Graph Saturation Problems with Colored Edges

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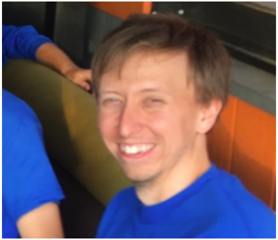
SIAM DM 18

June 7, 2018

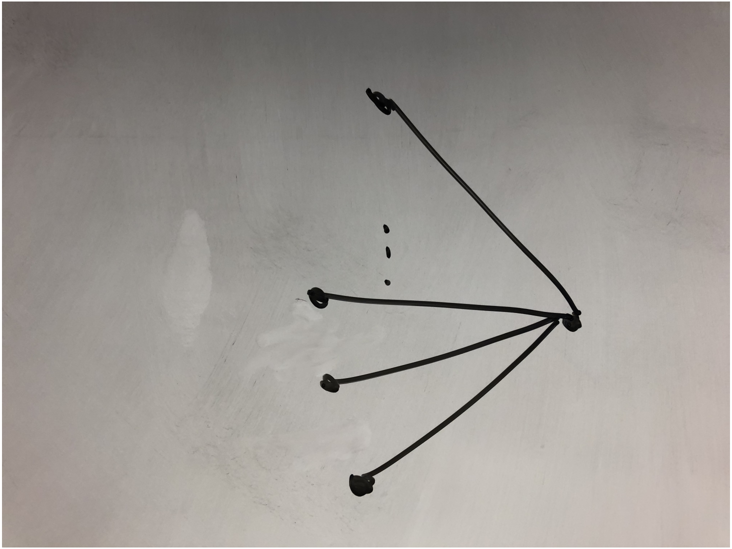

Calvin
COLLEGE




WALL of FAME!


Rainwater
Harvesting

Let $\mathcal{F}$ be a family of graphs. A graph G is called $\mathcal{F}$-saturated if it is $\mathcal{F}$-free but the addition of any edge creates a copy of some $F \in \mathcal{F}$.



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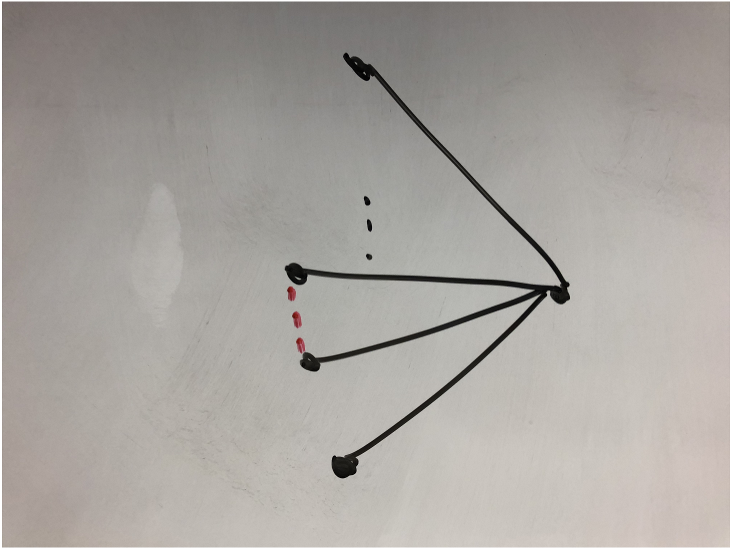


The saturation function is the minimum number of edges a graph can have and be $\mathcal{F}$-saturated

$\mathrm{sat}(n, \mathcal{F}) = \min$ number of edges in an $\mathcal{F}$-saturated graph on n vertices.

- Saturation function introduced by Erdős, Hajnal, and Moon in the 1960s.
- Long history of study, generalizations and variants.
- $\mathrm{sat}(n, \mathcal{F}) = O(n)$.
- We are interested in and edge-colored version of this problem.

t will always denote the number of colors. An edge coloring of G is a function $f : E(G) \to [t]$.

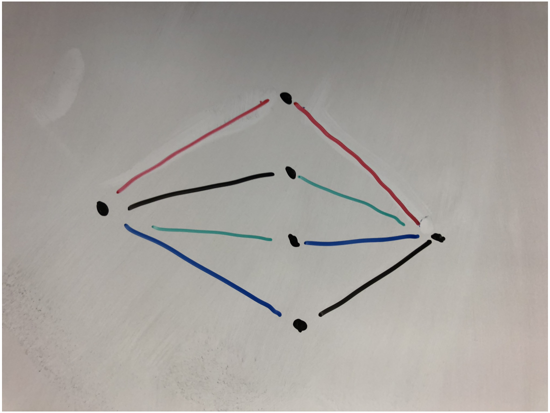
- An edge coloring is monochromatic if its image is only one value.
- An edge coloring is rainbow if it is an injection.
- $\mathcal{M}(H)$ denotes the set of all monochromatic edge colorings of H
- $\mathcal{R}(H)$ denotes the set of all rainbow edge colorings of H
- $\mathcal{C}_k(H)$ denotes the set of all edge colorings of H using exactly k colors.

Definition

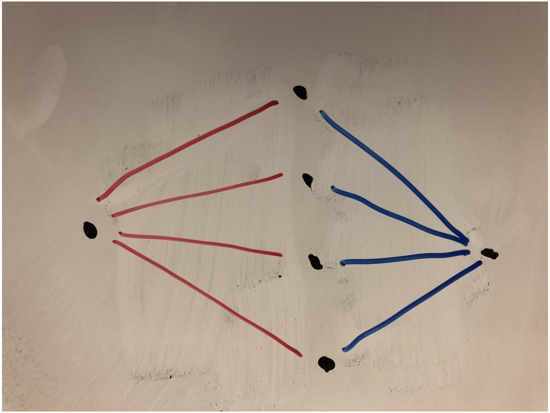
Given a family $\mathcal{C}$ of edge colored graphs, an edge-colored graph G is $(\mathcal{C}, t)$ saturated if it is $\mathcal{C}$ free but the addition of any edge in any color creates a copy of some graph in $\mathcal{C}$.

$$\mathrm{sat}_t(n, \mathcal{C})$$

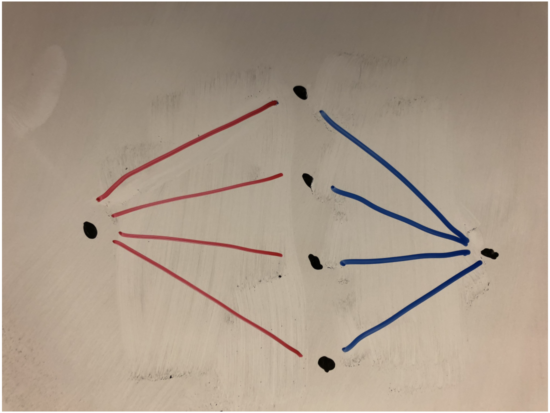
denotes the minimum number of edges in an n vertex $(\mathcal{C}, t)$ saturated graph.



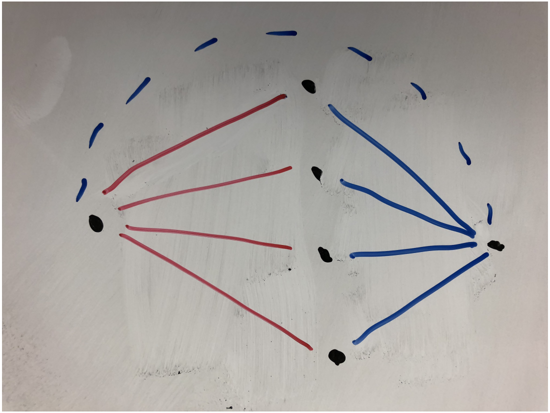
Not $(\mathcal{R}(K_3), 4)$ saturated



$(\mathcal{C}_2(K_3), 2)$ saturated



not $(\mathcal{M}(K_3), 2)$ saturated



not $(\mathcal{M}(K_3), 2)$ saturated

Hanson and Toft studied saturation for monochromatic cliques and showed that it behaves much like regular saturation. Barrus, Ferrara, Vandenbussche, and Wenger showed that the rainbow saturation problem is drastically different.

Theorem (Barrus-Ferrara-Vandenbussche-Wenger)

$$\Omega\left(\frac{n\log n}{\log\log n}\right) = \mathrm{sat}_t(n, \mathcal{R}(K_k)) = O\left(n \log n\right).$$

Conjecture

$$\mathrm{sat}_t(n, \mathcal{R}(K_k)) = \Theta(n \log n).$$

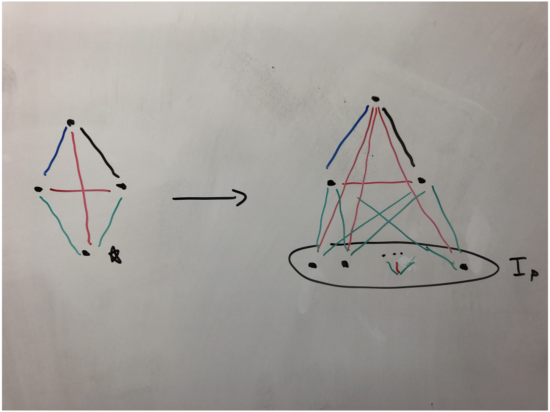
Theorem (GRWC4)

Let $k \geq 3$ and $t \geq c$ be fixed.

1. *If $c \geq \binom{k-1}{2} + 2$ then $\mathrm{sat}_t(n, \mathcal{C}_c(K_k)) = \Theta(n \log n)$.*
2. *If $c \leq \binom{k-1}{2} + 1$ then $\mathrm{sat}_t(n, \mathcal{C}_c(K_k)) = \Theta(n)$.*

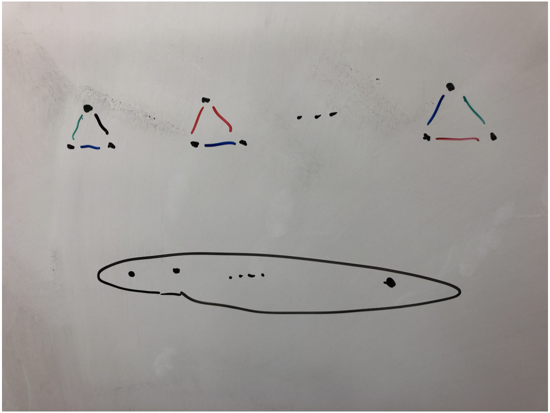
Corollary: Barrus-Ferrara-Vandenbussche-Wenger conjecture true!
Proved independently by Girão-Lewis-Popielarz and by Korándi

When $c \le \binom{k-1}{2} + 1$: we construct a graph which is $\mathcal{C}_c(K_k)$ saturated with $O(n)$ edges. Color a K_{k-1} with exactly $c-1$ colors and clone one vertex p times.



Note that if an edge is added to the independent set in any color not of the original $c-1$ colors, then there is a K_k with exactly c colors.

Do this for every possible K_{k-1} colored with $c-1$ colors and overlap the independent sets.



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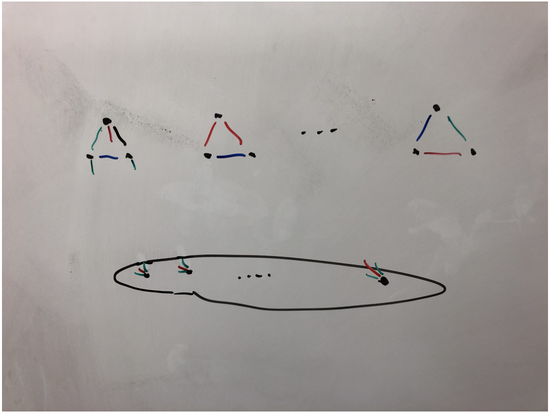

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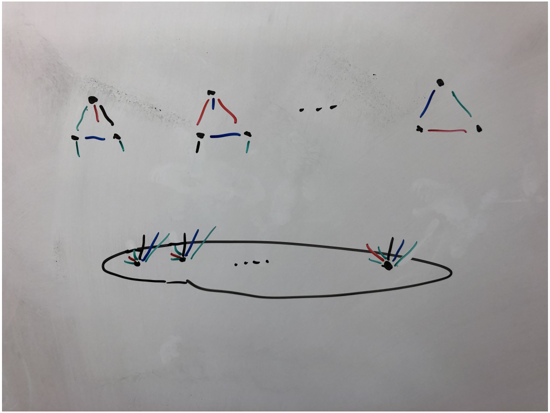


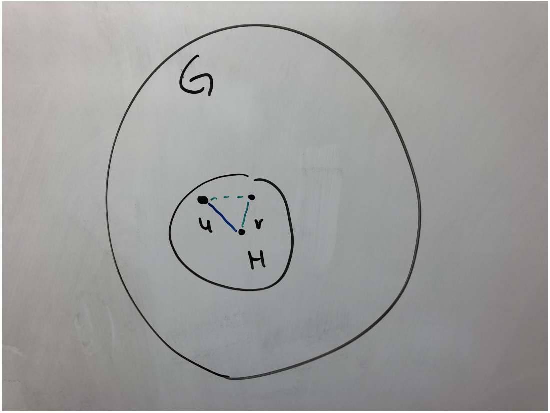
Theorem

Let $\mathcal{H}$ be any family of edge colored graphs where for each $H \in \mathcal{H}$ and for each edge $uv \in E(H)$ there is a rainbow path with 2 edges connecting u to v in H. Then for any $t \geq 3$ we have

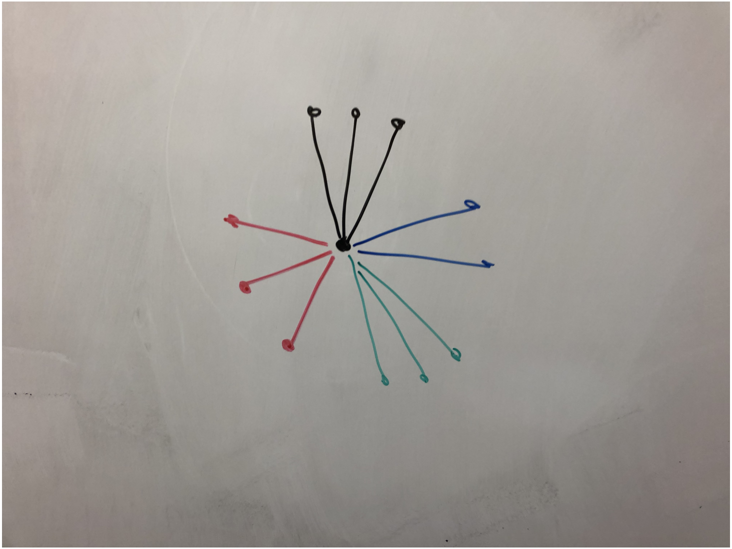
$$\left(\frac{1}{3} - o(1)\right)\frac{n \log n}{\log t} \leq \mathrm{sat}_t(n, \mathcal{H}).$$

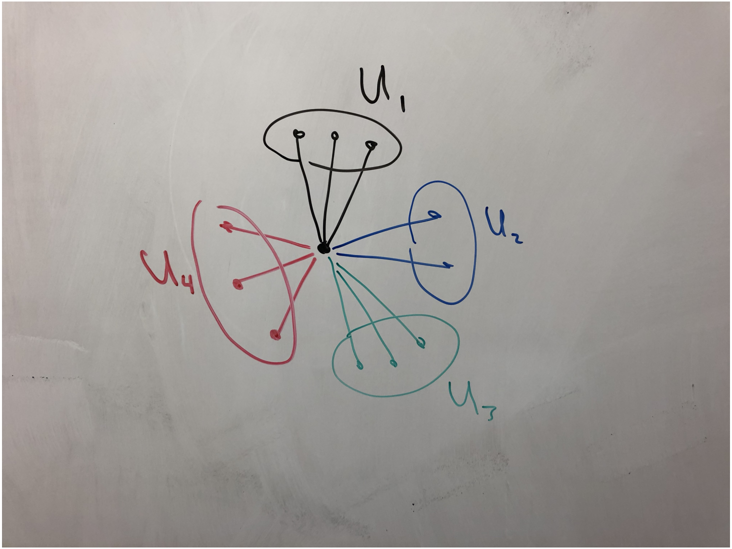
If $c \geq \binom{k-1}{2} + 2$ then any K_k edge colored with c colors satisfies this property.

Say an edge colored graph G is $\mathcal{H}$ saturated. Then the addition of any non edge uv creates a copy of $H \in \mathcal{H}$.



By assumption u and v are connected by a rainbow 2-path. Each rainbow 2-path can "cover" one non edge and all non edges must be "covered". Look at each vertex's neighborhood




U1
U2
U3
U4

This set of n auxiliary complete t-partite graphs must cover the edge set of the complement of G.

- Minimization problem: minimize the cost of a complete t-partite covering of G^c where the cost of a graph with partite sets $U_1, \cdots, U_t$ is $|U_1| + |U_2| + \cdots + |U_t|$.
- If G^c was a complete graph, this cost is at least $\frac{n \log n}{\log t}$ (prove a t-partite version of a theorem of Kászonyi and Tuza).
- G^c is almost a complete graph.

Theorem (GRWC4)

For $n \geq 11$,

$$\mathrm{sat}_t(\mathcal{C}_2(K_3)) = 2n - 4.$$

- $\delta = 1$ (easy)
- $\delta = 3$ (easy)
- $\delta = 2$ (hard)

1800 E Evans Ave, Denver, CO 80210

Half price pitcher night!


WALL
FAME!

Pikhurko showed that if H is an r-uniform hypergraph, then $\mathrm{sat}(n, H) = O(n^{r-1})$. When the hypergraph has "graph structure", its saturation number behaves like graph saturation numbers.

Definition

Let F be a graph and H a hypergraph. We say H is a *Berge-F* if there is a bijection $\phi : E(F) \to E(H)$ such that $e \subset \phi(e)$ for all $e \in E(F)$. Let $\mathcal{B}_r(F)$ be the set of r-uniform graphs which are a Berge-F.

Theorem (English-Gerbner-Methuku-Tait)

Let F be any graph. For $r \in \{3, 4, 5\}$, we have

$$\mathrm{sat}_r(n, \mathcal{B}_r(F)) = O(n).$$

Future Work

- For all of the colored saturation problems we considered, the families of edge colored graphs were invariant under a permutation of the colors.
- Hypergraph colored saturation?
- Berge hypergraph saturation for all uniformities.