Turán numbers for Berge-hypergraphs and related extremal problems

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Abstract

Let F be a graph. We say that a hypergraph H is a *Berge*-F if there is a bijection $f : E(F) \to E(H)$ such that $e \subseteq f(e)$ for every $e \in E(F)$. Note that Berge-F actually denotes a class of hypergraphs. The maximum number of edges in an n-vertex r-graph with no subhypergraph isomorphic to any Berge-F is denoted $\mathrm{ex}_r(n, \text{Berge-}F)$. In this paper we establish new upper and lower bounds on $\mathrm{ex}_r(n, \text{Berge-}F)$ for general graphs F, and investigate connections between $\mathrm{ex}_r(n, \text{Berge-}F)$ and other recently studied extremal functions for graphs and hypergraphs. One case of specific interest will be when $F = K_{s,t}$. Additionally, we prove a counting result for r-graphs of girth five that complements the asymptotic formula $\mathrm{ex}_3(n, \text{Berge-}\{C_2, C_3, C_4\}) = \frac{1}{6}n^{3/2} + o(n^{3/2})$ of Lazebnik and Verstraëte [*Electron. J. of Combin.* **10**, (2003)].

1 Introduction

Let F be a graph and H be a hypergraph. The hypergraph H is a *Berge*-F if there is a bijection $f : E(F) \to E(H)$ such that $e \subseteq f(e)$ for every $e \in E(F)$. Here we are following the presentation of Gerbner and Palmer [12]. This notion of a Berge-F extends Berge cycles and Berge paths, which have been investigated, to all graphs. In general, Berge-F is a family of graphs. Given an integer $r \geq 2$, write

$$\mathrm{ex}_r(n, \text{Berge-}F)$$

for the maximum number of edges in an r-uniform hypergraph (r-graph for short) on n vertices that does not contain a subhypergraph isomoprhic to a member of Berge-F. In the

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case that $r = 2$, Berge-F consists of a single graph, namely F, and $\mathrm{ex}_2(n, \text{Berge-}F)$ is the same as the usual Turán number $\mathrm{ex}(n, F)$.

By results of Győri, Katona and Lemons [14] and Davoodi, Győri, Methuku and Tompkins [6], we get tight bounds on $\mathrm{ex}_r(n, \text{Berge-}P_\ell)$ where P_ℓ is a path of length ℓ. When F is a cycle and $r \geq 3$, Győri and Lemons [15] determined

$$\mathrm{ex}_r(n, \text{Berge-}C_{2\ell}) = O(n^{1+1/\ell})$$

where the multiplicative constant depends on r and ℓ. This upper bound matches the order of magnitude in the graph case as given by the classical Even-Cycle Theorem of Bondy and Simonovits [5]. Unexpectedly, the same upper-bound holds in the odd case, i.e., for $r \geq 3$ it was shown in [15] that

$$\mathrm{ex}_r(n, \text{Berge-}C_{2\ell+1}) = O(n^{1+1/\ell}).$$

This differs significantly from the graph case where we may have $\lfloor n^2/4 \rfloor$ edges and no odd cycle.

Instead of a class of forbidden subhypergraphs, much effort has been spent on determining the Turán number of individual hypergraphs. One case closely related to the Berge question is the so-called expansion of a graph. Fix a graph F and let $r \geq 3$ be an integer. The *r-uniform expansion* of F is the r-uniform hypergraph F^+ obtained from F by enlarging each edge of F with $r-2$ new vertices disjoint from $V(F)$ such that distinct edges of F are enlarged by distinct vertices. More formally, we replace each edge $e \in E(F)$ with an r-set $e \cup S_e$ where the sets S_e have $r-2$ vertices and $S_e \cap S_f = \emptyset$ whenever e and f are distinct edges of H.

The r-graph F^+ has the same number of edges as F, but has $|V(F)| + |E(F)|(r-2)$ vertices. The special case when F is a complete graph K_k has been studied by Mubayi [26] and Pikhurko [28]. A series of papers [20, 21, 22] by Kostochka, Mubayi, and Verstraëte consider expansions for paths, cycles, trees, as well as other graphs. The survey of Mubayi and Verstraëte [27] discusses these results as well as many others. Given an integer $r \geq 3$ and a graph F, we write

$$\mathrm{ex}_r(n, F^+)$$

for the maximum number of edges in an n-vertex r-graph that does not contain a subhypergraph isomorphic to F^+. A representative theorem in [22] is that

$$\mathrm{ex}_3(n, K_{s,t}^+) = O(n^{3-3/s})$$

whenever $t \geq s \geq 3$. It is also shown that this bound is sharp when $t > (s-1)!$.

For a fixed graph F, both the Berge-F and expansion F^+ hypergraph problems are closely related to counting certain subgraphs in (ordinary) graphs with no subgraph isomorphic to F. Let G and F be graphs. Following Alon and Shikhelman [2], write

$$\mathrm{ex}(n, G, F)$$

for the maximum number of copies of G in an F-free graph with n vertices. A graph is *F-free* if it does not contain a subgraph isomorphic to F. The function $\mathrm{ex}(n, G, F)$ was studied in the case $(G,F)=(K_3,C_5)$ by Bollobás and Győri [4], and when $(G,F)=(K_3,C_{2\ell+1})$ by Győri and Li [16]. Later, Alon and Shikhelman [2] initiated a general study of $\mathrm{ex}(n,G,F)$. Among others, they proved

Theorem 1 (Alon, Shikhelman [2]). *If F is a graph with chromatic number $\chi(F)=k>r$, then*

$$\mathrm{ex}(n,K_r,F)=(1+o(1))\binom{k-1}{r}\left(\frac{n}{k-1}\right)^r.$$

Note that the famous Erdős-Stone theorem is the case when $r=2$.

The next proposition demonstrates a connection between the three extremal functions that we have defined so far.

Proposition 2. *If H is a graph and $r\geq 2$, then*

$$\mathrm{ex}(n,K_r,F)\leq \mathrm{ex}_r(n,\text{Berge-}F)\leq \mathrm{ex}_r(n,F^+).$$

One of the main questions that we consider in this work is the relationship between these functions for different graphs F. We will see that in some cases, all three are asymptotically equivalent, while in others they exhibit different asymptotic behavior. In light of the Erdős-Stone Theorem, it is not too surprising that the chromatic number of F plays a crucial role. When $\chi(F)>r$ (the so-called nondegenerate case) we have the following known result which was stated in [27]. We provide a proof in Section 3.1 for completeness. Given two functions $f,g:\mathbb{N}\to\mathbb{R}$, we write $f\sim g$ if $\lim\frac{f(n)}{g(n)}=1$.

Theorem 3. *Let $k>r\geq 2$ be integers and F be a graph. If $\chi(F)=k$, then*

$$\mathrm{ex}(n,K_r,F)\sim \mathrm{ex}_r(n,\text{Berge-}F)\sim \mathrm{ex}_r(n,F^+)\sim\binom{k-1}{r}\left(\frac{n}{k-1}\right)^r.$$

When $\chi(F)\leq r$ (the so-called degenerate case), we have the following.

Theorem 4. *Let $r\geq k\geq 3$ be integers. If F is a graph with $\chi(F)=k$, then*

$$\mathrm{ex}_r(n,F^+)=o(n^r).$$

It is important to mention that our proofs of Theorem 3 and Theorem 4 rely heavily on a well-known theorem of Erdős (see Theorem 11 in Section 2).

In the case that $\chi(F)\leq r$, the asymptotic equivalence between these three extremal functions need not hold. As an example, let us consider $K_{2,t}$. In [2], it is shown that for every fixed $t\geq 2$,

$$\mathrm{ex}(n,K_3,K_{2,t})=\left(\frac{1}{6}+o(1)\right)(t-1)^{3/2}n^{3/2}$$

as n tends to infinity. However, $\mathrm{ex}_3(n,\text{Berge-}K_{2,2})\geq\left(\frac{1}{3\sqrt{3}}-o(1)\right)n^{3/2}$ (see for instance Theorem 5 in [12]). Therefore,

$$\mathrm{ex}(n,K_3,K_{2,2})\nsim \mathrm{ex}_3(n,\text{Berge-}K_{2,2})$$

The next result implies that $\mathrm{ex}_3(n,\text{Berge-}K_{2,t})$ and $\mathrm{ex}(n,K_3,K_{2,t})$ have the same order of magnitude for all $t\geq 2$.

Theorem 5. *If $r \geq 3$ and $t \geq r-1$ are integers, then*

$$\mathrm{ex}_r(n, \text{Berge-}K_{2,t}) \leq \left(\frac{r-1}{t}\binom{t}{r-1} + 2t + 1\right)\mathrm{ex}(n, K_{2,t}).$$

We note that during the preparation of this manuscript we became aware of a very similar bound on $\mathrm{ex}_r(n, \text{Berge-}K_{2,t})$ given in a preprint of Gerbner, Methuku and Vizer [13]. The result of [13] gives a better constant than the one provided by Theorem 5, and shows that for all $t \geq 7$,

$$\mathrm{ex}(n, K_3, K_{2,t}) \sim \mathrm{ex}_3(n, \text{Berge-}K_{2,t}).$$

On the other hand, by taking all $\binom{n-1}{2}$ triples that contain a fixed element we get a 3-graph with $\Omega(n^2)$ edges that contains no $K_{2,t}^+$. For more on the Turán number of Berge-$K_{2,t}$, see [13, 31].

In the case that $3 \leq r \leq s \leq t$, we have the following upper bound which is a consequence of a more general result that is proved in Section 4.1.

Theorem 6. *For $3 \leq r \leq s \leq t$ and sufficiently large n,*

$$\mathrm{ex}_r(n, \text{Berge-}K_{s,t}) = O(n^{r - \frac{r(r-1)}{2s}}).$$

As for lower bounds, we use Projective Norm Graphs and a simple probabilistic argument to construct graphs with no $K_{s,t}$, but many copies of K_r.

Theorem 7. *Let $s \geq 3$ be an integer. If q is an even power of an odd prime, then*

$$\mathrm{ex}(2q^s, K_4, K_{s+1,(s-1)!+2}) \geq \left(\frac{1}{4} - o(1)\right) q^{3s-4}.$$

By Proposition 2, we have a lower bound on $\mathrm{ex}_4(2q^2, \text{Berge-}K_{s+1,(s-1)!+2})$. In the case when $s = 3$, this lower bound that is better than the standard construction using random graphs. This is discussed further in Section 4.2.

Our final result concerns counting r-graphs with no Berge-$\mathcal{F}$ where $\mathcal{F}$ is a family of graphs. Given an r-graph H, the *girth* of H is the smallest k such that H contains a Berge-C_k. When $k = 2$, C_2 is the graph with two parallel edges and H has girth at least 3 if and only if H is linear. In general, the girth of H is at least g if and only if H contains no Berge-C_k for $k \in \{2, 3, \ldots, g-1\}$. One of the seminal results in this area is the asymptotic formula

$$\mathrm{ex}_3(n, \text{Berge-}\{C_2, C_3, C_4\}) = \left(\frac{1}{6} + o(1)\right) n^{3/2}$$

of Lazebnik and Verstraëte [24]. This bound implies that there are at least

$$2^{(1/6+o(1))n^{3/2}}$$

n-vertex 3-graphs with girth 5. Our counting result provides an upper bound that matches this lower bound, up to a constant in the exponent, and holds for all $r \geq 2$.

Theorem 8. *Let $r \geq 2$. Then there exists a constant c_r such that the number of n-vertex r-graphs of girth at least 5 is at most $2^{c_r n^{3/2}}$.*

This is a consequence of a more general result that is given in Section 5. It was recently shown by Ergemlidze, Győri, and Methuku [9] that $\mathrm{ex}_3(n, \text{Berge-}\{C_2, C_4\}) = \left(\frac{1}{6} + o(1)\right) n^{3/2}$. We leave it as an open problem to determine if Theorem 8 holds under the weaker assumption that the graphs we are counting may have a Berge-C_3.

The rest of this paper is organized as follows. Section 2 gives the notation and some preliminary results that we will need. Section 3 contains the proof of Theorems 3 and 4. Section 4 focuses on the special case when $F = K_{s,t}$, while Section 5 contains the proof of Theorem 8 and related counting results.

2 Notation and preliminaries

In this section we introduce the notation that will be used throughout the paper. Additionally, we recall some known results that will be used in our arguments, and give a proof of Proposition 2.

For a graph G and a vertex $\in V(G)$, $k_m(G)$ is the number of copies of K_m in G and $\Gamma_G(v)$ is the subgraph of G induced by the neighbors of v. For positive integers r, m, and x, we write $K^r(x)$ for the complete r-partite r-graph with x vertices in each part. The graph $K_m(x)$ is the complete m-partite graph with x vertices in each part and we write K_m instead of $K_m(1)$.

In the previous section we defined the expansion F^+ of a graph. An important special case is when $F = K_k$ for some $k \geq 2$. By definition, the r-graph K_k^+ must contain a set of k vertices, say $\{v_1, \ldots, v_k\}$, such that every pair $\{v_i, v_j\}$ is contained in exactly one edge of K_k^+. We call this set the *core* of K_k^+. As $k \geq 2$, the core is uniquely determined since every vertex not in the core is contained in exactly one edge and every vertex in the core is contained in exactly $k-1$ edges. The r-graph K_k^+ has $\binom{k}{2}$ edges and $k + \binom{k}{2}(r-2)$ vertices.

Let H be an r-graph. We define ∂H to be the graph consisting of pairs contained in at least one r-edge of H, i.e.,

$$\partial H = \{\{x, y\} \subset V(H) : \{x, y\} \subset e \text{ for some } e \in H\}.$$

Given $\{x, y\} \in \partial H$, let

$$d(x, y) = |\{e \in H : \{x, y\} \subset e\}|.$$

The r-graph H is *d-full* if $d(x, y) \geq d$ for all $\{x, y\} \in \partial H$. If more than one hypergraph is present, we may write $d_H(x, y)$ instead of $d(x, y)$ to avoid confusion.

The first lemma is a very useful tool for Turán problems involving expansions (see [22, 27]).

Lemma 9 (Full Subgraph Lemma)**.** *For any positive integer d, the r-graph H has a d-full subgraph H_1 with*

$$e(H_1) \geq e(H) - (d-1)|\partial H|.$$

Proof. If H is not d-full, choose a pair $\{x,y\} \in \partial H$ for which $d(x,y) < d$. Remove all edges that contain the pair $\{x,y\}$ and let H' be the resulting graph. If H' is d-full, then we are done. Otherwise, we iterate this process which can continue for at most $|\partial H|$ steps. At each iteration, at most $d-1$ edges are removed. □

The next simple lemma is useful for finding pairs of vertices with bounded codegree in an r-graph with no Berge-F. See Lemma 3.2 of [20] for a similar result.

Lemma 10. *Let $r \geq 3$ be an integer and H be an r-graph with no Berge-F. If ∂H contains a copy of F, then there is a pair of vertices $\{x,y\}$ such that*

$$d_H(\{x,y\}) < e(F).$$

Proof. Suppose ∂H contains a copy of F, say with edges $e_1, \dots, e_m$ where $m = e(F)$. If every pair $e_i = \{x_i, y_i\}$ has

$$d_H(\{x_i, y_j\}) \geq e(F), \tag{1}$$

then we can choose $e(F)$ distinct edges $e'_i \in H$ for which $\{x_i, y_i\} \subset e'_i$ for all $1 \leq i \leq m$. This gives a Berge-F in H and so (1) cannot hold for all $\{x_i, y_j\}$. □

A consequence of Lemma 10 is that if H is an r-graph with no Berge-F and H' is a d-full subgraph of H with $d \geq e(F)$, then $\partial H'$ must be F-free. Lemma 10 will be used frequently in Section 4.1.

Lastly, we will need the following result of Erdős [7].

Theorem 11 (Erdős [7])**.** *Let r and x be positive integers. There is an $n_0 = n_0(r,x)$ and a positive constant $\alpha_{r,x}$ such that for all $n > n_0$, any n-vertex r-graph with more than $\alpha_{r,x} n^{r-1/x^{r-1}}$ edges must contain a complete r-partite r-graph with x vertices in each part.*

We conclude this section by providing a proof of Proposition 2.

Proof of Proposition 2. We begin the proof by showing that the first inequality holds. Let G be an n-vertex graph that is F-free and has $\mathrm{ex}(n, K_r, F)$ copies of K_r. Let H be the r-graph with the same vertex set as G, and an r-set e is an edge in H if and only if the vertices in e form a K_r in G. The number of edges in H is $\mathrm{ex}(n, K_r, F)$. Suppose that H has a Berge-F. Any pair of vertices $\{u,v\}$ that are contained in an edge of H are adjacent in G. Therefore, a Berge-F in H gives a copy of F in G. Namely, if $f : E(F) \to E(H)$ is an injection with the property that $\{x,y\} \subset f(\{x,y\})$ for all $\{x,y\} \in E(F)$, then these same pairs $\{x,y\}$ for which $\{x,y\} \in E(F)$ are edges of a copy of F in G. We conclude that H has no Berge-F.

The second inequality is trivial since F^+ is a particular Berge-F and so any r-graph that has no Berge-F has no F^+. □

3 General upper bounds

In this section, we prove an Erdős-Stone type result for r-graphs with no F^+. By Proposition 2 this gives general upper bounds on $\mathrm{ex}_r(n, \text{Berge-}F)$. We begin with the non-degenerate case, i.e., when $\chi(F) > r$.

3.1 Non-degenerate case and the proof of Theorem 3

In this section we prove Theorem 3. As mentioned in the introduction, this result was stated in Mubayi and Verstraëte's survey on Turán problems for expansions [27]. Let F be a graph with chromatic number $\chi(F) = k > r$. By Theorem 1 and Proposition 2 it is enough to show that $\mathrm{ex}_r(n, F) \sim \binom{k-1}{r}\left(\frac{n}{k-1}\right)^r$.

It was shown by Mubayi [26] (and later improved by Pikhurko [28]) that

$$\mathrm{ex}_r(n, K_k^+) \sim \binom{k-1}{r}\left(\frac{n}{k-1}\right)^r.$$

Therefore, in order to prove Theorem 3 it remains to prove the following lemma.

Lemma 12. *Let $k > r \geq 2$ be integers and F be a graph with f vertices. If $\chi(F) = k$ and $\epsilon > 0$, then for sufficiently large n, depending on k, r, f, and ϵ, we have*

$$\mathrm{ex}_r(n, F^+) < \mathrm{ex}_r(n, K_k^+) + \epsilon n^r.$$

Proof. Let F be a graph with f vertices and $\chi(F) = k$ where $k > r \geq 2$ are integers. Let $\epsilon > 0$ and G be an n-vertex r-graph with

$$e(G) \geq \mathrm{ex}_r(n, K_k^+) + \epsilon n^r.$$

By the Supersaturation Theorem of Erdős and Simonovits [8], there is a positive constant $c = c(\epsilon)$ such that G contains at least cn^m copies of K_k^+ where

$$m := k + \binom{k}{2}(r-2)$$

is the number of vertices in the r-graph K_k^+. Let Z be the m-graph with the same vertex set as G where e is an edge of Z if and only if there is a K_k^+ in G with vertex set e.

Fix a positive integer x large enough so that

$$x^k \geq \binom{m}{k}\alpha_{k,f}x^{k-1/f^k} \quad \text{and} \quad x > f^k$$

where $\alpha_{k,f}$ is the constant from Theorem 11. Note that x depends only on r, k, and f. For large enough n, depending on c and hence ϵ, we have

$$e(Z) \geq cn^m > \alpha_{m,x}n^{m-\frac{1}{x^{m-1}}}$$

so that Z contains a $K^m(x)$, say with parts $P_1, \ldots, P_m$. Therefore, for any

$$(p_1, \ldots, p_m) \in P_1 \times \cdots \times P_m,$$

there is a K_k^+ in G whose vertex set is $\{p_1, \ldots, p_m\}$.

A K_k^+ must contain k vertices that form the core and since

$$|P_1 \times \cdots \times P_m| = x^m,$$

there are at least $x^m/\binom{m}{k}$ copies of K_k^+ whose vertex sets are the edges of Z, and whose vertices in the core come from the same set of k P_i's. Without loss of generality, we may assume that we have $x^m/\binom{m}{k}$ copies of K_k^+ whose core vertices come from k-tuples in

$$P_1 \times \cdots \times P_k.$$

Let Y be the k-partite k-graph with vertex set $P_1 \cup \cdots \cup P_k$ whose edges are the k-tuples $(p_1, \ldots, p_k) \in P_1 \cup \cdots \cup P_k$ for which there is a K_k^+ in G whose vertices are an edge of Z, and whose core is $\{p_1, \ldots, p_k\}$. Given an edge $(p_1, \ldots, p_k)$ of Y, there are at most $x^{m-(k+1)}$ edges in Z that contain $\{p_1, \ldots, p_k\}$ so that

$$e(Y) \geq \frac{x^m/\binom{m}{k}}{x^{m-k}} = \frac{x^k}{\binom{m}{k}}.$$

We have chosen x large enough so that

$$\frac{x^k}{\binom{m}{k}} \geq \alpha_{k,f} x^{k-1/f^k}$$

holds. By Theorem 11, Y contains a $K^k(f)$, say with parts $R_1, \ldots, R_k$ where $R_i \subset P_i$ for $1 \leq i \leq k$.

Let us pause a moment to recapitulate what we have so far. For every k-tuple

$$(r_1, \ldots, r_k) \in R_1 \times \cdots \times R_k$$

and every $(m-k)$-tuple

$$(p_{k+1}, \ldots, p_m) \in P_{k+1} \times \cdots \times P_m,$$

there is a K_k^+ in G with vertex set $\{r_1, \ldots, r_k, p_{k+1}, \ldots, p_m\}$ whose core is $\{r_1, \ldots, r_k\}$. Since $x > f^k$ and each P_i has x vertices, we can choose f^k tuples

$$(p_{k+1}, \ldots, p_m) \in P_{k+1} \times \cdots \times P_m$$

such that the corresponding sets are pairwise disjoint. We then pair each one of these sets up with a k-tuple in $R_1 \times \cdots \times R_k$ in a 1-to-1 fashion. Each such pairing forms a K_k^+ in G and altogether, we have constructed a $K_k(f)^+$ in G. That is, we have an expansion of the complete k-partite Turán graph with f vertices in each part. As F is a subgraph of $K_k(f)$, F^+ is a subgraph of $K_k(f)^+$ and so G contains a copy of F^+. $\square$

3.2 The degenerate case and the proof of Theorem 4

In this section we prove Theorem 4, i.e., that if F is a graph with $\chi(F) \leq r$, then

$$\mathrm{ex}_r(n, F^+) = o(n^r).$$

As mentioned in the introduction, the proof is based on Theorem 11. It is an immediate corollary of the following.

Theorem 13. *If $r \geq 3$ is a fixed integer and F is a graph with $\chi(F) \leq r$, then there is a positive constant C, depending on r and F, such that*

$$\mathrm{ex}_r(n, F^+) \leq C n^{r - 1/x^{r-1}}$$

where $x = \binom{r}{2}|V(F)|^2 + |V(F)|$.

Proof. Assume that $|V(F)| = f$ so that $x = \binom{r}{2} f^2 + f$. Let H be an n-vertex r-graph with $e(H) \geq C n^{r-1/x^{r-1}}$ where C can be taken large as a function of r and F. We will show that H contains a subhypergraph isomorphic to F^+.

For large enough C, we have $e(H) > \alpha_{r,x} n^{r-1/x^{r-1}}$. By Theorem 11, H contains a $K^r(x)$. Here $K^r(x)$ is the complete r-partite r-graph with x vertices in each part. Let $W_1, \dots, W_r$ be the parts of the $K^r(x)$ in H. Partition each W_i into two sets U_i and D_i where $|U_i| = f$ and $|D_i| = \binom{r}{2} f^2$. We are going to construct a $K_r(f)^+$ in H one edge at a time. The vertices that lie in exactly one edge of the $K_r(f)^+$ will come from the sets $D_1 \cup \dots \cup D_r$, and the other vertices will come from $U_1 \cup \dots \cup U_r$.

Let $x \in U_1$ and $y \in U_2$. Choose exactly one vertex, say z_i, from D_i for $3 \leq i \leq r$ and make $\{x, y, z_3, \dots, z_r\}$ an edge. Next we pick a new pair $x' \in U_1$ and $y' \in U_2$ and choose exactly one vertex, say z_i', from $D_i \backslash \{z_i\}$ for $3 \leq i \leq r$. Make $\{x', y', z_3', \dots, z_r'\}$ an edge. We can continue this process and in the next round, we add an edge $\{x'', y'', z_3'', \dots, z_r''\}$ where $\{x'', y''\}$ is a new pair ($x'' \in U_1, y'' \in U_2$) and the sets $\{z_3, \dots, z_r\}$, $\{z_3', \dots z_r'\}$, and $\{z_3'', \dots, z_r''\}$ are all pairwise disjoint.

Since $|D_i| \geq f^2$, we can continue this process for all pairs of vertices in U_1 and U_2. Even more, since $|D_i| \geq \binom{r}{2} f^2$, this process can continue until we have considered all pairs U_i and U_j with $1 \leq i < j \leq r$. When the process is completed, we have constructed a $K_r(f)^+$ in H. Now since F is a subgraph of $K_r(f)$, we have that F^+ is a subgraph of $K_r(f)^+$ and this completes the proof of the theorem. □

4 Forbidding Berge-$K_{s,t}$

In this section we investigate the special case of forbidding the Berge-$K_{s,t}$.

4.1 Upper bounds and the proof of Theorems 5 and 6

We begin with an easy lemma.

Lemma 14. *If* $2 \le m \le s$, *then*

$$\mathrm{ex}(n, K_m, K_{1,s}) \le \left(\frac{n}{s}\right)\binom{s}{m}.$$

Proof. Let G be an n-vertex $K_{1,s}$-free graph. Every vertex of G has degree at most $s-1$ so

$$k_m(G) = \frac{1}{m}\sum_{v \in V(G)} k_{m-1}(\Gamma_G(v)) \le \frac{n}{m}\binom{s-1}{m-1} = \frac{n}{s}\binom{s}{m}.$$

□

We are now ready to prove Theorem 5.

Proof of Theorem 5. Fix integers $3 \le r \le t$ and let H be an n-vertex r-graph with no Berge-$K_{2,t}$. Let

$$H_0 = H,\ F_0 = \partial H_0,$$

and G_0 be the graph with no edges and vertex set $V(H_0)$. If the graph F_0 is not $K_{2,t}$-free, then by Lemma 10, there is a pair of vertices $\{x_1, y_1\}$ with

$$d_{H_0}(\{x_1, y_1\}) < 2t.$$

Now let H_1 be obtained from H_0 by removing all of the edges that contain $\{x_1, y_1\}$ and

$$F_1 = \partial H_1.$$

Let G_1 be the graph obtained by adding the edge $\{x_1, y_1\}$ to G_0.

Now we iterate this process. That is, for $i \ge 1$, we proceed as follows.

If F_{i-1} is not $K_{2,t}$-free, then by Lemma 10 there is a pair of vertices $\{x_i, y_i\}$ in H_{i-1} with

$$d_{H_{i-1}}(\{x_i, y_i\}) < 2t.$$

Let H_i be the r-graph obtained from H_{i-1} by removing all of the edges that contain the pair $\{x_i, y_i\}$, let

$$F_i = \partial H_i$$

and G_i be the graph obtained by adding the edge $\{x_i, y_i\}$ to G_{i-1}. Observe that

$$e(H_i) > e(H_{i-1}) - 2t.$$

Suppose that this can be done for $l := \delta e(H)$ steps where

$$\delta := \frac{1}{\frac{r-1}{t}\binom{t}{r-1} + 2t + 1}.$$

Consider the graph G_l. This graph has l edges and must be $K_{2,t}$-free otherwise, we find a $K_{2,t}$ in H since edges in G_i come from different edges in H. Thus,

$$\delta e(H) = e(G_l) \leq \mathrm{ex}(n, K_{2,t})$$

so

$$e(H) \leq \frac{1}{\delta}\mathrm{ex}(n, K_{2,t})$$

and we are done.

Now assume that this procedure terminates for some $l \in \{0, 1, \ldots, \delta e(H)\}$ where $l = 0$ is allowed. The graph F_l must be $K_{2,t}$-free so

$$|\partial H_l| = e(F_l) \leq \mathrm{ex}(n, K_{2,t}).$$

Let

$$d_t = \frac{r-1}{t}\binom{t}{r-1} + 1.$$

The values d_t and δ satisfy the equation

$$\frac{d_t}{1 - 2t\delta} = \frac{1}{\delta}.$$

If $e(H) \leq \frac{d_t}{1-2t\delta}\mathrm{ex}(n, K_{2,t})$, then we are done. For contradiction, suppose that

$$e(H) > \frac{d_t}{1 - 2t\delta}\mathrm{ex}(n, K_{2,t}). \tag{2}$$

Let H' be a d_t-full subgraph of H_l with

$$\begin{aligned} e(H') &\geq e(H_l) - d_t|\partial H_l| \geq e(H_0) - 2tl - d_t\mathrm{ex}(n, K_{2,t}) \\ &\geq e(H_0) - 2t\delta e(H) - d_t\mathrm{ex}(n, K_{2,t}) \\ &= (1 - 2t\delta)e(H) - d_t\mathrm{ex}(n, K_{2,t}) > 0 \end{aligned}$$

where the last inequality follows from (2).

Let $F' = \partial H'$. We now make a few observations about the graph F'. First note that F' contains edges since $e(H') > 0$. Second, F' is $K_{2,t}$-free. This is because H' is a subgraph of H_l and so F' is a subgraph of F_l, but F_l is $K_{2,t}$-free. Let v be a vertex of F' with positive degree. The subgraph of F' induced by the neighbors of v, which we denote by $\Gamma_{F'}(v)$, is $K_{1,t}$-free. Since $t \geq r - 1$, we have by Lemma 14 that

$$k_{r-1}(\Gamma_{F'}(v)) \leq \left(\frac{d_{F'}(v)}{t}\right)\binom{t}{r-1}. \tag{3}$$

Now we find a lower bound for $k_{r-1}(\Gamma_{F'}(v))$. Let w be a vertex in $\Gamma_{F'}(v)$. Since H' is d_t-full, there are at least d_t r-sets in H' which contain $\{v, w\}$. Now if e is an r-set in H' that contains

$\{v,w\}$, then the $(r-1)$-set $e\backslash\{v\}$ forms a $(r-1)$-clique in $\Gamma_{F'}(v)$. Therefore, this holds for any of the $d_{F'}(v)$ vertices in $\Gamma_{F'}(v)$ and so

$$k_{r-1}(\Gamma_{F'}(v)) \geq \frac{1}{r-1} d_{F'}(v) d_t. \tag{4}$$

Combining (3) and (4) gives

$$\frac{1}{r-1} d_{F'}(v) d_t \leq k_{r-1}(\Gamma_{F'}(v)) \leq \binom{d_{F'}(v)}{t}\binom{t}{r-1}.$$

As $d_{F'}(v) > 0$, the above inequality implies

$$d_t \leq \frac{r-1}{t}\binom{t}{r-1}$$

which is a contradiction since $d_t = \frac{r-1}{t}\binom{t}{r-1}+1$. We conclude that (2) cannot hold and this completes the proof. □

We now prove a general upper bound that implies Theorem 6. A similar result was proved in [13]. We have chosen to use notation similar to that of [13] to highlight the correspondence.

Theorem 15. *Suppose F is a bipartite graph and that there is a vertex $x \in V(F)$ such that for all $m \geq 1$,*

$$\mathrm{ex}(m, K_{r-1}, F-x) \leq cm^i$$

for some positive constant c and integer $i \geq 1$. If $r \geq 3$ is an integer, v_F is the number of vertices of F, and e_F is the number of edges of F, then for large enough n, depending on r and F,

$$\mathrm{ex}_r(n, \text{Berge-}F) \leq 4c(r-1)2^{i-1}\frac{\mathrm{ex}(n,F)^i}{n^{i-1}} + 4(v_F+e_F)n^2.$$

Proof. Let F be a bipartite graph satisfying the assumptions of the theorem. Let H be an n-vertex r-graph with no Berge-F. If $e(H) \leq 4(v_F+e_F)n^2$, then we are done. Assume otherwise and that θ satisfies

$$e(H) = 4(v_F+e_F)n^{r-\theta}.$$

Note that $r-\theta \geq 2$ since $e(H) > 4(v_F+e_F)n^2$. Let H_1 be a (v_F+e_F)-full subgraph of H with

$$\begin{aligned} e(H_1) &\geq e(H) - (v_F+e_F)|\partial H| \geq 4(v_F+e_F)n^{r-\theta} - (v_F+e_F)n^2 \\ &\geq 3(v_F+e_F)n^{r-\theta}. \end{aligned}$$

If ∂H_1 contains a copy of F, then since H_1 is (v_F+e_F)-full, we have a Berge-F in H_1 (and thus H) by Lemma 10; a contradiction Thus, ∂H_1 is F-free and therefore $|\partial H_1| \leq \mathrm{ex}(n,F)$. Let

$$d = \frac{(v_F+e_F)n^{r-\theta}}{\mathrm{ex}(n,F)}.$$

Let H_2 be a d-full subgraph of H_1 with

$$\begin{aligned} e(H_2) &\geq e(H_1) - d|\partial H_1| \geq 3(v_F + e_F)n^{r-\theta} - d \cdot \mathrm{ex}(n, F) \\ &= 2(v_F + e_F)n^{r-\theta}. \end{aligned}$$

Let H_3 be the subgraph of H_2 obtained by removing all isolated vertices and let $G = \partial H_3$.

The graph G is F-free as it is a subgraph of ∂H_1, so $e(G) \leq \mathrm{ex}(n, F)$. Let v be a vertex of G with

$$d_G(v) \leq \frac{2\mathrm{ex}(n, F)}{n}. \tag{5}$$

Let $\Gamma_G(v)$ be the subgraph of G induced by the neighbors of v in G. As H_3 is d-full, we have that there are at least d edges in H_3 that contain both v and w for any vertex $w \in \Gamma_G(v)$. Each such edge in H_3 gives rise to a K_{r-1} in $\Gamma_G(v)$ that contains w. Therefore,

$$k_{r-1}(\Gamma_G(v)) \geq \frac{d_G(v)d}{r-1}.$$

However, G is F-free and so $\Gamma_G(v)$ is $(F - x)$-free where x is any vertex in F. We conclude that

$$\frac{d_G(v)d}{r-1} \leq k_{r-1}(\Gamma_G(v)) \leq \mathrm{ex}(d_G(v), K_{r-1}, F - x)$$

for any $x \in V(F)$. Using our hypothesis and the definition of d, this inequality can be rewritten as

$$\frac{d_G(v)(v_F + e_F)n^{r-\theta}}{(r-1)\mathrm{ex}(n, F)} \leq c d_G(v)^i.$$

We can cancel a factor of $d_G(v)$ and rearrange the above inequality to get, using (5), that

$$(v_F + e_F)n^{r-\theta} \leq c(r-1)\mathrm{ex}(n, F)\left(\frac{2\mathrm{ex}(n, F)}{n}\right)^{i-1}.$$

Since $e(H) = 4(v_F + e_F)n^{r-\theta}$,

$$e(H) \leq 4c(r-1)2^{i-1}\frac{\mathrm{ex}(n, F)^i}{n^{i-1}}.$$

□

We complete this section by using Theorem 15 to prove Theorem 6. We must show that

$$\mathrm{ex}_r(n, \text{Berge-}K_{s,t}) = O(n^{r - \frac{r(r-1)}{2s}})$$

for $3 \leq r \leq s \leq t$.

Proof of Theorem 6. Let $3 \le r \le s \le t$ be integers. By a result of Alon and Shikhelman (see Lemma 4.2 [2]),

$$\mathrm{ex}(m, K_{r-1}, K_{s-1,t}) \le \left(\frac{1}{(r-1)!} - o_m(1)\right)(t-1)^{\frac{(r-1)(r-2)}{2(s-1)}} m^{r-1-\frac{(r-1)(r-2)}{2(s-1)}}.$$

We apply Theorem 15 with c sufficiently large as a function of r, s, and t, with

$$i = r - 1 - \frac{(r-1)(r-2)}{2(s-1)},$$

and use the well-known bound $\mathrm{ex}(n, K_{s,t}) = O(n^{2-1/s})$ to get that for large enough n,

$$\mathrm{ex}_r(n, \text{Berge-}K_{s,t}) = O(n^{(2-1/s)i-i+1}).$$

Here the implied constant depends only on r, s, and t. A short calculation shows that

$$(2 - 1/s)i - i + 1 = r - \frac{r(r-1)}{2s}$$

and this completes the proof. □

4.2 Lower Bounds and the proof of Theorem 7

By Proposition 2,

$$\mathrm{ex}(n, K_r, F) \le \mathrm{ex}_r(n, \text{Berge-}F) \le \mathrm{ex}_r(n, F^+).$$

We can use this inequality together with the results of [2] to immediately obtain lower bounds on $\mathrm{ex}_r(n, \text{Berge-}F)$ and $\mathrm{ex}_r(n, F^+)$.

Theorem 16 (Alon, Shikhelman [2])**.** *For $r \ge 2$, $s \ge 2r - 2$, and $t \ge (s-1)! + 1$,*

$$\left(\frac{1}{r!} + o(1)\right) n^{r-\frac{r(r-1)}{2s}} \le \mathrm{ex}(n, K_r, K_{s,t}).$$

For $s \ge 2$ and $t \ge (s-1)! + 1$,

$$\left(\frac{1}{6} + o(1)\right) n^{3-\frac{3}{s}} \le \mathrm{ex}(n, K_3, K_{s,t}).$$

Kostochka, Mubayi, and Verstraëte [22] proved that for any $3 \le s \le t$,

$$\mathrm{ex}_3(n, K_{s,t}^+) = O(n^{3-3/s}).$$

It follows from Proposition 2 that all three of the functions

$$\mathrm{ex}(n, K_3, K_{s,t}),\ \mathrm{ex}_3(n, \text{Berge-}K_{s,t}),\ \text{and } \mathrm{ex}_3(n, K_{s,t}^+)$$

are $O(n^{3-3/s})$, and in the case that $t \ge (s-1)! + 1$, they are $\Theta(n^{3-3/s})$.

Before giving our lower bounds we introduce some notation. Let G be a graph and A and B be disjoint subsets of $V(G)$. Write $G[A]$ for the subgraph of G induced by A and $G(A, B)$ for the spanning subgraph of G whose edges are those with one endpoint in A and the other in B.

Lemma 17. *Let $3 \leq s \leq t$ be integers. Let G be a graph and $V(G) = A \cup B$ be a partition of the vertex set of G. If $G[A]$ is $K_{2,2}$-free, $G[B]$ is $K_{2,2}$-free, and $G(A, B)$ is $K_{s,t}$-free, then G is $K_{s+1,t+1}$-free.*

Proof. For contradiction, suppose that

$$\{x_1, \ldots, x_{s+1}\} \text{ and } \{y_1, \ldots, y_{t+1}\}$$

are parts of a $K_{s+1,t+1}$ in G. Assume first that A contains at least s of the x_i's. Since $s > 2$ and $G[A]$ is $K_{2,2}$-free, A can contain at most one y_j so that B contains at least t of the y_j's. This, however, gives a $K_{s,t}$ in $G(A, B)$ which is a contradiction. By symmetry, B cannot contain s of the x_i's and so we may assume that A contains at least two x_i's and B contains at least two x_i's. Here we are using the fact that $s + 1 \geq 4$. As $G[A]$ and $G[B]$ are $K_{2,2}$-free, each of A and B can contain at most one y_j which is a contradiction since $t + 1 > 2$. □

Our construction will make use of the Projective Norm Graphs of Alon, Kollár, Rónyai, and Szabó [1, 18]. Let q be a power of an odd prime, $s \geq 2$ be an integer, and $N : \mathbb{F}_{q^{s-1}} \to \mathbb{F}_q$ be the norm function defined by

$$N(X) = X^{1+q+q^2+\cdots+q^{s-2}}.$$

The Projective Norm Graph, which we denote by $H(s, q)$, is the graph with vertex set $\mathbb{F}_{q^{s-1}} \times \mathbb{F}_q^*$ where (x_1, x_2) is adjacent to (y_1, y_2) if $N(x_1 + y_1) = x_2 y_2$. We will use a bipartite version of this graph. Let $H^b(s, q)$ be the bipartite graph whose parts are A and B where A and B are disjoint copies of $\mathbb{F}_{q^{s-1}} \times \mathbb{F}_q^*$, and $(x_1, x_2)_A$ in A is adjacent to $(y_1, y_2)_B$ in B if

$$N(x_1 + y_1) = x_2 y_2.$$

It is shown in [1] that $H(s, q)$ is $K_{s,(s-1)!+1}$-free. A similar argument gives that $H^b(s, q)$ is $K_{s,(s-1)!+1}$-free.

Lemma 18. *Let $s \geq 3$ be a fixed integer. The graph $H^b(s, q)$ has at least*

$$(1 - o(1))\frac{q^{4(s-1)}}{4}$$

copies of $K_{2,2}$ where $o(1) \to 0$ as $q \to \infty$.

Proof. We will use a known counting argument to obtain a lower bound on the number of $K_{2,2}$'s in a d-regular bipartite graph with n vertices in each part.

Suppose that F is a d-regular bipartite graph with parts X and Y where $|X| = |Y| = n$. Write $X^{(2)}$ for the set of all subsets of size 2 in X and write $\hat{d}(\{x, x'\})$ for the number of vertices that are adjacent to both x and x'. We have

$$\sum_{\{x,x'\} \in X^{(2)}} \hat{d}(\{x, x'\}) = \sum_{y \in Y} \binom{d(y)}{2} = n\binom{d}{2}. \tag{6}$$

The number of $K_{2,2}$'s in F is

$$\sum_{\{x,x'\}\in X^{(2)}} \binom{\hat{d}(\{x,x'\})}{2} \geq \binom{n}{2}\binom{\binom{n}{2}^{-1}\sum_{\{x,x'\}\in X^{(2)}}\hat{d}(\{x,x'\})}{2} \geq \binom{n}{2}\binom{n\binom{d}{2}/\binom{n}{2}}{2}$$

where the first inequality is by convexity and the second is by (6). Therefore, the number of $K_{2,2}$'s in F is at least

$$\frac{1}{2}n\binom{d}{2}\left(\frac{n\binom{d}{2}}{\binom{n}{2}}-1\right) = \frac{nd(d-1)}{4}\left(\frac{d(d-1)}{n-1}-1\right).$$

The graph $H^b(s,q)$ has $q^{s-1}(q-1)$ vertices in each part and is $(q^{s-1}-1)$-regular. For $s \geq 3$, we have that the number of $K_{2,2}$'s in $H^b(s,q)$ is at least

$$(1-o(1))\frac{q^{4s-4}}{4}$$

where $o(1) \to 0$ as $q \to \infty$. □

Let q be a power of an odd prime and R_q be the graph with vertex set $\mathbb{F}_q \times \mathbb{F}_q$ where (a_1, a_2) is adjacent to (b_1, b_2) if and only if $a_1 + b_1 = a_2 b_2$. The graph R_q has q^2 vertices. It is easy to check (see [25]) that R_q has $\frac{1}{2}q^2(q-1)$ edges and no copy of $K_{2,2}$.

We now have all of the tools that we need in order to prove Theorem 7. We must show that for $s \geq 3$ and q an even power of an odd prime,

$$\mathrm{ex}(2q^s, K_4, K_{s+1,(s-1)!+2}) \geq \left(\frac{1}{4}-o(1)\right)q^{3s-4}.$$

Proof of Theorem 7. Let A and B be disjoint sets of q^s vertices each. Choose $A' \subset A$ and $B' \subset B$ arbitrarily with $|A'| = |B'| = q^{s-1}(q-1)$. Put a copy of $H^b(s,q)$ between A' and B'. Finally, pick two independent random copies of $R_{q^{s/2}}$ on vertex sets A and B and let G be the resulting graph. Observe that a given pair in A (or B) is adjacent with probability $q^{-s/2}$. By Lemma 18 and independence, the expected number of copies of K_4 in G is at least

$$\left(\frac{1}{4}-o(1)\right)q^{4(s-1)}\left(\frac{1}{q^{s/2}}\right)^2 = \left(\frac{1}{4}-o(1)\right)q^{3s-4}.$$

Fix a graph G_q with at least this many copies of K_4. Clearly $G_q[A]$ and $G_q[B]$ are both $K_{2,2}$-free and the edges of $G_q(A,B)$ form a $H^b(s,q)$ which is $K_{s,(s-1)!+1}$-free. By Lemma 17, G_q is $K_{s+1,(s-1)!+2}$-free. □

A density of primes argument, Theorem 7, and Theorem 6 give the following result for 4-graphs.

Corollary 19. *If $s \geq 3$ is an integer, then for sufficiently large n, there are positive constants c_s and C_s such that*

$$c_s n^{3-4/s} \leq \mathrm{ex}_4(n, \text{Berge-}K_{s+1,(s-1)!+2}) \leq C_s n^{4-6/(s+1)}.$$

In particular, there is a positive constant c such that

$$cn^{5/3} \leq \mathrm{ex}(n, K_4, K_{4,4}) \tag{7}$$

provided n is sufficiently large. This lower bound is better than what one obtains using a simple expected value argument and random graphs. Indeed, suppose G is a random n-vertex graph where a pair forms an edge with probability p, independently of the other edges. Let X be the number of 4-cliques in G and Y be the number of $K_{4,4}$'s in G. We have

$$\mathbb{E}(X - Y) \geq \left(\frac{n}{4}\right)^4 p^6 - n^8 p^{16}.$$

If $p = \left(\frac{3}{2^{11}}\right)^{1/10} n^{-2/5}$, then

$$\mathbb{E}(X - Y) \geq 0.00004 n^{8/5}.$$

This implies that there is an n-vertex graph for which we can remove one edge from each $K_{4,4}$ and have a subgraph that is $K_{4,4}$-free and has at least $0.00004n^{8/5}$ copies of K_4. While simple, this argument does not improve (7).

5 Counting r-graphs of girth 5 and the proof of Theorem 8

For a family of forbidden subgraphs $\mathcal{F}$, denote by $F_r(n, \mathcal{F})$ the family of all r-uniform simple hypergraphs on n vertices which do not contain any member of $\mathcal{F}$ as a subgraph and let $F_r(n, \mathcal{F}, m)$ denote those graphs in $F_r(n, \mathcal{F})$ which have m edges. Let

$$f_r(n, \mathcal{F}) = |F_r(n, \mathcal{F})|$$
$$f_r(n, \mathcal{F}, m) = |F_r(n, \mathcal{F}, m)|.$$

It is clear that

$$f_r(n, \mathcal{F}) \geq 2^{\mathrm{ex}_r(n, \mathcal{F})}. \tag{8}$$

In this section, we will study the quantities $f_r(n, \mathcal{F})$ and $f_r(n, \mathcal{F}, m)$ when $\mathcal{F}$ is the family of Berge cycles of length at most 4. Let $\mathcal{B}_k = \{\text{Berge-}C_2, \ldots, \text{Berge-}C_k\}$. Note that when a hypergraph is Berge-C_2-free, this means that any two hyperedges share at most one vertex (i.e., the hypergraph is linear). Throughout this section, when we say a hypergraph of *girth* g, we mean an r-uniform hypergraph that is $\mathcal{B}_{g-1}$-free, i.e, it contains no Berge-C_k for $k < g$.

Lazebnik and Verstraëte [24] examined girth 5 hypergraphs and gave the following bounds for $r = 3$

$$\mathrm{ex}_3(n, \mathcal{B}_4) = \frac{1}{6}n^{3/2} + o(n^{3/2})$$

and for general r (with n large enough),

$$\frac{1}{4}r^{-4r/3}n^{4/3} \leq \mathrm{ex}_r(n, \mathcal{B}_4) \leq \frac{1}{r(r-1)}n^{3/2} + O(n).$$

Our main result in this section is the next theorem.

Theorem 20. *Let $r \geq 2$ and n be large enough. Then*

$$f_r(n, \mathcal{B}_4, m) \leq \exp\left(n^{4/3}\log^3 n\right)\left(\frac{n^3}{m^2}\right)^m.$$

Theorem 20 yields the following two corollaries, the first of which implies Theorem 8.

Corollary 21. *Let $r \geq 2$. Then there exists a constant C such that*

$$f_r(n, \mathcal{B}_4) \leq 2^{Cn^{3/2}}.$$

The first group to consider extremal problems in random graphs was probably Babai-Simonovits-Spencer [3]. Among others they asked: what is the maximum number of edges of a C_4-free subgraph of the random graph $G_{n,p}$ when $p = 1/2$? Here we give a partial answer to the corresponding question in Berge-hypergraph setting. Let $G_{n,p}^{(r)}$ be the random r-uniform hypergraph on n vertices, each edge being present independently with probability p.

Corollary 22. *Let $0 < p < \frac{1}{(r(r-1))^2}$. Then there exists an $\epsilon > 0$ such that with probability tending to 1,*

$$\mathrm{ex}_r(G_{n,p}^{(r)}, \mathcal{B}_4) < (1-\epsilon)\mathrm{ex}_r(n, \mathcal{B}_4).$$

Theorem 20 implies Corollary 21 by noting that $(n^3/m^2)^m = 2^{O(n^{3/2})}$ and Corollary 22 by a simple first moment argument combined with the fact [24] that $\mathrm{ex}_r(n, \mathcal{B}_4) \leq \frac{1+o(1)}{r(r-1)}n^{3/2}$.

Proof of Theorem 20. For a graph H and a natural number d, let $\mathrm{ind}(H, d)$ denote the number of independent sets of size exactly d in H. We adapt the proofs of Kleitman's and Winston's upper bound on the number of C_4-free graphs [17] (see also [29] for a nice exposition) and Füredi's extension to graphs with m edges [11]. The rough idea of the proof is that any hypergraph of girth 5 can be decomposed into a sequence of subhypergraphs satisfying mild conditions, and that the number of such sequences is bounded.

If G is any hypergraph, we may successively peel off vertices of minimum degree. Specifically, let v_n be a vertex such that $d_G(v_n) = \delta(G)$. Once $v_n, v_{n-1}, \ldots, v_{k+1}$ are chosen, let v_k satisfy

$$|\Gamma(v_k) \setminus \{v_n, \ldots, v_{k+1}\}| = \delta(G \setminus \{v_n, \ldots, v_{k+1}\}).$$

For each i, let $G_i = G[\{v_1, \ldots, v_i\}]$. This sequence of subhypergraphs has the property that for all i,

$$\delta(G_{i-1}) \geq \delta(G_i) - 1 = d_{G_i}(v_i) - 1.$$

That is, $\delta(G_i) \leq \delta(G_{i-1})+1$. Now, if G is $\mathcal{B}_4$-free, then each G_i is also $\mathcal{B}_4$-free. To summarize, any hypergraph of girth 5 may be constructed one vertex at a time such that

1. At each step, the subhypergraph is $\mathcal{B}_4$-free.

2. When adding the i'th vertex v_i, we have that the minimum degree of the graph which v_i is being added to is at least $d_{G_i}(v_i) - 1$.

The crux of the upper bound is that one cannot add a vertex to a graph of high minimum degree and keep it $\mathcal{B}_4$-free in too many ways. To formalize this, let $g_i(d)$ be the maximum number of ways to attach a vertex of degree d to a $\mathcal{B}_4$-free graph on i vertices with minimum degree at least $d-1$, such that the resulting graph remains $\mathcal{B}_4$-free, and let $g_i = \max_{d \le i} g_i(d)$. Note that

$$g_i(d) \le \binom{i}{(r-1)d}((r-1)d)! \tag{9}$$

for all d, so g_i is well-defined. Now let us count the number of sequences of subhypergraphs $G_1, \ldots, G_n$ that can come from a hypergraph of girth 5 with m edges, G. Note that each G of girth 5 creates (once the vertices are ordered) a unique sequence $G_1, \ldots, G_n$. First, we trivially bound the number of ways to order the vertices $(v_1, \ldots, v_n)$ by $n!$, and we also trivially bound the number of degree sequences $\{d_{G_1}(v_1), \ldots, d_{G_n}(v_n)\}$ by $n!$. By the way we have constructed the sequence $\{G_1, \ldots, G_n\}$ and by the definition of $g_i(d)$, we have that

$$f_r(n, \mathcal{B}_4, m) \le n!n! \max \prod_{i=1}^{n} g_i(d_i),$$

where the maximum is taken over all degree sequences such that $\sum d_i = m$.

If $d_i \le i^{1/3} \log i$, we use (9) and have that, for large i,

$$g_i(d_i) \le i^{i^{1/3} \log^2 i}.$$

From now on we will assume $d_i \ge i^{1/3} \log i$. Assume that G_i is a hypergraph of girth 5 on i vertices with minimum degree at least d. We construct an auxiliary graph H_i with vertex set $V(H_i) = V(G_i)$ and $xy \in E(H_i)$ if and only if there is a path of length 2 from x to y in the hypergraph G_i.

Now we observe that in order to attach v_{i+1} to G_i and have the resulting graph G_{i+1} remain $\mathcal{B}_4$-free, the neighborhood of v_{i+1} must be an independent set in H_i. To see this, if $v_{i+1} \sim x$ and $v_{i+1} \sim y$ where $xy \in E(H_i)$, then there is a path of length 2 in G_i from x to y. Now, if there exists a hyperedge $e \in E(G_{i+1})$ such that $\{x, y, v_{i+1}\} \subset e$, this creates a Berge-$C_3$ in G_{i+1}. Otherwise, the vertex v_{i+1} creates a Berge-C_4 in G_{i+1}.

Therefore to bound $g_i(d_i)$ it suffices to give a uniform upper bound on $\mathrm{ind}(H_i, d_i)$. To do this, we use a lemma of Kleitman and Winston, which is the original inspiration for the container method [17].

Lemma 23 (Kleitman and Winston (cf [19, 29])**.** *Let G be a graph on n vertices. Let $\beta \in (0, 1)$, q an integer, and R a real number satisfy*

1. $R \ge e^{-\beta q} n$.

2. For all subsets $U \subset V(G)$ with $|U| \geq R$,

$$e_G(U) \geq \beta \binom{|U|}{2}.$$

Then for all $m \geq q$,

$$\mathrm{ind}(G, m) \leq \binom{n}{q} \binom{R}{m-q}.$$

We now give an upper bound on $\mathrm{ind}(H_i, d)$. Let $B \subset V(H_i)$. Then (with floors and ceilings omitted)

$$\begin{aligned} e_{H_i}(B) &\geq \sum_{z \in V(G_i)} \binom{|\Gamma_{G_i}(z) \cap B|/(r-1)}{2} \\ &\geq i \binom{\frac{1}{(r-1)i} \sum_{z \in V(G_i)} |\Gamma_{G_i}(z) \cap B|}{2} \\ &\geq i \binom{\frac{1}{(r-1)i} \sum_{y \in B} \frac{d(y)}{r}}{2} \\ &\geq i \binom{\frac{|B|\delta(G_i)}{r^2 i}}{2} \geq i \binom{\frac{|B|(d_i - 1)}{r^2 i}}{2} \\ &\geq \frac{|B|^2 d_i^2}{8 r^4 i}, \end{aligned}$$

where the last inequality holds for i large enough. This quantity is bigger than

$$i^{-1/3} \log i \binom{|B|}{2}$$

for i large enough since $d_i \geq i^{1/3} \log i$. Now we let $\beta = i^{-1/3} \log i$ (which is in $(0,1)$ for i large enough), $R = \frac{i}{d_i}$, and $q = i^{1/3}$. Note that $R > 1$ and $e^{-\beta q} i = 1$. Therefore by Lemma 23, we have

$$\mathrm{ind}(H_i, d_i) \leq \binom{i}{i^{1/3}} \binom{\frac{i}{d_i}}{d_i - i^{1/3}}.$$

Since $d_i - i^{1/3} \geq \frac{1}{2} d_i$ for i large enough, we have

$$\mathrm{ind}(H_i, d_i) \leq \left(\frac{2ei}{d_i^2}\right)^{d_i} (i^{2/3})^{i^{1/3}}.$$

Thus

$$\begin{aligned} f_r(n, \mathcal{B}_4, m) &\leq n! n! \max \prod \left(\frac{2ei}{d_i^2}\right)^{d_i} (n^{2/3})^{2n^{1/3} \log^2 n} \\ &\leq \exp\left(n^{4/3} \log^3 n + (\log n + O(1)) \sum d_i - 2 \sum d_i \log d_i\right) \end{aligned}$$

for n large enough. Next we note that $\sum d_i = m$ and by convexity $\sum d_i \log d_i \geq m \log(m/n)$. Rearranging gives the result. □

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