Distinct edge weights on graphs

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Overview

1. Sidon sets
2. Sum-injective labelings
3. Product-injective labelings
   - Lower bound
   - Upper bound
4. Sum-injective labelings
   - An idea

Sidon sets

Definition

Given an abelian group (or monoid) Γ, a *Sidon set* A is a set $A \subset \Gamma$ such that $a, b, c, d \in A$ and

$$a + b = c + d$$

implies that $\{a, b\} = \{c, d\}$.

In this talk we will consider Sidon subsets of $[N] := \{1, 2, ..., N\}$ of integers under either "+" or "$*$".

Sidon sets

Having the Sidon property forces a Sidon set to be “thin”.

No progressions of length more than two.

How thin does such a set have to be?

Sidon sets

Erdős and Turán (1941) showed that if $A \subset [n]$ is a Sidon set (with addition), then

$$|A| < n^{1/2} + O(n^{1/4}).$$

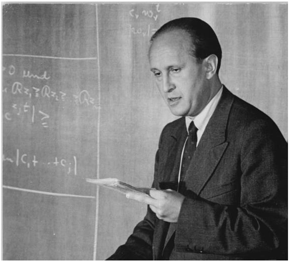

Figure: Erdős and Turán

Sidon sets

There are still open questions about Sidon sets (with addition). How big can they be? What is the structure of a Sidon set of large size?

Denote by $f(n)$ the largest integer k for which there is a sequence $1 \leqslant a_1 < \cdots < a_k \leqslant n$ so that all the sums $a_i + a_j$ are distinct. Turán and I conjectured about 40 years ago [5] that

$$f(n) = n^{1/2} + O(1). \tag{1}$$

The conjecture seems to be very deep and I offered long ago a prize of 500 dollars for a proof or disproof of (1). The sharpest known results in the direction of (1) state [5]

$$n^{1/2} - n^{1/2-c} < f(n) < n^{1/2} + n^{1/4} + 1. \tag{2}$$

Figure: 500 USD Erdős question

Sidon sets: Generalizations

- $B_h[g]$ sets: The number of solutions to

$$a_1 + \cdots + a_h = b_1 + \cdots + b_h$$

  is bounded by g. Very little is understood about these sets when $h > 2$.

- k-fold Sidon sets:

$$a + b \neq i(c + d)$$

  for $1 \leq i \leq k$. Asymptotics not known for $k \geq 2$.

- Restricted Sidon sets: taking only squares, cubes, 5th powers, etc.

$$a^5 + b^5 = c^5 + d^5 \quad ?$$

Sidon sets on graphs

A generalization for graph theorists:

Given a graph G let $\chi : V(G) \to \mathbb{N}$ be an injective function (i.e. label the vertices with distinct natural numbers).

Definition

A *sum-injective coloring* of graph G is an injection $\chi : V(G) \to \mathbb{Z}$ such that $\chi(x) + \chi(y) \neq \chi(u) + \chi(v)$ for distinct edges $xy, uv \in E(G)$.

We weight an edge with the sum of its endpoints and require that all the edges have distinct weights.

Sidon sets on graphs

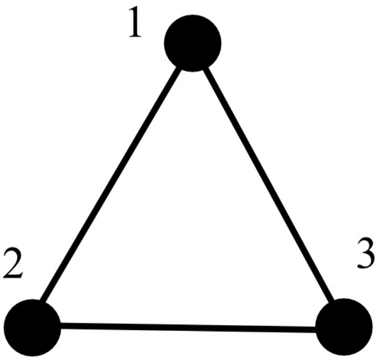

Figure: A sum-injective labeling of K_3

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Sidon sets on graphs

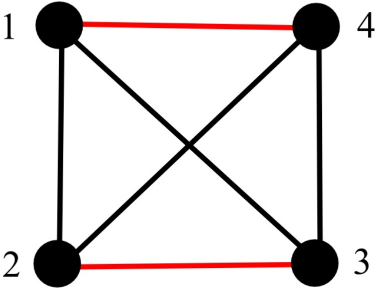

Figure: Not a sum-injective labeling of K_4

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Sidon sets on graphs

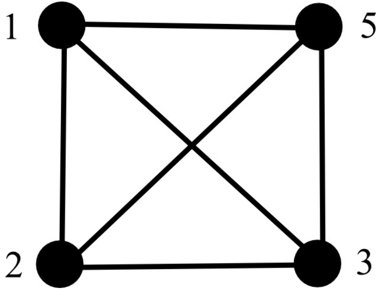

Figure: A sum-injective labeling of K_4

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Sidon sets on graphs

Note that any graph admits a sum-injective labeling by using a Sidon set.We denote by $S(G)$ the minimum N

such that G admits a sum-injective coloring $\chi : V(G) \to [N]$.
$S(K_3) = 3$, $S(K_4) = 5$.

$$S(G) \leq S(K_n) \leq \text{Largest integer in a Sidon set of size } n$$

Denote by $D(G)$ the minimum N such that G admits a difference-injective coloring $\chi : V(G) \to [N]$

Sidon sets on graphs

Sidon sets of $[N]$ have size at most $(1 + o(1))\sqrt{N}$

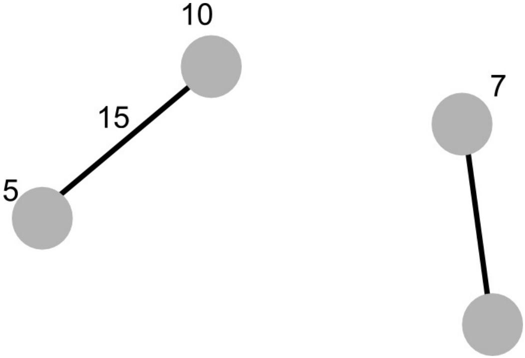
$$\Updownarrow$$

$$S(K_n) = D(K_n) = (1 + o(1))n^2.$$

What about other graphs?

Sum-Injective Coloring

A greedy algorithm gives an upper bound.



The unlabelled vertex cannot receive color 8 (or colors $5, 7, 10$). There may be a restricted color for each neighbor and each edge in G.

Upper bound

Theorem

Let G be a graph with maximum degree Δ. Then

$$S(G) \leq \Delta|E(G)| + n - 1 \leq \frac{\Delta^2 n}{2} + n.$$

Note that $S(G) \leq S(K_n) \leq (1 + o(1))n^2$ by coloring with a Sidon set. Therefore this upper bound is trivial unless Δ is less than $\sqrt{n}$.

Lower bounds

Theorem (Bollobás and Pikhurko 2005)

Let G be a random graph with expected degree d. Then almost surely

$$S(G) \geq \begin{cases} c_1 n^2 & \text{if } d \geq n^{1/2} \log n \\ c_2 \frac{d^2 n}{\log n} & \text{if } d = o\left(\sqrt{n \log n}\right). \end{cases}$$

$$D(G) \geq \begin{cases} (1 - o(1))n^2 & \text{if } d \geq n^{1/2} \log n \\ c_3 \frac{d^2 n}{\log n} & \text{if } d = o\left(\sqrt{n \log n}\right). \end{cases}$$

It is surprising that graphs much less dense than K_n have $S(G) = \Omega\left(n^2\right)$ and $D(G) \sim n^2$.
The Sidon condition is too strong. For most graphs with only $n^{3/2+o(1)}$ edges, labeling with a Sidon set is asymptotically best possible.

Multiplication

Let's switch from $(\mathbb{Z}, +)$ to $(\mathbb{Z}, *)$.

What is a good Sidon subset (under multiplication) of $[N]$? i.e. pick a large subset of natural numbers that has no nontrivial solutions to

$$a \cdot b = c \cdot d.$$

The primes up to N?

$$\pi(N) \sim \frac{N}{\log N}$$

Multiplication

Theorem (Erdős, 1938)

Choosing primes is asymptotically best possible. If $A \subset [N]$ has the property that $a, b, c, d \in A$ and $ab = cd$ implies that $\{a, b\} = \{c, d\}$, then

$$|A| \leq (1 + o(1)) \frac{N}{\log N}.$$

Is the restriction that $ab \neq cd$ for all $\{a, b\} \neq \{c, d\}$ too strong?

Product-injective labelings

Definition

A *product-injective coloring* of graph G is an injection $\chi : V(G) \to \mathbb{Z}$ such that $\chi(x) \cdot \chi(y) \neq \chi(u) \cdot \chi(v)$ for distinct edges $xy, uv \in E(G)$.

We weight an edge with the product of its endpoints and require that all the edges have distinct weights.

Denote by $P(G)$ the minimum N such that G admits a product-injective coloring $\chi : V(G) \to [N]$

Product-injective labelings

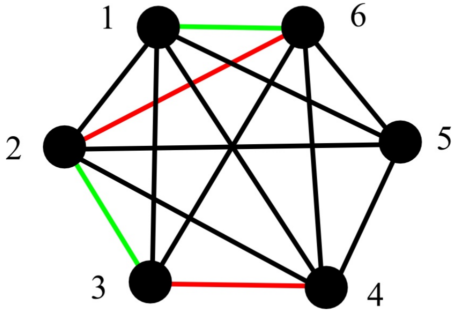


Figure: Not a product-injective labeling of K_6

$P(K_6) > 6.$

Product-injective labelings

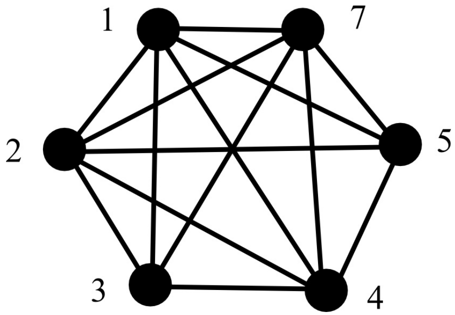


Figure: A product-injective labeling of K_6

$P(K_6) = 7.$

Product-injective labelings

Erdős' result says that $P(K_n) \sim n \log n$.
For all graphs G on n vertices

$$P(G) \leq P(K_n) \leq (1 + o(1)) n \log n.$$

Recall $D(K_n) \sim n^2$ but there are graphs G much sparser that also have $D(G) \sim n^2$.

An analogous result for products?

Product-injective labelings

Theorem (MT and Verstraëte)

Let G be a random graph with expected degree at least $\sqrt{n}(\log n)^5$. Then

$$P(G) \sim n \log n$$

almost surely.

Erdős: $P(K_n) \sim n \log n$.
Proof: If A is a subset of N, with $|A| = (1+\varepsilon)\frac{N}{\log N}$, then A has a nontrivial solution to $ab = cd$.

An auxiliary graph

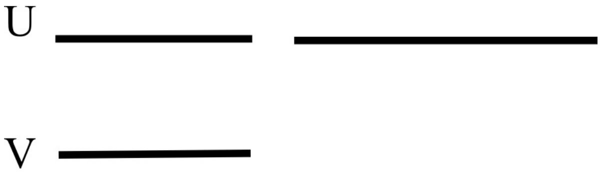


Figure: $U = [N^{2/3}] \cup \text{primes up to } n \qquad V = [N^{2/3}]$.

- Every $a \in [N]$ can be written as $a = u \cdot v$ with $u \in U$, $v \in V$, and $v \leq u$.
- For each $a \in A$, pick such a representation.

An auxiliary graph

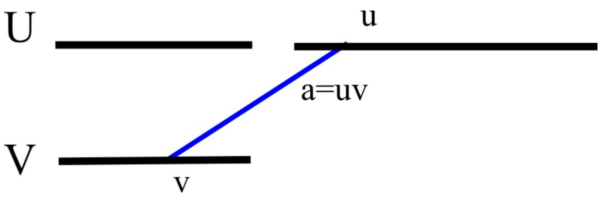


Figure: $U = [N^{2/3}] \cup$ primes up to n $\qquad V = [N^{2/3}]$.

- Every $a \in [N]$ can be written as $a = u \cdot v$ with $u \in U$, $v \in V$, and $v \leq u$.
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- The number of edges in this graph is $|A|$.

An auxiliary graph

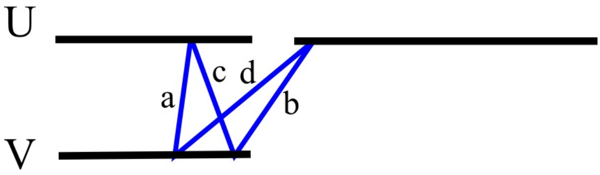


Figure: $U = [N^{2/3}] \cup$ primes up to n $\qquad V = [N^{2/3}]$.

- Every $a \in [N]$ can be written as $a = u \cdot v$ with $u \in U$, $v \in V$, and $v \leq u$.
- For each $a \in A$, pick such a representation.
- The number of edges in this graph is $|A|$.
- Each C_4 yields a nontrivial solution to $ab = cd$.

Erdős showed there is at least one C_4 in this graph.

A Lemma

In fact there are many C_4's in this graph.

Lemma

If $A \subset [N]$ with $|A| \geq (1+\varepsilon)\frac{N}{\log N}.$, then the number of nontrivial solutions to $ab = cd$ in A is

$$\Omega\left(\frac{n^2}{(\log n)^8}\right).$$

How do we use this to prove the lower bound?

Lower Bound

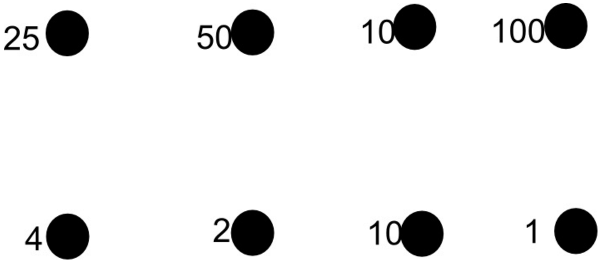
We show that if G is a random graph with expected degree at least $n^{1/2}(\log n)^5$, then $P(G) \geq (1-\epsilon)n \log n$ almost surely.

Strategy:

- Fix a coloring χ from $[(1-\epsilon)n \log n]$.
- Show that the probability that G is product-injectively colored by χ is $o\,(1/\text{number of colorings})$. (use the lemma here)
- The union bound gives that $P(G) \geq (1-\epsilon)n \log n$.

Lower Bound

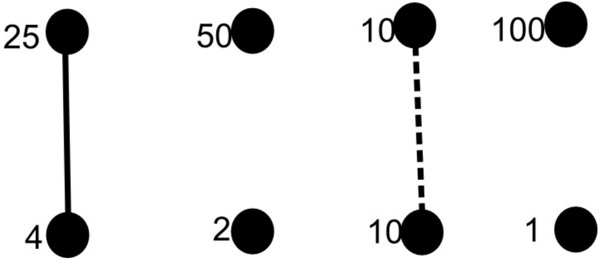
Fix a coloring χ from $[(1-\epsilon)n\log n]$. Look at solutions to $\chi(x)\chi(y) = \chi(u)\chi(v)$.



Lemma: At least $\delta n^2(\log n)^{-8}$ such solutions.

Lower Bound

Fix a coloring χ from $[(1-\epsilon)n\log n]$. Look at solutions to $\chi(x)\chi(y) = \chi(u)\chi(v)$.



For each picture like this, at most one edge can be generated.

Upper bound

What about an upper bound? Recall

$$S(G) \leq \Delta|E(G)| + n.$$

This bound also holds for $P(G)$, but it is very poor.

Upper bound

Theorem (MT and Verstraëte)

Let G be any graph with maximum degree less than $\sqrt{n}(\log n)^{-1}$. Then

$$P(G) \sim n.$$

Recall $n \leq P(G) \leq n$'th prime number $\sim n \log n$.

Upper bound

Almost all graphs	$\Delta > \sqrt{n}(\log n)^5$	$P(G) \sim n \log n$
All graphs	$\Delta < \sqrt{n}(\log n)^{-1}$	$P(G) \sim n$

Upper Bound

Let G be a graph with maximum degree $\Delta \leq \sqrt{n}(\log n)^{-1}$. We will label it with maximum label $(1+o(1))n$ such that no two edges have the same product.

Strategy:

- Throw away highly divisible numbers.

Theorem (Hardy and Ramanujan 1917)

Let $\Omega(k)$ be the number of prime power divisors of k. Then for ω any function that tends to infinity

$$\left|\left\{x \leq N : |\Omega(x) - \log\log N| > \omega\sqrt{\log\log N}\right\}\right| = o(N).$$

Almost every number up to n has less than $\log n$ divisors.

Upper Bound

Strategy:

- Throw away highly divisible numbers.
- Color from a set of size $n + w$, choosing n colors randomly.
- Choose w strategically so that the probability that any two edges share a weight is small, but w is still $o(n)$.
- Local Lemma?

Local Lemma

- For edges uv and xy, let $A_{uv,xy}$ be the event that $\chi(u)\chi(v) = \chi(x)\chi(y)$.
- We've chosen w large enough so that $\mathbb{P}(A_{uv,xy})$ is small.
- If $A_{uv,xy}$ does not occur for any pair uv and xy, then χ is a product-injective labeling.
- However, all of the pairs of events are dependent.

Local Lemma

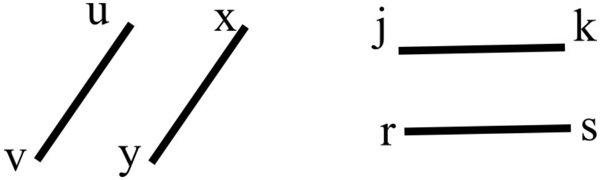


Figure: Almost independent events

- If $\{uv, xy\}$ and $\{jk, rs\}$ are disjoint, then $A_{uv,xy}$ and $A_{jk,rs}$ are dependent but only superficially.

Local Lemma

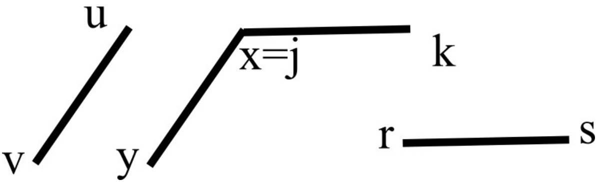


Figure: Highly dependent events

- If $\{uv, xy\}$ and $\{jk, rs\}$ are disjoint, then $A_{uv,xy}$ and $A_{jk,rs}$ are dependent but only superficially.
- If $\{uv, xy\}$ and $\{jk, rs\}$ are not disjoint, then $A_{uv,xy}$ and $A_{jk,rs}$ are highly dependent.
- Let $K_{uv,xy}$ be all of the not highly dependent events for $A_{uv,xy}$.

Let K be an arbitrary subset of $K_{uv,xy}$. Then $\mathbb{P}(A_{uv,xy}|K)$ is still small enough.The proof of the Local Lemma goes through.

Upper Bound

Strategy:

- Throw away highly divisible numbers.
- Color from a set of size $n + w$, choosing n colors randomly.
- Choose w strategically so that the probability that any two edges share a weight is small, but w is still $o(n)$.
- Apply the Modified Local Lemma to show that there is a positive probability that none of the edges have the same product. □

Sums again

Back to working in $(\mathbb{Z}, +)$. Edges have weight the sum of their endpoints.

Theorem (Bollobás and Pikhurko 2005)

Let G be a random graph with expected degree $d = o(\sqrt{n \log n})$, then

$$S(G) = \Omega\left(\frac{d^2 n}{\log n}\right)$$

almost surely.

Recall that a greedy algorithm gives $S(G) \leq \Delta^2 n + n$.

Which bound?

Which bound is closer?

Theorem (Bollobás and Pikhurko 2005)

Let G be a random graph with expected degree $d \gg \log n$. Then

$$S(G) \leq (1 + o(1)) \frac{d^2 n}{\log d}.$$

Is there an analogous result for general graphs of maximum degree d?

Sum-injective coloring

The greedy upper bound $S(G) \leq \Delta^2 n + n$ should be improved, as many of the restrictions are the same.

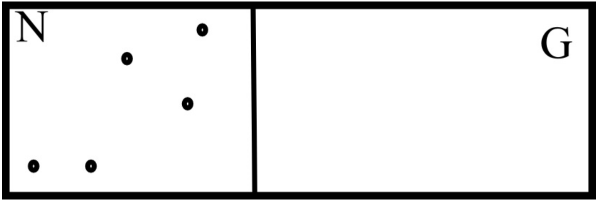
Conjecture

Let G be a graph with maximum degree d. Then

$$S(G) = O\left(\frac{d^2 n}{\log d}\right).$$

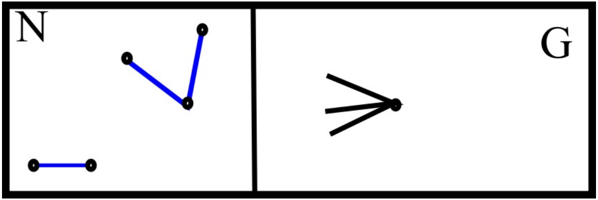
Proof Idea: $d^2 n$ restrictions but many are repeated. Use a semi-random method to color.

Rödl nibble



- Label about $\frac{n}{\log d}$ vertices at a time and label randomly.

Rödl nibble



- Label about $\frac{n}{\log d}$ vertices at a time and label randomly.
- Work in $\mathbb{Z}_n$ so that all weights are equally likely.
- Both the weights and the labels of a vertex's neighbors are uniformly distributed.
- The labels that a vertex is restricted from using should look uniformly distributed.
- We can always find a label for a vertex unless $C\frac{d^2 n}{\log d}$ unique restrictions have been made.

Coupon Collector Problem

The expected time to collect n coupons drawing uniformly, independently, and with replacement is asymptotic to $n \log n$.

Theorem (Erdős and Rényi, 1961)

Let T_n be the time to collect n coupons. Then

$$\mathbb{P}(T_n < n \log n + cn) \to e^{-e^{-c}}$$

as $n \to \infty$.

Heuristically, it should be very unlikely that there is enough time to "collect" all $C\frac{d^2 n}{\log d}$ "coupons". There is not enough time to run out of available labels.

Open problems

1. Prove conjecture: $S(G) = O\left(\frac{d^2 n}{\log d}\right)$.
2. Sidon sets (with addition) of squares, cubes.
3. k-fold Sidon sets.

Thank you!