{d(x,y)}{2} \geq \overline{e}\binom{\frac{1}{\overline{e}}\sum_{\{x,y\}\notin E(G)} d(x,y)}{2} \tag{10}$$

where $\alpha(k,t) := (2k-2)(t-1)((2k-2)(t-1)-1)$. We can rewrite the sum $\sum_{\{x,y\}\notin E(G)} d(x,y)$ as

$$\sum_{\{x,y\}\notin E(G)} d(x,y) = \sum_{z\in V(G)} \left(\binom{d(z)}{2} - e(\Gamma(z))\right) = \sum_{z\in V(G)} \binom{d(z)}{2} - 3t_3(G). \tag{11}$$

By convexity,

$$\sum_{z\in V(G)} \binom{d(z)}{2} \geq n\binom{2e/n}{2} \tag{12}$$

where e is the number of edges of G. By a result of Győri and Li [29], since G is C_{2k+1}-free the number of triangles in G is at most $(c_k/3)n^{1+1/k}$. Here c_k is a constant depending only on k. This fact, together with (11) and (12), give

$$\sum_{\{x,y\}\notin E(G)} d(x,y) \geq n\binom{2e/n}{2} - c_k n^{1+1/k}.$$

Combining this with (10) leads to

$$\frac{\alpha(k,t)}{2}\binom{n}{2} \geq \bar{e}\left(\frac{\frac{1}{\bar{e}}\left(n\binom{2e/n}{2} - c_k n^{1+1/k}\right)}{2}\right)$$

$$\geq \frac{\bar{e}}{2}\left(\frac{n}{\bar{e}}\binom{2e/n}{2} - \frac{c_k n^{1+1/k}}{\bar{e}} - 1\right)^2$$

$$= \frac{1}{2\bar{e}}\left(n\binom{2e/n}{2} - c_k n^{1+1/k} - \bar{e}\right)^2.$$

Using the trivial estimate $\bar{e} \leq \binom{n}{2}$, we have

$$\alpha(k,t)\binom{n}{2}^2 \geq \left(n\binom{2e/n}{2} - c_k n^{1+1/k} - \bar{e}\right)^2.$$

A straightforward calculation gives

$$\left(\alpha(k,t)^{1/2} + 1\right)\binom{n}{2} + c_k n^{1+1/k} \geq \frac{2e^2}{n}$$

from which it follows that

$$\left(\alpha(k,t)^{1/2} + 1\right)^{1/2}\frac{n^{3/2}}{2} + \sqrt{\frac{c_k}{2}}n^{1+1/2k} \geq e.$$

∎

5   Concluding remarks

In this paper, we introduced a new function $\mathrm{ex}(n, \{H, F\text{-ind}\})$, and established its interesting behavior, including a proof of the asymptotic bound of Kővári, Sós, and Turán for induced copies of $K_{s,t}$, as long as any other subgraph is also forbidden. That result relied on Theorem 1.2, which bounded the number of fixed-order cliques in K_r-free graphs with no induced copy of $K_{s,t}$. Our bound in that theorem is probably not tight, and although it served our purposes in this paper, it is an independently interesting question (along the lines of Alon and Shikhelman's problem in [2]) to resolve its asymptotic behavior. A close look at the proof of Theorem 1.2 shows that if G is an n-vertex graph with no copy of K_r and no induced copy of $K_{s,t}$, then

$$
\begin{aligned}
(m+1)t_{m+1}(G) &\leq i_s(G)^{1/s} t_m(G)^{1-1/s} R(r,t)^{1/s}\binom{R(r,t)}{m-1}^{1/s} 2R(r-m,s) \\
&+ 2st_m(G)R(r-m,s)
\end{aligned}
\tag{13}
$$

where $i_s(G)$ is the number of independent sets in G with s vertices, and $t_k(G)$ is the number of cliques in G with k vertices. In proving Theorem 1.2, we used the trivial estimate $t_k(G) \leq n^k$ which is valid for all n and k. We can do better in the case that n tends to infinity. Indeed, suppose $k \geq 2$ is a fixed integer and $t_k(G) \geq cn^k$ holds for some fixed constant $c > 0$. In this case, G must contain a

complete k-partite graph with at least $c^k \log n$ vertices in each part. If two of these parts have an independent set of size $s+t$, then we find an induced $K_{s,t}$. Therefore, at least one of these parts must have a clique of size at least $\left(\frac{c^k \log n}{s+t}\right)^{1/(s+t)}$. If n is large enough, then this clique will contain a copy of H. We conclude that $t_k(G) = o(n^k)$ so by (13),

$$t_{m+1}(G) = o(n^{m+1-m/s} + n^m).$$

One can also obtain polynomial improvements using a recursive argument. For example, when $s = 2$, we have a bound of the form $k_2(G) = O(n^{2-1/s})$ and so with (13), we get $k_3(G) = O(n^{7/4})$ for an H-free n-vertex graph G with no induced $K_{2,t}$. These improvements become negligible as s and t tend to infinity.

Apart from its natural interest, another side effect of an improvement could also potentially translate into a constant factor improvement in the Erdős-Hajnal problem for forbidden $K_{s,t}$. (The conjecture for that forbidden subgraph was known since the original Erdős-Hajnal paper [19], which covered the more general case of *cographs*.) In connection with the Erdős-Hajnal conjecture, we note the following corollary of Theorem 1.1, which complements the work of Gyárfás, Hubenko, and Solymosi in [28] and Theorem 1.3.

Corollary 5.1 *If G has average degree d and no copy of $K_{s,t}$ as an induced subgraph, then*

$$\omega(G) = \Omega\left(\left(\frac{d^s}{n^{s-1}}\right)^{\frac{1}{t-1+s-1}}\right).$$

Proof. Let G be an n-vertex graph with average degree d and no copy of $K_{s,t}$ as an induced subgraph. Let $\omega = \omega(G)$ and note that G contains no copy of K_{w+1}. By (4),

$$\frac{e(G)^s}{n^{s-1}4^{s^2}} \leq R(\omega,t)n^s + (s+1)t_{s+1}(G).$$

Next we replace $e(G)$ with $\frac{nd}{2}$ and use (13) with $r = \omega + 1$ and $m = s$ to get

$$\frac{d^s}{2^s 4^{s^2} n^{s-1}} \leq R(\omega,t) + R(\omega,t)^{1/s}\binom{R(\omega,t)}{s-1}^{1/s} 2R(\omega,s) + 2sR(\omega,s).$$

For large enough ω, we have $R(\omega,t) < \omega^{t-1}$ and $R(\omega,s) < \omega^{s-1}$ so

$$\frac{d^s}{2^s 4^{s^2} n^{s-1}} \leq \omega^{t-1} + 2\omega^{\frac{t-1}{s}} \omega^{(t-1)\frac{s-1}{s}} 2\omega^{s-1} + 2s\omega^{s-1} \leq s\omega(G)^{t-1+s-1}.$$

∎

It also remains open to estimate $\mathrm{ex}(n, \{H, F\text{-ind}\})$ with greater accuracy. The bounds would likely depend on the structures of H and F. The results from the later sections of our paper start this investigation by proving some bounds in the case of odd cycles and $K_{2,t}$-induced. It would be interesting if the behavior of this function is sometimes determined by natural parameters of H and F, as in the case of the ordinary Turán problem.

Finally, we note that using the same technique as in the proof of Theorem 1.5, the main result of [6], and a result of Maclaurin now called the Fisher-Ryan inequalities [24, 37], one can show

$$\text{ex}(n, \{C_{2k+1}, K_{s,s}\text{-ind}\}) \leq \frac{4^s(s-1)^{1/s}(2k-3)^{1/s}}{(s!)^{1/s}}n^{2-1/s} + o(n^{2-1/s}),$$

where $k \geq s \geq 3$. Whereas Theorem 1.5 has the correct dependence on t, we do not know if the above equation has the correct dependence on s.

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