Degree Ramsey numbers for even cycles

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Abstract

Let $H \xrightarrow{s} G$ denote that any s-coloring of $E(H)$ contains a monochromatic G. The degree Ramsey number of a graph G, denoted by $R_\Delta(G,s)$, is $\min\{\Delta(H) : H \xrightarrow{s} G\}$. We consider degree Ramsey numbers where G is a fixed even cycle. Kinnersley, Milans, and West showed that $R_\Delta(C_{2k},s) \geq 2s$, and Kang and Perarnau showed that $R_\Delta(C_4,s) = \Theta(s^2)$. Our main result is that $R_\Delta(C_6,s) = \Theta(s^{3/2})$ and $R_\Delta(C_{10},s) = \Theta(s^{5/4})$. Additionally, we substantially improve the lower bound for $R_\Delta(C_{2k},s)$ for general k.

1 Introduction

Theorems in Ramsey theory state that if a structure is in some suitable sense "large enough", then it must contain a fixed substructure. The classical *Ramsey number* of a graph G, denoted by $R(G,s)$, is the smallest n such that $K_n \xrightarrow{s} G$, where $H \xrightarrow{s} G$ denotes that any s-coloring of the edges of H produces a monochromatic subgraph isomorphic to G. Classical Ramsey numbers may be thought of in a more general setting, as just one type of parameter Ramsey number. Note that the classical Ramsey number of G is $\min\{|V(H)| : H \xrightarrow{s} G\}$. For any monotone graph parameter ρ, one may define the *ρ-Ramsey number* of G, denoted by $R_\rho(G,s)$, to be

$$\min\{\rho(H) : H \xrightarrow{s} G\}.$$

This generalizes the classical Ramsey number as $R_\rho(G,s)$ is the Ramsey number for G when $\rho(H)$ denotes the number of vertices in H. The study of parameter Ramsey numbers dates back to the 1970s [7]. Since then, many researchers have studied this quantity when $\rho(H)$ is the clique number of H [12, 23, 25] (giving way to the study of Folkman numbers), when $\rho(H)$ is the number of edges in H [3, 4, 9, 10, 11, 13, 27] (now called the *size Ramsey number*), when $\rho(H) = \chi(H)$ [7, 28, 29] or when it is the circular chromatic number [15]. In this note we are interested in the *degree Ramsey number*, which is when $\rho(H) = \Delta(H)$.

Burr, Erdős, and Lovász [7] showed that $R_\Delta(K_n,s) = R(K_n,s)-1$. Kinnersley, Milans, and West [18] and Jiang, Milans, and West [16] proved several theorems regarding the degree Ramsey numbers of trees and cycles. Horn, Milans, and Rödl showed that the family of closed blowups of trees is R_Δ-bounded (that their degree

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Ramsey number is bounded by a function of the maximum degree of the graph and s). The main open question in this area is whether the set of all graphs is R_Δ-bounded (see [8]).

The main result of this note is to determine the order of magnitude of $R_\Delta(C_6, s)$ and $R_\Delta(C_{10}, s)$. Kang and Perarnau [17] showed that $R_\Delta(C_4, s) = \Theta(s^2)$, and for general k, the best lower bound on $R_\Delta(C_{2k}, s)$ is by Kinnersley, Milans, and West [18] who show $R_\Delta(C_{2k}, s) \geq 2s$. We substantially improve this lower bound in Theorem 1.3.

As the determination of Ramsey numbers for C_{2k} is closely related to the Turán number for C_{2k}, one may find it natural that the order of magnitude for $R_\Delta(C_{2k}, s)$ should be able to be determined for $k \in \{2, 3, 5\}$ but in no other cases. This is also the current state of affairs for the Turán numbers $\mathrm{ex}(n, C_{2k})$ as well as for the classical Ramsey numbers, where Li and Lih [22] showed that $R(C_{2k}, s) = \Theta\left(s^{k/(k-1)}\right)$ for $k \in \{2, 3, 5\}$.

Before we state our theorems, we need some preliminary definitions. For graphs F and G, a *locally injective homomorphism* from F to G is a graph homomorphism $\phi : V(F) \to V(G)$ such that for every $v \in V(F)$, the restriction of ϕ to the neighborhood of v is injective. Let $\mathcal{L}_F$ denote the family of all graphs H such that there is a locally injective homomorphism from F to H. We say that a graph is $\mathcal{L}_F$-free if it does not contain any $H \in \mathcal{L}_F$. We now state our main theorem.

Theorem 1.1 *$R_\Delta(C_6, s) = \Theta(s^{3/2})$ and $R_\Delta(C_{10}, s) = \Theta(s^{5/4})$.*

To prove this theorem, we first show that the complete graph can be partitioned "efficiently" into graphs coming from the generalized quadrangle and generalized hexagon. Then we use the following general theorem, which is implicit in the work of Kang and Perarnau [17].

Theorem 1.2 (Kang-Perarnau [17]) *Let F be a graph with at least one cycle and $\epsilon > 0$ be fixed and let G be a graph of maximum degree Δ. If the edges of K_n can be partitioned into $O\left(n^{1-\epsilon}\right)$ $\mathcal{L}_F$-free graphs, then G can be partitioned into $O\left(\Delta^{1-\epsilon}\right)$ graphs which are F-free.*

They did not state their theorem in this way, and for completeness we sketch its proof in Section 3. Stating it in this general way allows us to improve the result of Kinnersley, Milans, and West [18].

Theorem 1.3 *Let k be fixed and $\delta = 0$ if k is odd and $\delta = 1$ if k is even. Then*

$$R_\Delta(C_{2k}, s) = \Omega\left(\left(\frac{s}{\log s}\right)^{1+\frac{2}{3k-5+\delta}}\right).$$

We prove our main theorem in Section 2. We sketch the proof of Theorem 1.2 and use it to prove Theorem 1.3 in Section 3.

2 Proof of Theorem 1.1

The theorem follows from Theorem 1.2 and the following proposition which we prove after the proof of Theorem 1.1.

Proposition 2.1 *The edge set of K_n may be partitioned into $O(n^{2/3})$ graphs of girth 8 or $O(n^{4/5})$ graphs of girth 12.*

Proof of Theorem 1.1. Showing that $R_\Delta(G,s) \geq k$ is equivalent to showing that any graph of maximum degree at most k may be partitioned into s graphs each of which are G-free. We notice that if there is a locally injective homomorphism from C_n to a graph H, then H must contain a cycle of length at most n. Therefore, if a graph has girth g, then it is $\mathcal{L}_{C_n}$-free for any $n \leq g$.

Therefore, by Proposition 2.1 and Theorem 1.2, if G is a graph of maximum degree Δ, then G can be partitioned into $O(\Delta^{2/3})$ graphs which are C_6-free or $O(\Delta^{4/5})$ graphs which are C_{10}-free. This shows that $R_\Delta(C_6,s) = \Omega(s^{3/2})$ and $R_\Delta(C_{10},s) = \Omega(s^{5/4})$.

The upper bound follows from the classical even-cycle theorem of Erdős, that $\mathrm{ex}(n, C_{2k}) = O(n^{1+1/k})$ (cf [5]). If $E(K_n)$ is colored with s colors, then one color class contains at least $\binom{n}{2}/s$ edges. Therefore, for a constant c_k depending only on k, if $c_k n^{1-1/k} > s$, then any s-coloring of K_n contains a monochromatic C_{2k}. This implies

$$R_\Delta(C_{2k},s) \leq \left(\frac{s}{c_k}\right)^{1+\frac{1}{k-1}} - 1.$$

We note that the best constant c_k that is known comes from the current best-known upper bound for $\mathrm{ex}(n, C_{2k})$ by Bukh and Jiang [6]. ■

Proof of Proposition 2.1. We use the generalized quadrangle and the generalized hexagon to partition K_n efficiently into graphs of girth 8 and 12 respectively. Let $q \geq 5$ be a prime and $\mathbb{F}_q$ the field of order q. We define bipartite graphs Q and H. Let $V(Q) = \mathcal{P}_Q \cup \mathcal{L}_Q$ and $V(H) = \mathcal{P}_H \cup \mathcal{L}_H$ where $\mathcal{P}_Q = \{(p_1,p_2,p_3) : p_i \in \mathbb{F}_q\}$, $\mathcal{L}_Q = \{(l_1,l_2,l_3) : l_i \in \mathbb{F}_q\}$, $\mathcal{P}_H = \{(p_1,p_2,p_3,p_4,p_5) : p_i \in \mathbb{F}_q\}$, and $\mathcal{L}_H = \{(l_1,l_2,l_3,l_4,l_5) : l_i \in \mathbb{F}_q\}$. Now define $E(Q)$ by $(p_1,p_2,p_3) \sim (l_1,l_2,l_3)$ if and only if

$$\begin{aligned} l_2 - p_2 &= l_1 p_1 \\ l_3 - 2p_3 &= -2l_1 p_2 \end{aligned}$$

and $E(H)$ by $(p_1,p_2,p_3,p_4,p_5) \sim (l_1,l_2,l_3,l_4,l_5)$ if and only if

$$\begin{aligned} l_2 - p_2 &= l_1 p_1 \\ l_3 - 2p_3 &= -2l_1 p_2 \\ l_4 - 3p_4 &= -3l_1 p_3 \\ 2l_5 - 3p_5 &= 3l_3 p_2 - 3l_2 p_3 + l_4 p_1 \end{aligned}$$

The graphs Q and H are q-regular bipartite graphs on $2q^3$ and $2q^5$ vertices respectively. In [20] (Theorems 2.1 and 2.5), Lazebnik and Ustimenko showed that Q has girth 8 and H has girth 12. Q and H are large induced subgraphs of the incidence graph of the generalized quadrangle and generalized hexagon (see also [19]). We first show that we may partition K_{p^3,p^3} with disjoint copies of Q and K_{p^5,p^5} with disjoint copies of H.

Let (α_2,α_3) be an arbitrary pair in $\mathbb{F}_q^2$ and $(\beta_2,\beta_3,\beta_4,\beta_5)$ an arbitrary quadruple in $\mathbb{F}_q^4$. Define the graph Q_{α_2,α_3} to be the graph with vertex set $V(Q)$ and $(p_1,p_2,p_3) \sim (l_1,l_2,l_3)$ if and only if

$$
\begin{aligned}
l_2 - (p_2 + \alpha_2) &= l_1 p_1 \\
l_3 - 2(p_3 + \alpha_3) &= -2l_1(p_2 + \alpha_2)
\end{aligned}
$$

and $H_{\beta_2,\beta_3,\beta_4,\beta_5}$ to be the graph with vertex set $V(H)$ and $(p_1, p_2, p_3, p_4, p_5) \sim (l_1, l_2, l_3, l_4, l_5)$ if and only if

$$
\begin{aligned}
l_2 - (p_2 + \beta_2) &= l_1 p_1 \\
l_3 - 2(p_3 + \beta_3) &= -2l_1(p_2 + \beta_2) \\
l_4 - 3(p_4 + \beta_4) &= -3l_1(p_3 + \beta_3) \\
2l_5 - 3(p_5 + \beta_5) &= 3l_3(p_2 + \beta_2) - 3l_2(p_3 + \beta_3) + l_4 p_1
\end{aligned}
$$

Then Q_{α_2,α_3} is isomorphic to Q with the explicit isomorphism given by $(p_1, p_2, p_3) \mapsto (p_1, p_2 - \alpha_2, p_3 - \alpha_3)$ and $(l_1, l_2, l_3) \mapsto (l_1, l_2, l_3)$ and $H_{\beta_2,\beta_3,\beta_4,\beta_5}$ is isomorphic to H with isomorphism given by $(p_1, p_2, p_3, p_4, p_5) \mapsto (p_1, p_2 - \beta_2, p_3 - \beta_3, p_4 - \beta_4, p_5 - \beta_5)$ and $(l_1, l_2, l_3, l_4, l_5) \mapsto (l_1, l_2, l_3, l_4, l_5)$.

Now we claim that the family $\{Q_{\alpha_2,\alpha_3}\}_{\alpha_i \in \mathbb{F}_q}$ covers K_{q^3,q^3} and $\{H_{\beta_2,\beta_3,\beta_4,\beta_5}\}_{\beta_i \in \mathbb{F}_q}$ covers K_{q^5,q^5}. Since each graph is q regular and there are q^2 and q^4 of them respectively, each cover is also a partition. To show that the edges of the complete bipartite graph are all covered, let (p_1, p_2, p_3) and (l_1, l_2, l_3) be arbitrary and fixed. We must show that there is a choice of α_2 and α_3 such that

$$
\begin{aligned}
l_2 - (p_2 + \alpha_2) &= l_1 p_1 \\
l_3 - 2(p_3 + \alpha_3) &= -2l_1(p_2 + \alpha_2)
\end{aligned}
$$

It is clear that there is a unique solution α_2, α_3 to this triangular system. Similarly, for $(p_1, p_2, p_3, p_4, p_5)$ and $(l_1, l_2, l_3, l_4, l_5)$ arbitrary and fixed, there is a unique solution $\beta_2, \beta_3, \beta_4, \beta_5$ to the system

$$
\begin{aligned}
l_2 - (p_2 + \beta_2) &= l_1 p_1 \\
l_3 - 2(p_3 + \beta_3) &= -2l_1(p_2 + \beta_2) \\
l_4 - 3(p_4 + \beta_4) &= -3l_1(p_3 + \beta_3) \\
2l_5 - 3(p_5 + \beta_5) &= 3l_3(p_2 + \beta_2) - 3l_2(p_3 + \beta_3) + l_4 p_1
\end{aligned}
$$

Therefore, K_{q^3,q^3} may be covered by q^2 graphs each of girth 8, and K_{q^5,q^5} may be covered with q^4 graphs each of girth 12. Now we must show that we can use a partition of $K_{n,n}$ to find an efficient covering of K_n where we only lose a constant multiplicative factor in the number of graphs used. Li and Lih showed this in [22], and we include the details for completeness.

By standard results on density of the primes, we have shown above that we may partition $K_{n,n}$ into $(1+o(1))n^{2/3}$ graphs of girth 8 or $(1+o(1))n^{4/5}$ graphs of girth 12. Break the vertex set of K_n into two parts V_1 and V_2 with sizes as equal as possible. We may cover the edges between V_1 and V_2 with $(1+o(1))(n/2)^{2/3}$ graphs of girth 8 or $(1+o(1))(n/2)^{4/5}$ graphs of girth 12. Now break V_1 and V_2 into equal size pieces U_1, U_2, U_3, U_4 each of size $n/4$. Since the disjoint union of two graphs of girth g still has girth g, we may cover the edges between U_1 and U_2 and the edges between U_3 and U_4 with $(1+o(1))(n/4)^{2/3}$ graphs of girth 8 or $(1+o(1)(n/4)^{4/5}$ graphs of girth 12. Repeating this procedure allows us to cover all of the edges in K_n with

$$\sum_{i=1}^{O(\log n)} (1+o(1)) \left(\frac{n}{2^i}\right)^{2/3} = O(n^{2/3})$$

graphs of girth 8 or

$$\sum_{i=1}^{O(\log n)} (1+o(1)) \left(\frac{n}{2^i}\right)^{4/5} = O(n^{4/5})$$

graphs of girth 12.

■

We note that this section shows that $R(\{C_3, \cdots C_{2k}\}, s) = \Theta(s^{k/(k-1)})$ for $k \in \{2,3,5\}$ and implies the main result of [22]. We also note that we have seen in this section that by Theorem 1.2, giving good lower bounds on degree Ramsey numbers for a graph F can be reduced to giving good lower bounds on the classical Ramsey number for $\mathcal{L}_F$. In the case that $F = K_{a,b}$, we have that $\mathcal{L}_F = \{K_{a,b}\}$. Using the projective norm graphs, Alon, Rónyai, and Szabó [1] showed that for $a > (b-1)!$, $R(K_{a,b}, s) = \Theta(s^b)$. Along with Theorem 1.2, this shows that for $a > (b-1)!$, one also has $R_\Delta(K_{a,b}, s) = \Theta(s^b)$.

3 Proof of Theorem 1.2

Throughout this section, assume that F is a graph with at least one cycle, that $\epsilon > 0$ is fixed, and that the edges of K_n can be partitioned into $O(n^{1-\epsilon})$ graphs which are $\mathcal{L}_F$-free. Call a coloring of a graph a *proper rainbow coloring* if the coloring is proper, and the restriction of the coloring to any neighborhood is an injection (ie each vertex sees a rainbow). To prove Theorem 1.2 we need the following lemma, which appears in [17]. Similar lemmas appear in [26] and [24].

Sketch of Proof of Theorem 1.2.

Lemma 3.1 *Let G be a graph of sufficiently large maximum degree Δ and minimum degree $\delta \geq \log^2 \Delta$. Then there is a spanning subgraph H of G and a proper rainbow coloring $\chi(H)$ using at most 200Δ colors such that for all $v \in V(G)$, $d_H(v) \geq \frac{1}{10} d_G(v)$.*

This lemma allows us to use the partition of K_n to partition a large piece of our graph G (viz H) into F-free subgraphs.

Proposition 3.2 *Let G be a graph of sufficiently large maximum degree Δ and minimum degree $\delta \geq \log^2 \Delta$. There exist $l = O(\Delta^{1-\epsilon})$ disjoint spanning subgraphs $H_1, \cdots, H_l$, all of which are F-free, such that for all $v \in V(G)$*

$$\sum_{i=1}^{l} d_{H_i}(v) \geq \frac{1}{10} d_G(v).$$

Proof. Recall that the edge set of $K_{200\Delta}$ can be partitioned into $l = O(\Delta^{1-\epsilon})$ graphs which are $\mathcal{L}_F$-free. Denote these graphs by $G_1, \cdots, G_l$. Let H be the spanning subgraph of G with coloring χ from Lemma 3.1. Recall that χ is a proper rainbow coloring using at most 200Δ colors and that $d_H(v) \geq \frac{1}{10} d_G(v)$ for all v. For $1 \leq i \leq l$, define graphs H_i which are subgraphs of H by $V(H_i) = V(G)$ and $uv \in E(H_i)$ if and only if

$$\chi(u)\chi(v) \in E(G_i) \text{ and } uv \in E(H).$$

Since $G_1, \cdots, G_l$ is a partition of $E(K_{200\Delta})$, we have that $H_1, \cdots, H_l$ is a partition of H, and thus the minimum degree condition is satisfies. To see that each H_i is F-free, we claim that for each i, χ is a locally injective homomorphism from H_i to G_i. To see this, note that the definition of $E(H_i)$ guarantees that χ is a homomorphism from H_i to G_i, and χ being a rainbow coloring implies that the homomorphism is locally injective. Since G_i is $\mathcal{L}_F$-free, we have that H_i is F-free. ■

Let G_1 be the graph obtained from G by sequentially removing any vertex of degree less than $\log^2 \Delta$, and let G_2 be the graph whose edges are $E(G) \setminus E(G_1)$. Since G_2 has degeneracy at most $\log^2 \Delta$, it has arboricity at most $\log^2 \Delta$ and so we may partition $E(G_2)$ into that many forests. Since F contains a cycle, each of these are F-free. Now we may apply Proposition 3.2 to G_1, and have therefore covered a large piece of G with at most $O(\Delta^{1-\epsilon}) + \log^2 \Delta$ graphs which are F-free. Removing these edges decreases the maximum degree by a multiplicative factor of at least $\frac{9}{10}$. We may repeat this procedure until the maximum degree of part of the graph which is not yet covered is less than $\log^2 \Delta$. Since the maximum degree decreases by a constant multiplicative factor at each step, this will take at most $O(\log \Delta)$ steps. Therefore, the total number of graphs used to cover $E(G)$ is

$$\sum_{i=0}^{O(\log \Delta)} 200 \left(\left(\frac{9}{10} \right)^i \Delta \right)^{1-\epsilon} + \log^2 \Delta = O\left(\Delta^{1-\epsilon}\right) + O(\log^3 \Delta).$$

■

Proof of Theorem 1.3. Let k be fixed and let $\delta = 0$ if k is odd and 1 if k is even. Showing that

$$R_\Delta(C_{2k}, s) = \Omega\left(\left(\frac{s}{\log s} \right)^{1+\frac{2}{3k-5+\delta}} \right)$$

is equivalent to showing that any graph of maximum degree Δ can be partitioned into $O(\Delta^{1-\frac{2}{3k-3+\delta}} \log \Delta)$ graphs each with no copy of C_{2k}. By the same argument in the proof of Theorem 1.1 and by Theorem 1.2, it suffices to show that K_n can be partitioned into $O(n^{1-\frac{2}{3k-3+\delta}} \log n)$ graphs of girth greater than $2k$. Lazebnik, Ustimenko, and Woldar [21] showed that there are graphs on n vertices and $\epsilon_k n^{1+\frac{2}{3k-3+\delta}}$ edges that have girth at least $2k+2$, where ϵ_k is a constant depending only on k. For C a constant to be chosen later, place $Cn^{1-\frac{2}{3k-3+\delta}} \log n$ copies of this graph onto K_n, each time permuting the vertices with a permutation $\sigma \in S_n$ chosen uniformly at random and independently. For each pair u, v, let $X_{u,v}$ be the random variable that counts how many times the edge uv in K_n is covered. We are done if we can show that all of the X_{uv} are positive with positive probability. Since the expected value of each X_{uv} is greater than $\epsilon_k C \log n$, the Chernoff bound (cf [2]) gives that

$$\mathbb{P}(X_{uv} = 0) \leq \exp\left(\frac{-\epsilon_k C \log n}{3}\right).$$

For C large enough this is $o(n^{-2})$, and the union bound gives that every edge is covered with probability tending to 1. ■

Indeed, this proof shows a more general theorem:

Theorem 3.3 *Let F be a fixed graph containing at least one cycle and let* $\mathrm{ex}(n, \mathcal{L}_F) = \Omega(n^{1+\eta})$. *Then if $\eta < 1$,*

$$R_\Delta(F, s) = \Omega\left(\frac{s}{\log s}\right)^{\frac{1}{1-\eta}},$$

and if $\eta = 1$, then

$$R_\Delta(F, s) = 2^{\Omega(s^{1/4})}.$$

4 Concluding Remarks

Determining the order of magnitude for $R_\Delta(C_{2k}, s)$ for $k \notin \{2, 3, 5\}$ is out of reach with the current state of knowledge, as any improvement to the best-known exponents would yield a corresponding improvement in the best-known exponents for $\mathrm{ex}(n, C_{2k})$. One should be able to remove the logarithmic factor in the denominator of Theorem 1.3 but we could not see an easy way to do this. Probably the most interesting open question in the area of degree Ramsey numbers is whether or not $R_\Delta(G, s)$ is bounded by some function of $\Delta(G)$ and s. Horn, Milans, and Rödl [14] showed that this is true for the family of closed blowups of trees. However, it is not clear that it should be true for general G.

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References

[1] Noga Alon, Lajos Rónyai, and Tibor Szabó. Norm-graphs: variations and applications. *Journal of Combinatorial Theory, Series B*, 76(2):280–290, 1999.

[2] Noga Alon and Joel H Spencer. *The probabilistic method*. John Wiley & Sons, 2000.

[3] József Beck. On size ramsey number of paths, trees, and circuits i. *Journal of Graph Theory*, 7(1):115–129, 1983.

[4] József Beck. On size ramsey number of paths, trees, and circuits ii. *Algorithms and Combinatorics*, 5:34–45, 1983.

[5] John A Bondy and Miklós Simonovits. Cycles of even length in graphs. *Journal of Combinatorial Theory, Series B*, 16(2):97–105, 1974.

[6] Boris Bukh and Zilin Jiang. A bound on the number of edges in graphs without an even cycle. *arXiv preprint arXiv:1403.1601*, 2014.

[7] Stefan A Burr, Paul Erdős, and Laszlo Lovász. On graphs of ramsey type. *Ars Combinatoria*, 1(1):167–190, 1976.

[8] David Conlon, Jacob Fox, and Benny Sudakov. Recent developments in graph ramsey theory. *Surveys in Combinatorics*, pages 49–118, 2015.

[9] Domingos Dellamonica. The size-ramsey number of trees. *Random Structures & Algorithms*, 40(1):49–73, 2012.

[10] Jair Donadelli, Penny E Haxell, and Yoshiharu Kohayakawa. A note on the size-ramsey number of long subdivisions of graphs. *RAIRO-Theoretical Informatics and Applications*, 39(1):191–206, 2005.

[11] Paul Erdős, Ralph J Faudree, Cecil C Rousseau, and Richard H Schelp. The size ramsey number. *Periodica Mathematica Hungarica*, 9(1-2):145–161, 1978.

[12] Jon Folkman. Graphs with monochromatic complete subgraphs in every edge coloring. *SIAM Journal on Applied Mathematics*, 18(1):19–24, 1970.

[13] Joel Friedman and Nicholas Pippenger. Expanding graphs contain all small trees. *Combinatorica*, 7(1):71–76, 1987.

[14] Paul Horn, Kevin G Milans, and Vojtěch Rödl. Degree ramsey numbers of closed blowups of trees. *The Electronic Journal of Combinatorics*, 21(2):P2–5, 2014.

[15] Fang-Kai Jao. *On vertex degrees, graph decomposition, and circular chromatic Ramsey number*. PhD thesis, University of Illinois at Urbana Champaign, 2013.

[16] Tao Jiang, Kevin G Milans, and Douglas B West. Degree ramsey numbers for cycles and blowups of trees. *European Journal of Combinatorics*, 34(2):414–423, 2013.

[17] Ross J Kang and Guillem Perarnau. Decomposition of bounded degree graphs into C_4-free subgraphs. *European Journal of Combinatorics*, 44:99–105, 2015.

[18] William B Kinnersley, Kevin G Milans, and Douglas B West. Degree ramsey numbers of graphs. *Combinatorics, Probability and Computing*, 21(1-2):229–253, 2012.

[19] Felix Lazebnik and Shuying Sun. Some families of graphs, hypergraphs, and digraphs defined by systems of equations: a survey. *to appear*.

[20] Felix Lazebnik and Vasiliy A Ustimenko. New examples of graphs without small cycles and of large size. *European Journal of Combinatorics*, 14(5):445–460, 1993.

[21] Felix Lazebnik, Vasiliy A Ustimenko, and Andrew J Woldar. A new series of dense graphs of high girth. *Bulletin of the American Mathematical Society*, 32(1):73–79, 1995.

[22] Yusheng Li and Ko-Wei Lih. Multi-color ramsey numbers of even cycles. *European Journal of Combinatorics*, 30(1):114–118, 2009.

[23] Qizhong Lin and Yusheng Li. A folkman linear family. *SIAM Journal on Discrete Mathematics*, 29(4):1988–1998, 2015.

[24] Michael Molloy, Benny Sudakov, and Jacques Verstraëte. On generalized Turán numbers. *preprint*.

[25] Jaroslav Nešetřil and Vojtěch Rödl. The ramsey property for graphs with forbidden complete subgraphs. *Journal of Combinatorial Theory, Series B*, 20(3):243–249, 1976.

[26] Guillem Perarnau and Bruce Reed. Existence of spanning $\mathcal{F}$-free subgraphs with large minimum degree. *arXiv preprint arXiv:1404.7764*, 2014.

[27] Vojtěch Rödl and Endre Szemerédi. On size ramsey numbers of graphs with bounded degree. *Combinatorica*, 20(2):257–262, 2000.

[28] Xuding Zhu. Chromatic ramsey numbers. *Discrete mathematics*, 190(1):215–222, 1998.

[29] Xuding Zhu. The fractional version of Hedetniemi's conjecture is true. *European Journal of Combinatorics*, 32(7):1168–1175, 2011.