The Zarankiewicz problem in 3-partite graphs

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“How many edges can be in a 3-partite C_4-free graph?”

Turán numbers

The *Turán number* of a graph F is the maximum number of edges that an n vertex graph may have under the condition that it does not contain F as a subgraph, denoted

$$\mathrm{ex}(n, F).$$

Theorem (Erdős-Stone 1946)

Let $\chi(F) \geq 2$ be the chromatic number of F. Then

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1}\right)\binom{n}{2} + o(n^2).$$

Theorem (Kővári-Sós-Turán 1954)

For integers $2 \leq s \leq t$,

$$\mathrm{ex}(n, K_{s,t}) \leq \frac{1}{2}(t-1)^{1/s}n^{2-1/s} + \frac{1}{2}(s-1)n.$$

$\mathrm{ex}(n, F) < n^{2-\epsilon}$ for bipartite F.

The Zarankiewicz problem

Given integers m, n, s, t, define

$$z(m, n, s, t)$$

to be the maximum number of 1s in a $0 - 1$ matrix with

- size $m \times n$
- having no $s \times t$ submatrix of all 1s.

Equivalent to asking for the maximum number of edges in an $m \times n$ bipartite graph with no $K_{s,t}$.

$$2\text{ex}(n, K_{s,t}) \leq z(n, n, s, t) \leq \text{ex}(2n, K_{s,t}).$$

General question

Given a graph F and an integer $k \geq 2$ define

$$\mathrm{ex}_{\chi \leq k}(n, F)$$

to be the maximum number of edges in an n-vertex F-free graph with chromatic number at most k.

$$\mathrm{ex}_{\chi \leq 2}(n, F) \leq \mathrm{ex}_{\chi \leq 3}(n, F) \leq \cdots \leq \mathrm{ex}_{\chi \leq n}(n, F) = \mathrm{ex}(n, F).$$

Casey's Question: What is $\mathrm{ex}_{\chi \leq 3}(n, C_4)$?

Not just a novelty!

Conjecture (Erdős-Simonovits 1982)

Given any finite family of graphs $\mathcal{F}$ there exists an ℓ such that

$$\mathrm{ex}(n, \mathcal{F} \cup C_{2\ell+1}) \sim \mathrm{ex}_{\chi \leq 2}(n, \mathcal{F}).$$

Theorem (Erdős-Simonovits 1982)

$$\mathrm{ex}(n, \{C_4, C_5\}) \sim \mathrm{ex}_{\chi \leq 2}(n, C_4) \sim \frac{1}{2\sqrt{2}} n^{3/2}.$$

Conjecture (Erdős 1975)

$$\mathrm{ex}(n, \{C_4, C_3\}) \sim \mathrm{ex}_{\chi \leq 2}(n, C_4).$$

Craig

Results

Theorem (Tait-Timmons)

Let $2 \leq s \leq t$ be integers. Then

$$\mathrm{ex}_{\chi\leq 3}(n, K_{s,t}) \leq \left(\frac{1}{3}\right)^{1-1/s}\left(\frac{t-1}{2}+o(1)\right)^{1/s} n^{2-1/s}.$$

$$\mathrm{ex}_{\chi\leq 3}(n, K_{2,2t+1}) = \sqrt{\frac{t}{3}}n^{3/2} + o(n^{3/2}).$$

$\mathrm{ex}(n, K_{s,t}) \leq \frac{1}{2}(t-s+1+o(1))^{1/s}n^{2-1/s}$

$\mathrm{ex}(n, K_{2,2t+1}) = \frac{\sqrt{2t}}{2}n^{3/2} + o(n^{3/2})$ $\quad (\frac{1}{\sqrt{3}} < \frac{\sqrt{2}}{2})$

$\mathrm{ex}_{\chi\leq 2}(n, K_{2,2t+1}) = \frac{\sqrt{t}}{2}n^{3/2} + o(n^{3/2})$ $\quad (\frac{1}{2} < \frac{1}{\sqrt{3}})$

Sunny

Allen, Keevash, Sudakov, and Verstraëte gave a nontrivial upper bound for $\mathrm{ex}_{\chi\leq k}(n,\mathcal{F})$ for any “smooth” family using sparse regularity.

Theorem (Allen-Keevash-Sudakov-Verstraëte 2014)

There are $K_{2,2t+1}$ and triangle free graphs on n vertices with

$$\frac{t+1}{\sqrt{t(t+2)}}\mathrm{ex}_{\chi\leq 2}(n,K_{2,2t+1})$$

edges.

Conjecture (Allen-Keevash-Sudakov-Verstraëte 2014)

Erdős's conjecture is false, ie

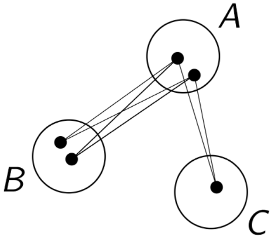
$$\mathrm{ex}(n,\{C_3,C_4\})\not\sim \mathrm{ex}_{\chi\leq 2}(n,C_4).$$

Upper bound

To prove the upper bound: do the obvious thing! Let the partite sets be A, B, C, then

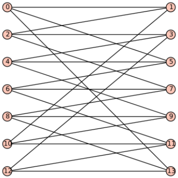
$$(t-1)\binom{|A|}{s} \geq \sum_{v\in B}\binom{d_A(v)}{s} + \sum_{v\in C}\binom{d_A(v)}{s}.$$

Use convexity and optimize!



Lower bound

How to construct dense $K_{2,t}$ free graphs? $X = Y = \mathbb{F}_q \times \mathbb{F}_q$, $(x_1, x_2) \sim (y_1, y_2)$ if and only if $x_1y_1 + x_2y_2 = 1$.

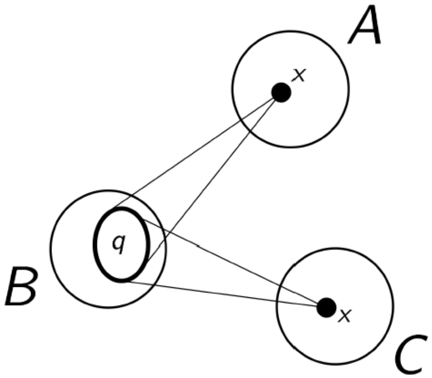


Füredi's idea: "mod out" by a subgroup. H a subgroup of $\mathbb{F}_q^*$ of size t.

- Let $X = Y = (\mathbb{F}_q \times \mathbb{F}_q \setminus (0,0))/H$,
- $(x_1, x_2) \sim (y_1, y_2)$ if and only if $x_1y_1 + x_2y_2 \in H$.
- $\frac{q^2-1}{t}$ vertices, degree q, no $K_{2,t+1}$.

Lower bound

Put copies of Füredi's graph between parts? Too symmetric.



Lower bound

We construct a similar bipartite graph to put between parts that breaks the symmetry. Let $A \subset \mathbb{Z}_{q^2-1}$ be a Bose-Chowla Sidon set. This means that if $a+b=c+d$ for $a,b,c,d \in A$ then $\{a,b\}=\{c,d\}$. Let $t|q^2-1$ and let H be a subgroup of $\mathbb{Z}_{q^2-1}$ of order t. Define a bipartite graph with

- Partite sets $X = Y = \mathbb{Z}_{q^2-1}/H$
- $x \sim y$ if and only if $x - y \in A$
- $|A| = q$ regular, $K_{2,t+1}$ free.

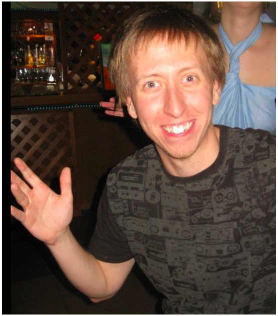
The non-bipartite version of this graph is similar to the non-bipartite version of Füredi's graph. When $q = 19$ and $t \in \{1,2,3,6\}$ our graph has one more edge than Füredi's.

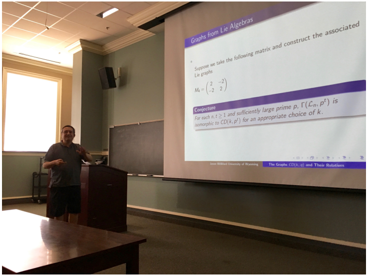
Lower bound

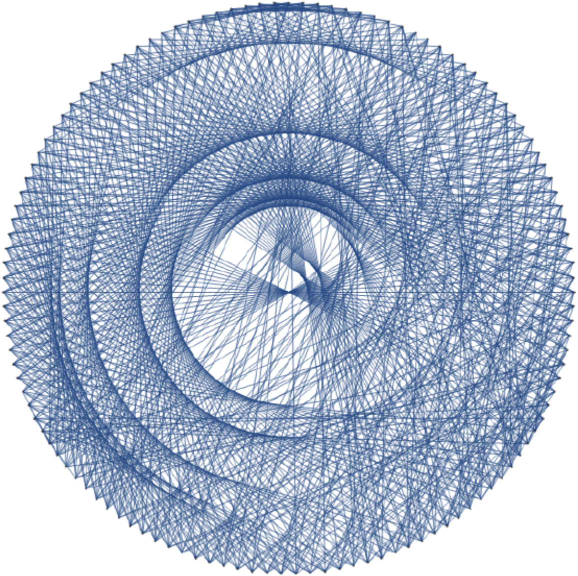
- Put this bipartite graph between parts in a “directed triangle”.
- Symmetry broken! This graph is $K_{2,2t+1}$ free.
- The common neighborhood of a pair of vertices is determined by how many solutions there are to $a + b = h$ with $a, b \in A$ and $h \in H$.

Forbidding C_4

$$\frac{n^{3/2}}{2\sqrt{2}} \leq \mathrm{ex}_{\chi \leq 3}(n, C_4) \leq \frac{n^{3/2}}{\sqrt{6}}.$$



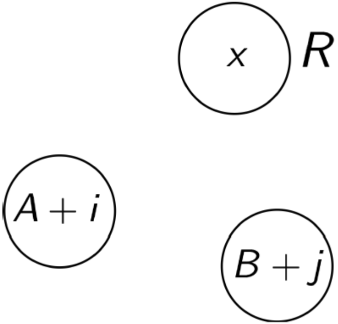




- A (v, k, λ)-difference family in a group Γ of order v is a collection of sets $\{D_1, \cdots, D_t\}$ each of size k such that $(D_1 - D_1) \cup \cdots \cup (D_t - D_t)$ contains every nonzero element of Γ exactly λ times.
- $A = \{0, 1\}$ and $2A$ is a $(5, 2, 1)$ difference family in $\mathbb{Z}_5$.
  $A = \{1, 10, 16, 18, 37\}$ and $9A$ is a $(41, 5, 1)$ difference family in $\mathbb{Z}_{41}$.
- These difference families yield constructions where the counting in the upper bound is tight! $\mathrm{ex}_{\chi\leq 3}(15, C_4) = 30$ and $\mathrm{ex}_{\chi\leq 3}(123, C_4) = 615$.

Theorem (Tait-Timmons-Williford)

Let R be a finite ring, $A \subset R$ an additive Sidon set, and $c \in R$ invertible. Let $B = cA = \{ca : a \in A\}$.
Then if $(A - A) \cap (B - B) = \{0\}$ there exists a 3-partite, C_4 free graph on $3|R|$ vertices which is $|A|$ regular between each pair of parts.



$$A + i \sim B + j \quad \text{if and only if} \quad -cj + i \in A$$

- If there is an infinite family of $(2k^2 - 2k + 1, k, 1)$-difference families in $\mathbb{Z}_{2k^2-2k+1}$ where the blocks are translates of each other this would yield an infinite family of graphs where the upper bound is (exactly!) tight.
- No $(61, 6, 1)$-difference family exists in $\mathbb{F}_{61}$.
- *Exact* difference families too restrictive and not necessary for an asymptotic result.

Open Problems

- Constructions for $k > 3$. Can't break symmetry!
- "Approximate" designs
- $K_{3,3}$ free 3-partite graphs?