Name: ______________________________________

Instructions: You have 50 minutes to complete this exam. Show your work and justify all of your responses. No calculators, notes, or other external aids are allowed.

1. (10 points) Let d be a fixed natural number. Give the best upper and lower bounds you can for $\text{ex}(n, K_{1,d+1})$.

Solution: Note that if $n \le d+1$ then a complete graph is $K_{1,d+1}$ free and so $\text{ex}(n, K_{1,d+1}) = \binom{n}{2}$, so from now on we will assume $n > d+1$. Let G be a graph which is $K_{1,d+1}$ free. This means that the maximum degree of G satisfies $\Delta \le d$ (otherwise, a vertex with $d+1$ neighbors is a $K_{1,d+1}$).

Then we have

$$2e(G) = \sum_{v \in V(G)} d(v) \le \sum \Delta \le \sum d = nd.$$

This shows that

$$\text{ex}(n, K_{1,d+1}) \le \frac{nd}{2}.$$

Note that if n and d are both odd we must have $\text{ex}(n, K_{1,d+1}) \le \lfloor \frac{nd}{2} \rfloor$ since the Turán number is an integer.

The disjoint union of complete graphs on $d+1$ vertices is $K_{1,d+1}$ free, so we have

$$\text{ex}(n, K_{1,d+1}) \ge \left\lfloor \frac{n}{d+1} \right\rfloor \binom{d+1}{2}.$$

This matches the upper bound if $(d+1)|n$. If not, we must be a bit more careful to get an exact answer. A construction matching the upper bound must satisfy either

- Every vertex has degree d if n or d is even, or
- $n-1$ vertices have degree d and one vertex has degree $d-1$ if both n and d are odd.

If d is even arrange the vertices of G on a circle and make a vertex adjacent to the $d/2$ vertices on its left and right.

If d is odd and n is even, arrange $V(G)$ on a circle and make a vertex adjacent to the $\lfloor d/2 \rfloor$ vertices on its left and right as well as the vertex antipodal to it (there is such a vertex because n is even).

Now assume that d and n are both odd. Let H be the graph on $d+2$ vertices given by $(d-1)/2$ disjoint edges and a disjoint $K_{1,2}$. Let H' be the graph $K_{d+2} \setminus E(H)$. Then H' satisfies the property that all but one of its vertices have degree d and the remaining vertex has degree $d-1$. For n odd at least than $2d+4$, we may take a copy of H' and for the remaining (even number of) vertices, do the above construction in the even number of vertices case. It is left as an exercise to do the case analysis for n between $d+2$ and $2d+4$.

Solution: (Alternate [better] lower bound): We will construct a graph with $\lfloor \frac{nd}{2} \rfloor$ edges with maximum degree d. We will add edges subject to the condition that the maximum degree is at most d. Add an edge between the vertex of smallest degree and that of second smallest degree (break ties arbitrarily). At any point in the process, the minimum degree will be at most 1 smaller than the second smallest degree. If the second smallest degree is strictly smaller than d, then there is still an edge we may add. Therefore, when the process terminates there is at most one vertex of degree strictly less than d, and this vertex has degree $d-1$. This is best possible if n and d are both odd by the solution above. If either n or d is even, then by the handshaking lemma we may not have $n-1$ vertices of degree d and one of degree $d-1$, and therefore we obtain a d-regular graph.

2. (10 points) Let G be a connected graph, regular of degree d. Show that the eigenvalue d has multiplicity 1. (Hint: Use the eigenvalue-eigenvector equation and look at the vertex with largest eigenvector entry).

Solution: Let A be the adjacency matrix of G. Since G is d-regular, every row of A has exactly d 1s in it, and so

$$A \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = d \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

So d is an eigenvalue. Next we show that any eigenvector for d is a constant vector, and so is a multiple of the above eigenvector, showing that d is an eigenvalue with multiplicity 1. Let $A\mathbf{x} = d\mathbf{x}$ and let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Assume that $\mathbf{x}$ is normalized so that the maximum entry is equal to 1 and let z be a vertex with $x_z = 1$. The eigenvector-eigenvalue equation gives that

$$d = d \cdot x_z = \sum_{i \sim z} x_i \leq \sum_{i \sim z} x_z = d(z) = d.$$

Therefore we have equality in the inequality, which implies that for all $i \sim z$, we have $x_i = x_z = 1$. Repeating the argument shows that all of the vertices that are neighbors of neighbors of z must have eigenvector entry 1 as well. If we continually apply this argument, we will eventually see every vertex in the graph since G is connected. Therefore, once normalized, every eigenvector entry is 1.

3. (10 points) A group of n adults check their hats and jackets at a restaurant. How many ways are there to distribute back the hats and jackets such that no person gets back their original outfit (that is, they can receive either their correct hat OR their correct jacket OR neither, but they cannot receive their correct hat and jacket)? Your answer may contain a sum.

Solution: Let S_n be the set of permutations on n elements. We represent the ways that the adults can receive their hats and jackets back by $S_n \times S_n$ (for $(\sigma, \tau) \in S_n \times S_n$, person i gets back person $\sigma(i)$'s hat and person $\tau(i)$'s jacket). Let $X = S_n \times S_n$ and let

$$A_i := \{(\sigma, \tau) \in X : \sigma(i) = \tau(i) = i\},$$

for $1 \leq i \leq n$. Then the quantity we wish to count is exactly

$$\left| X \setminus \left(\bigcup_{i=1}^{n} A_i \right) \right|.$$

We do this by inclusion-exclusion:

$$\left| X \setminus \left(\bigcup_{i=1}^{n} A_i \right) \right| = \sum_{S \subset [n]} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right|.$$

Given a fixed $S \subset [n]$, we have $\bigcap_{i \in S} A_i$ is exactly the set of pairs of permutations whose points are fixed on S, and therefore this set has size $((n-|S|)!)^2$. Since this value depends only on the size of S and not on the set itself, we have

$$\left| X \setminus \left(\bigcup_{i=1}^{n} A_i \right) \right| = \sum_{k=0}^{n} (-1)^k \binom{n}{k} ((n-k)!)^2.$$

4. (10 points) A graph is called triangle saturated if it does not contain any triangles but changing any non-edge to an edge creates a triangle. The extremal number $\mathrm{ex}(n, K_3)$ is the maximum number of edges in a triangle saturated graph on n vertices. In this problem we will be interested in finding the *minimum* number of edges in a triangle saturated graph. This quantity is denoted

$$\mathrm{sat}(n, K_3).$$

Give an exact formula for $\mathrm{sat}(n, K_3)$ (ie, make a construction giving an upper bound, and then prove that any triangle saturated graph on n vertices must have at least that many edges).

Solution: Note that the star $K_{1,n-1}$ is triangle saturated. This gives

$$\mathrm{sat}(n, K_3) \le n - 1.$$

To show the lower bound, we claim that a disconnected graph cannot be triangle saturated. To prove this let G be a disconnected graph, and let $u, v \in V(G)$ with no path connecting them. In particular, there is no path with 2 edges connecting them and uv is not already an edge. Then the addition of the edge uv to G cannot create a triangle (because there was no path of length 2 connecting u and v). Therefore, for any triangle saturated graph G, G must be connected, and therefore it must have at least $n - 1$ edges, showing $\mathrm{sat}(n, K_3) \ge n - 1$.