The Zarankiewicz problem in 3-partite graphs

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Abstract

Let F be a graph, $k \geq 2$ be an integer, and write $\mathrm{ex}_{\chi \leq k}(n, F)$ for the maximum number of edges in an n-vertex graph that is k-partite and has no subgraph isomorphic to F. The function $\mathrm{ex}_{\chi \leq 2}(n, F)$ has been studied by many researchers. Finding $\mathrm{ex}_{\chi \leq 2}(n, K_{s,t})$ is a special case of the Zarankiewicz problem. We prove an analogue of the Kővári-Sós-Turán Theorem by showing

$$\mathrm{ex}_{\chi \leq 3}(n, K_{s,t}) \leq \left(\frac{1}{3}\right)^{1-1/s} \left(\frac{t-1}{2} + o(1)\right)^{1/s} n^{2-1/s}$$

for $2 \leq s \leq t$. Using Sidon sets constructed by Bose and Chowla, we prove that this upper bound in asymptotically best possible in the case that $s = 2$ and $t \geq 3$ is odd, i.e., $\mathrm{ex}_{\chi \leq 3}(n, K_{2,2t+1}) = \sqrt{\frac{t}{3}} n^{3/2} + o(n^{3/2})$ for $t \geq 1$.

1 Introduction

Let G and F be graphs. We say that G is *F-free* if G does not contain a subgraph that is isomorphic to F. The *Turán number of F* is the maximum number of edges in an F-free graph with n vertices. This maximum is denoted $\mathrm{ex}(n, F)$ and an F-free graph with n vertices and $\mathrm{ex}(n, F)$ edges is called an *extremal graph* for F. One of the most well-studied cases is when $F = C_4$, a cycle of length four. This problem was considered by Erdős [7] in 1938, and lies in the intersection of extremal graph theory, finite geometry via projective planes and difference sets, and combinatorial number theory via Sidon sets. Roughly 30 years later, Brown [3], and Erdős, Rényi, and Sós [8, 9] independently showed that $\mathrm{ex}(n, C_4) = \frac{1}{2} n^{3/2} + o(n^{3/2})$. They constructed, for each prime power q, a C_4-free graph with $q^2 + q + 1$ vertices and $\frac{1}{2} q(q + 1)^2$ edges. These graphs are examples of orthogonal polarity graphs which have since been studied and applied to other problems in combinatorics. Answering a question of Erdős, Füredi [11, 12] showed that for $q > 13$, orthogonal polarity graphs are the only extremal graphs for C_4 when the number of vertices is $q^2 + q + 1$. Füredi [13] also used finite fields to construct, for each $t \geq 1$, $K_{2,t+1}$-free graphs with n vertices and $\sqrt{\frac{t}{2}} n^{3/2} + o(n^{3/2})$ edges.

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This construction, together with the famous upper bound of Kövári, Sós, and Turán [17], shows that $\mathrm{ex}(n, K_{2,t+1}) = \sqrt{\frac{t}{2}}n^{3/2} + o(n^{3/2})$ for all $t \geq 1$.

Because of its importance in extremal graph theory, variations of the bipartite Turán problem have been considered. One such instance is to find the maximum number of edges in an F-free n by m bipartite graph. Write $\mathrm{ex}(n, m, F)$ for this maximum. Estimating $\mathrm{ex}(n, n, K_{s,t})$ is the "balanced" case of the Zarankiewicz problem. The results of [13, 17] show that $\mathrm{ex}(n, n, K_{2,t+1}) = \sqrt{t}n^{3/2} + o(n^{3/2})$ for $t \geq 1$. The case when F is a cycle of even length has also received considerable attention. Naor and Verstraëte [18] studied the case when $F = C_{2k}$. More precise estimates were obtained by Füredi, Naor, and Verstraëte when $F = C_6$ [14]. For more results along these lines, see [4, 5, 16] and the survey of Füredi and Simonovits [15] to name a few.

Now we introduce the extremal function that is the focus of this work. For an integer $k \geq 2$, define

$$\mathrm{ex}_{\chi \leq k}(n, F)$$

to be the maximum number of edges in an n-vertex graph G that is F-free and has chromatic number at most k. Thus, $\mathrm{ex}_{\chi \leq 2}(n, F)$ is the maximum number of edges in an F-free bipartite graph with n vertices (the part sizes need not be the same). Trivially,

$$\mathrm{ex}_{\chi \leq k}(n, F) \leq \mathrm{ex}(n, F)$$

for any k. In the case that $k = 2$,

$$\mathrm{ex}_{\chi \leq 2}(n, K_{2,t}) = \frac{\sqrt{t-1}}{2\sqrt{2}}n^{3/2} + o(n^{3/2})$$

by [13, 17]. Our focus will be on $\mathrm{ex}_{\chi \leq 3}(n, K_{2,t})$ and our first result gives an upper bound on $\mathrm{ex}_{\chi \leq 3}(n, K_{s,t})$.

Theorem 1.1 *For $n \geq 1$ and $2 \leq s \leq t$,*

$$\mathrm{ex}_{\chi \leq 3}(n, K_{s,t}) \leq \left(\frac{1}{3}\right)^{1-1/s} \left(\frac{t-1}{2} + o(1)\right)^{1/s} n^{2-1/s}.$$

When $s = 2$, Theorem 1.1 improves the trivial bound

$$\mathrm{ex}_{\chi \leq 3}(n, K_{2,t}) \leq \mathrm{ex}(n, K_{2,t}) = \frac{\sqrt{t-1}}{2}n^{3/2} + o(n^{3/2}).$$

Allen, Keevash, Sudakov, and Verstraëte [1] constructed 3-partite graphs with n vertices that are $K_{2,3}$-free and have $\frac{1}{\sqrt{3}}n^{3/2} - n$ edges. This construction shows that Theorem 1.1 is asymptotically best possible in the case that $s = 2$, $t = 3$. Our next result shows that this asymptotic formula holds for $K_{2,2t+1}$ for all $t \geq 1$.

Theorem 1.2 *For any integer $t \geq 1$,*

$$\mathrm{ex}_{\chi \leq 3}(n, K_{2,2t+1}) = \sqrt{\frac{t}{3}}n^{3/2} + o(n^{3/2}).$$

We believe that the most interesting remaining open case is determining the behavior when forbidding $K_{2,2} = C_4$.

Problem 1.3 *Determine the asymptotic behavior of*

$$\mathrm{ex}_{\chi \leq 3}(n, C_4).$$

In particular it would be very interesting to know whether or not $\mathrm{ex}_{\chi \leq 2}(n, C_4) \sim \mathrm{ex}_{\chi \leq 3}(n, C_4)$. We discuss this in more detail in Section 4. A random partition into k parts of an n-vertex C_4-free graph with $\frac{1}{2}n^{3/2} + o(n^{3/2})$ edges gives a lower bound of

$$\mathrm{ex}_{\chi \leq k}(n, C_4) \geq \left(1 - \frac{1}{k}\right)\frac{n^{3/2}}{2} - o(n^{3/2}).$$

When $k \geq 4$, this is a better lower bound than the one provided by

$$\frac{1}{2\sqrt{2}}n^{3/2} + o(n^{3/2}) \leq \mathrm{ex}_{\chi \leq 2}(n, C_4) \leq \mathrm{ex}_{\chi \leq k}(n, C_4)$$

which holds for all $k \geq 2$.

In the next section we prove Theorem 1.1 and in Section 3 we prove Theorem 1.2.

2 Proof of Theorem 1.1

In this section we prove Theorem 1.1. The proof is based on the standard double counting argument of Kövári, Sós, and Turán [17].

Proof of Theorem 1.1. Let G be an n-vertex 3-partite graph that is $K_{s,t}$-free. Let A_1, A_2, and A_3 be the parts of G, and define δ_i by

$$\delta_i n = |A_i|.$$

By the Kövári-Sós-Turán Theorem [17], there is a constant $\beta_{s,t} > 0$ such that the number of edges with one end point in A_1 and the other in A_2 is at most $\beta_{s,t}n^{2-1/s}$. If there are $o(n^{2-1/s})$ edges between A_1 and A_2, then we may remove these edges to obtain a bipartite graph G' that is $K_{s,t}$-free which gives

$$e(G) \leq e(G') - o(n^{2-1/s}) \leq \mathrm{ex}_{\chi \leq 2}(n, K_{s,t}).$$

In this case, we may apply the upper bound of Füredi [10] to see that the conclusion of Theorem 1.1 holds. Therefore, we may assume that there is a positive constant $c_{1,2}$ so that the number of edges between A_1 and A_2 is $c_{1,2}n^{2-1/s}$. Similarly, let $c_{1,3}n^{2-1/s}$ and $c_{2,3}n^{2-1/s}$ be the number of edges between A_1 and A_3, and between A_2 and A_3, respectively.

Using the assumption that G is $K_{s,t}$-free and convexity, we have

$$\begin{aligned}(t-1)\binom{|A_1|}{s} &\geq \sum_{v\in A_2}\binom{d_{A_1}(v)}{s}+\sum_{v\in A_3}\binom{d_{A_1}(v)}{s}\\ &\geq |A_2|\binom{\frac{1}{|A_2|}e(A_1,A_2)}{s}+|A_3|\binom{\frac{1}{|A_3|}e(A_1,A_3)}{s}\\ &\geq \frac{\delta_2 n}{s!}\left(\frac{e(A_1,A_2)}{|A_2|}-s\right)^s+\frac{\delta_3 n}{s!}\left(\frac{e(A_1,A_3)}{|A_3|}-s\right)^s.\end{aligned}$$

After some simplification we get

$$(t-1)\frac{(\delta_1 n)^s}{s!}\geq \frac{\delta_2 n}{s!}\left(\frac{c_{1,2}n^{2-1/s}}{\delta_2 n}-s\right)^s+\frac{\delta_3 n}{s!}\left(\frac{c_{1,3}n^{2-1/s}}{\delta_3 n}-s\right)^s.$$

For $j\in\{2,3\}$, we can assume that $\dfrac{c_{1,j}n^{2-1/s}}{\delta_j n}>s$ otherwise

$$e(A_1,A_j)=c_{1,j}n^{2-1/s}\leq s\delta_j n\leq sn=o(n^{2-1/s}).$$

From the inequality $(1+x)^s\geq 1+sx$ for $x\geq -1$, we now have

$$\begin{aligned}(t-1)\delta_1^s n^s &\geq \delta_2 n\left(\frac{c_{1,2}n^{2-1/s}}{\delta_2 n}\right)^s-\delta_2 n s^2\left(\frac{c_{1,2}n^{2-1/s}}{\delta_2 n}\right)^{s-1}\\ &+ \delta_3 n\left(\frac{c_{1,3}n^{2-1/s}}{\delta_3 n}\right)^s-\delta_3 n s^2\left(\frac{c_{1,3}n^{2-1/s}}{\delta_3 n}\right)^{s-1}.\end{aligned}$$

Dividing through by n^s and rearranging gives

$$(t-1)\delta_1^s\geq \frac{c_{1,2}^s}{\delta_2^{s-1}}+\frac{c_{1,3}^s}{\delta_3^{s-1}}-\frac{s^2c_{1,2}^{s-1}}{\delta_2^{s-2}n^{1-1/s}}-\frac{s^2c_{1,3}^{s-1}}{\delta_3^{s-2}n^{1-1/s}}.$$

Multiplying through by $\delta_2^{s-1}\delta_3^{s-1}$ leads to

$$(t-1)\delta_1^s\delta_2^{s-1}\delta_3^{s-1}\geq c_{1,2}^s\delta_3^{s-1}+c_{1,3}^s\delta_2^{s-1}-\frac{s^2\delta_3^{s-1}\delta_2c_{1,2}^{s-1}}{n^{1-1/s}}-\frac{s^2\delta_2^{s-1}\delta_3c_{1,3}^{s-1}}{n^{1-1/s}}.$$

Since δ_2 and δ_3 are both at most 1 and $c_{1,j}$ is at most $\beta_{s,t}$,

$$(t-1)\delta_1^s\delta_2^{s-1}\delta_3^{s-1}\geq c_{1,2}^s\delta_3^{s-1}+c_{1,3}^s\delta_2^{s-1}-\frac{2s^2\beta_{s,t}^{s-1}}{n^{1-1/s}}.$$

By symmetry between the parts A_1, A_2, and A_3,

$$(t-1)\delta_2^s\delta_1^{s-1}\delta_3^{s-1}\geq c_{1,2}^s\delta_3^{s-1}+c_{2,3}^s\delta_1^{s-1}-\frac{2s^2\beta_{s,t}^{s-1}}{n^{1-1/s}}$$

and

$$(t-1)\delta_3^s\delta_1^{s-1}\delta_2^{s-1}\geq c_{1,3}^s\delta_2^{s-1}+c_{2,3}^s\delta_1^{s-1}-\frac{2s^2\beta_{s,t}^{s-1}}{n^{1-1/s}}.$$

Add these three inequalities together and divide by 2 to obtain

$$\frac{t-1}{2}\delta_1^{s-1}\delta_2^{s-1}\delta_3^{s-1}(\delta_1+\delta_2+\delta_3) \geq c_{1,2}^s\delta_3^{s-1} + c_{1,3}^s\delta_2^{s-1} + c_{2,3}^s\delta_1^{s-1} - \frac{3s^2\beta_{s,t}^{s-1}}{n^{1-1/s}}.$$

Now $n = |A_1| + |A_2| + |A_3| = (\delta_1 + \delta_2 + \delta_3)n$ so we may replace $\delta_1 + \delta_2 + \delta_3$ with 1. This leads us to the optimization problem of maximizing

$$c_{1,2} + c_{1,3} + c_{2,3}$$

subject to the constraints

$$0 \leq \delta_i, \qquad 0 \leq c_{i,j} \leq 1, \qquad \delta_1 + \delta_2 + \delta_3 = 1,$$

and

$$\frac{t-1}{2}\delta_1^{s-1}\delta_2^{s-1}\delta_3^{s-1} \geq \delta_3^{s-1}c_{1,2}^s + \delta_2^{s-1}c_{1,3}^s + \delta_1^{s-1}c_{2,3}^s.$$

This can be done using the method of Lagrange Multipliers (see the Appendix) and gives

$$c_{1,2} + c_{1,3} + c_{2,3} \leq \left(\frac{1}{3}\right)^{1-1/s}\left(\frac{t-1}{2}\right)^{1/s}.$$

We conclude that the number of edges of G is at most

$$\left(\frac{1}{3}\right)^{1-1/s}\left(\frac{t-1}{2}\right)^{1/s} n^{2-1/s} + o(n^{2-1/s}).$$

■

3 Proof of Theorem 1.2

In this section we construct a 3-partite $K_{2,2t+1}$-free graph with many edges. The construction is inspired by Füredi's construction of dense $K_{2,t}$-free graphs [13].

Let $t \geq 1$ be an integer. Let q be a power of a prime chosen so that t divides $q-1$ and let θ be a generator of the multiplicative group $\mathbb{F}_{q^2}^* := \mathbb{F}_{q^2}\backslash\{0\}$. Let $A \subset \mathbb{Z}_{q^2-1}$ be a Bose-Chowla Sidon set [2]. That is,

$$A = \{a \in \mathbb{Z}_{q^2-1} : \theta^a - \theta \in \mathbb{F}_q\}$$

and note that $|A| = q$. Let H be the subgroup of $\mathbb{Z}_{q^2-1}$ generated by $(\frac{q-1}{t})(q+1)$. Thus,

$$H = \left\{0, \left(\frac{q-1}{t}\right)(q+1), 2\left(\frac{q-1}{t}\right)(q+1), \dots, (t-1)\left(\frac{q-1}{t}\right)(q+1)\right\}$$

and furthermore, H is contained in the subgroup of $\mathbb{Z}_{q^2-1}$ generated by $q+1$. Let $G_{q,t}$ be the bipartite graph whose parts are X and Y where each of X and Y is a disjoint copy of the quotient group $\mathbb{Z}_{q^2-1}/H$. A vertex $x + H \in X$ is adjacent to $x + a + H \in Y$ for all $a \in A$.

We will need the following lemma, which was proved in [20].

Lemma 3.1 *[Lemma 2.2 of [20]] Let $A \subset \mathbb{Z}_{q^2-1}$ be a Bose-Chowla Sidon set. Then*

$$A - A = \mathbb{Z}_{q^2-1} \setminus \{q+1, 2(q+1), 3(q+1), \ldots, (q-2)(q+1)\}.$$

In particular, Lemma 3.1 implies that $(A - A) \cap H = \emptyset$.

Lemma 3.2 *If $t \geq 1$ is an integer and q is a power of a prime for which t divides $q-1$, then the graph $G_{q,t}$ is a bipartite graph with $\frac{q^2-1}{t}$ vertices in each part, is $K_{2,t+1}$-free, and has $q\left(\frac{q^2-1}{t}\right)$ edges.*

Proof. It is clear that $G_{q,t}$ is bipartite and has $\frac{q^2-1}{t}$ vertices in each part. Let $x + H$ be a vertex in X. The neighbors of $x + H$ are of the form $x + a + H$ where $a \in A$. We show that these vertices are all distinct. If $x + a + H = x + b + H$ for some $a, b \in H$, then $a - b \in H$. By Lemma 3.1

$$(A - A) \cap H = \{0\}$$

where $A - A = \{a - b : a, b \in A\}$. We conclude that $a = b$ and so the degree of $x + H$ is $|A| = q$. This also implies that $G_{q,t}$ has $q\left(\frac{q^2-1}{t}\right)$ edges and to finish the proof, we must show that $G_{q,t}$ has no $K_{2,t+1}$.

We consider two cases depending on which part contains the part of size two of the $K_{2,t+1}$. First suppose that $x + H$ and $y + H$ are distinct vertices in X and let $z + H$ be a common neighbor in Y. Then $z + H = x + a + H$ and $z + H = y + b + H$ for some $a, b \in A$. Therefore, $z = x + a + h_1$ and $z = y + b + h_2$ for some $h_1, h_2 \in H$. From this pair of equations we get $a - b = y - x + h_2 - h_1$. Since H is a subgroup, $h_2 - h_1 = h_3$ for some $h_3 \in H$ and now we have

$$a - b = y - x + h_3. \tag{1}$$

The right hand side of (1) is not zero since $x + H$ and $y + H$ are distinct vertices in A. As A is a Sidon set and $y - x + h_3 \neq 0$, there is at most one ordered pair $(a, b) \in A^2$ for which $a - b = y - x + h_3$. There are t possibilities for h_3 and so t possible ordered pairs $(a, b) \in A^2$ for which

$$z + H = x + a + H = y + b + H$$

is a common neighbor of $x + H$ and $y + H$. This shows that $x + H$ and $y + H$ have at most t common neighbors.

Now suppose $x+H$ and $y+H$ are distinct vertices in Y and $z+H$ is a common neighbor in X. There are elements $a, b \in A$ such that $z+a+H = x+H$ and $z+b+H = y+H$. Thus, $z+a+h_1 = x$ and $z+b+h_2 = y$ for some $h_1, h_2 \in H$. Therefore, $x-a-h_1 = y-b-h_2$ so $a-b = x-y+h_2-h_1$. We can then argue as before that there are at most t ordered pairs $(a, b) \in A^2$ such that $z + H$ is a common neighbor of $z + a + H = x + H$ and $z + b + H = y + H$. ■

Once again, let $t \geq 1$ be an integer and q be a power of a prime for which t divides $q - 1$. Let $\Gamma_{q,t}$ be the 3-partite graph with parts X, Y, and Z where each part is a copy of the quotient group $\mathbb{Z}_{q^2-1}/H$. Here H is the subgroup generated by $(\frac{q-1}{t})(q+1)$. A vertex $x + H \in X$ is adjacent to $x + a + H \in Y$ for all $a \in A$. Similarly, a vertex $y + H \in Y$ is adjacent to $y + a + H \in Z$ for all $a \in A$, and a vertex $z + H \in Z$ is adjacent to $z + a + H \in X$ for all $a \in A$.

Lemma 3.3 *The graph $\Gamma_{q,t}$ is $K_{2,2t+1}$-free.*

Proof. By Lemma 3.2, a pair of vertices in one part of $\Gamma_{q,t}$ have at most t common neighbors in each of the other two parts. Thus, there cannot be a $K_{2,2t+1}$ in $\Gamma_{q,t}$ where the part of size two is contained in one part.

Now let $x+H$ and $y+H$ be vertices in two different parts. Without loss of generality, assume $x+H \in X$ and $y+H \in Y$. Suppose $z+H \in Z$ is a common neighbor of $x+H$ and $y+H$. There are elements $a, b \in A$ such that $z+H = y+a+H$ and $z+b+H = x+H$ so we have

$$z = y + a + h_1 \quad \text{and} \quad z + b = x + h_2$$

for some $h_1, h_2 \in H$. This pair of equations implies

$$a + b = x - y + h_2 - h_1$$

and since H is a subgroup, $h_2 - h_1 \in H$. Let $h_2 - h_1 = h_3$ where $h_3 \in H$ so

$$a + b = x - y + h_3.$$

There are t possibilities for h_3. Given h_3, the equation $a + b = x - y + h_3$ uniquely determines the pair $\{a, b\}$ since A is a Sidon set. There are two ways to order a and b and so $x+H$ and $y+H$ have at most $2t$ common neighbors in Z. ∎

Proof of Theorem 1.2. By Theorem 1.1,

$$\mathrm{ex}_{\chi \leq 3}(n, K_{2,2t+1}) = \sqrt{\frac{1}{3}} \left(\frac{2t+1-1}{2} + o(1) \right)^{1/2} n^{3/2} = \sqrt{\frac{t}{3}} n^{3/2} + o(n^{3/2}).$$

As for the lower bound, if q is any power of a prime for which t divides $q-1$, then by Lemmas 3.2 and 3.3, the graph $\Gamma_{q,t}$ is a 3-partite graph with $\frac{q^2-1}{t}$ vertices in each part, is $K_{2,2t+1}$-free, and has $3q\left(\frac{q^2-1}{t}\right)$ edges. Thus,

$$\mathrm{ex}_{\chi \leq 3}\left(\frac{3(q^2-1)}{t}, K_{2,2t+1} \right) \geq 3q \left(\frac{q^2-1}{t} \right).$$

If $n = \frac{3(q^2-1)}{t}$, then the above can be rewritten as

$$\mathrm{ex}_{\chi \leq 3}(n, K_{2,2t+1}) \geq n \left(\sqrt{\frac{nt}{3} + 1} \right) \geq \sqrt{\frac{t}{3}} n^{3/2} - n.$$

A standard density of primes argument finishes the proof. ∎

4 Concluding Remarks

We may consider a similar graph to $G_{q,t}$ and $\Gamma_{q,t}$ which does not necessarily have bounded chromatic number. Let Γ be a finite abelian group with a subgroup H of order t. Let $A \subset \Gamma$ be a Sidon set such that $(A - A) \cap H = \{0\}$. Then we may construct a graph G with vertex set Γ/H where $x + H \sim y + H$ if and only if $x + y = a + h$ for some $a \in A$ and $h \in H$. Then the proof of Lemma 3.2 shows that G is a $K_{2,t+1}$-free graph on $|\Gamma|/|H|$ vertices and every vertex has degree $|A|$ or $|A| - 1$.

When $\Gamma = \mathbb{Z}_{q^2-1}$, t divides $q - 1$, and A is a Bose-Chowla Sidon set, the resulting graph G is similar to the one constructed by Füredi in [13]. When $t = 1$ the main result of [19] shows that these two graphs are isomorphic. However, in general these graphs are not isomorphic. When $q = 19$ and $t \in \{3, 6\}$ the graph constructed above has one more edge than the graph constructed by Füredi.

Turning to the question of determining $\mathrm{ex}_{\chi\leq 3}(n, C_4)$, Theorem 1.1 shows that

$$\mathrm{ex}_{\chi\leq 3}(n, C_4) \lesssim \frac{n^{3/2}}{\sqrt{6}}.$$

Furthermore, the optimization shows that if this bound is tight asymptotically, then a construction would have to be 3-partite with each part of size asymptotic to $\frac{n}{3}$ and average degree asymptotic to $\sqrt{\frac{n}{6}}$ between each part. The following construction is due to Jason Williford [21].

Theorem 4.1 *Let R be a finite ring, $A \subset R$ an additive Sidon set and $B = cA = \{ca : a \in A\}$. Then if $(A - A) \cap (B - B) = \{0\}$ where c is invertible, there is a graph on $3|R|$ vertices which is 3-partite, C_4-free and is $|A|$-regular between parts.*

Proof. We construct a graph with partite sets S_1, S_2, S_3 where $S_1 = R$, $S_2 = \{A + i\}_{i\in R}$ and $S_3 = \{B + j\}_{j\in R}$. A vertex in S_1 is adjacent to a vertex in S_2 or S_3 by inclusion. The vertex $A + j \in S_2$ is adjacent to $B + i \in S_3$ if $-cj + i \in A$. Since c is invertible, we have that both A and B are Sidon sets. Therefore, the bipartite graphs between S_1 and S_2, and between S_1 and S_3 are incidence graphs of partial linear spaces, and thus do not contain C_4.

If there were a C_4 with $A + i, A + j \in S_2$ and $B + k, B + l \in S_3$, it implies that there exist $a_1, a_2, a_3, a_4 \in A$ such that

$$\begin{aligned} -ci + k &= a_1 \\ -ci + l &= a_2 \\ -cj + k &= a_3 \\ -cj + l &= a_4. \end{aligned}$$

This means that $k - l = a_1 - a_2 = a_3 - a_4$. Since A is a Sidon set this means that $a_1 = a_2$ or $a_1 = a_3$, which implies that either $k = l$ or $i = j$.

If there were a C_4 with $i \in S_1$, $A+j, A+k \in S_2$ and $B+l \in S_3$, it means that there are $a_1, a_2, a_3, a_4 \in A$ such that

$$\begin{aligned} i &= a_1 + j \\ i &= a_2 + k \\ -cj + l &= a_3 \\ -ck + l &= a_4. \end{aligned}$$

This means that $c(j-k) = c(a_2-a_1) = a_4-a_3$. Since $B = cA$ we have that $b_2-b_1 = a_4-a_3$ for some $b_1, b_2 \in B$, and therefore $b_2 - b_1 = a_4 - a_3 = 0$. This implies that $j = k$. The case when there are two vertices in S_3 and one each in S_1 and S_2 is similar. ■

The condition that $(A - A) \cap (B - B) = \{0\}$ and A is a Sidon set implies that $2|A|(|A| - 1) \leq |R| - 1$. In $\mathbb{Z}_5$, if $A = \{0, 1\}$ and $B = 2A = \{0, 2\}$, we have $(A - A) \cap (B - B) = \{0\}$ and $(A - A) \cup (B - B) = \mathbb{Z}_5$. This gives a 3-partite graph on 15 vertices which is C_4-free and is 4-regular. In $\mathbb{Z}_{41}$, the set $A = \{1, 10, 16, 18, 37\}$ and $B = 9A$ have the same property that $(A - A) \cap (B - B) = \{0\}$ and $(A - A) \cup (B - B) = \mathbb{Z}_{41}$. This gives a 3-partite C_4-free graph on 123 vertices which is 10 regular.

In general, a (v, k, λ)-difference family in a group Γ of order v is a collection of sets $\{D_1, \dots, D_t\}$ each of size k such that the multiset

$$(D_1 - D_1) \cup \cdots \cup (D_t - D_t)$$

contains every nonzero element of Γ exactly λ times. If one could find an infinite family of $(2k^2-2k+1, k, 1)$-difference families in $\mathbb{Z}_{2k^2-2k+1}$ where the two blocks are multiplicative translates of each other by a unit, then the resulting graph would match the upper bound in Theorem 1.1. The sets $A = \{0, 1\}$ and $2A$ in $\mathbb{Z}_5$, and $A = \{1, 10, 16, 18, 37\}$ and $9A$ in $\mathbb{Z}_{41}$ are examples of this for $k = 2$ and $k = 5$ respectively. We could not figure out how to extend this construction in general, and in [6] it is shown that no $(61, 6, 1)$-difference family exists in $\mathbb{F}_{61}$.

To show Theorem 1.1 is tight asymptotically it would suffice to find something weaker than a $(2k^2-2k+1, k, 1)$-difference family where the two blocks are multiplicative translates of each other. We do not need every nonzero element of the group to be represented as a difference of two elements, just a proportion of them tending to 1.

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6 Appendix

Here we solve the optimization problem of Theorem 1.1 using the method of Lagrange Multipliers. For convenience, we write x for $c_{1,2}$, y for $c_{1,3}$, and z for $c_{2,3}$. Recall that δ_1, δ_2, and δ_3 are positive real numbers that satisfy $\delta_1 + \delta_2 + \delta_3 = 1$. Let

$$f(x, y, z) = x + y + z$$

and

$$g(x, y, z) = \frac{t-1}{2}\delta_1^{s-1}\delta_2^{s-1}\delta_3^{s-1} - \delta_3^{s-1}x^s - \delta_2^{s-1}y^s - \delta_1^{s-1}z^s.$$

For a parameter λ, let $L(x, y, z, \lambda) = f(x, y, z) + \lambda g(x, y, z)$. Taking partial derivatives, we get

$$L_x = 1 - s\lambda\delta_3^{s-1}x^{s-1} = 0, \tag{2}$$

$$L_y = 1 - s\lambda\delta_2^{s-1}y^{s-1} = 0, \tag{3}$$

$$L_z = 1 - s\lambda\delta_1^{s-1}z^{s-1} = 0, \tag{4}$$

$$\lambda\left(\frac{t-1}{2}\delta_1^{s-1}\delta_2^{s-1}\delta_3^{s-1} - \delta_3^{s-1}x^s - \delta_2^{s-1}y^s - \delta_1^{s-1}z^s\right) = 0. \tag{5}$$

Note that $\lambda \neq 0$ otherwise we contradict (2) so by (5),

$$\frac{t-1}{2}\delta_1^{s-1}\delta_2^{s-1}\delta_2^{s-1} = \delta_3^{s-1}x^s + \delta_2^{s-1}y^s + \delta_1^{s-1}z^s. \tag{6}$$

From (2), (3), and (4) we have

$$\left(\frac{1}{2\lambda}\right)^{\frac{1}{s-1}} = \delta_3 x = \delta_2 y = \delta_1 z. \tag{7}$$

Combining this with (6) and using $\delta_3 = 1 - \delta_1 - \delta_2$, we get an equation that can be solved for x to obtain

$$x = \left(\frac{(t-1)\delta_1^s\delta_2^s}{2(\delta_1(1-\delta_1) + \delta_2(1-\delta_2) - \delta_1\delta_2)}\right)^{1/s}.$$

Using (7), we can then solve for y and z and get

$$x + y + z = \frac{(t-1)^{1/s}}{2^{1/s}}\left(\delta_1(1-\delta_1) + \delta_2(1-\delta_2) - \delta_1\delta_2\right)^{1-1/s}.$$

The maximum value of

$$\delta_1(1-\delta_1)+\delta_2(1-\delta_2)-\delta_1\delta_2$$

over all $\delta_1, \delta_2 \geq 0$ for which $0 \leq \delta_1+\delta_2 \leq 1$ is $\frac{1}{3}$ and it is obtained only when $\delta_1 = \delta_2 = \frac{1}{3}$. Therefore,

$$x+y+z \leq \frac{(t-1)^{1/s}}{2^{1/s}}\left(\frac{1}{3}\right)^{1-1/s}.$$