Math 301: Homework 8

Due Wednesday November 8 at noon on Canvas

1. In this problem, we will "smash together" the two partite sets of the incidence graph of a projective plane and give an asymptotic formula for $\mathrm{ex}(n, C_4)$. Let V be a 3-dimensional vector space over a finite field $\mathbb{F}_q$. We define a graph G_π^o where $V(G_\pi^o)$ is the set of one-dimensional subspaces of V. There are q^2+q+1 of these (to see this, think of a vector in V having 3 coordinates, and then for each subspace it is defined by a vector which you can normalize so that the first non-zero coordinate is 1). Two vertices are adjacent if and only if the subspaces are orthogonal to each other.

   (a) Show that each vertex has degree $q+1$ (Hint: V is a 3-dimensional vector space. Given a fixed 1-dimensional subspace, the set of vectors orthogonal to it is 2-dimensional. How many 1-dimensional subspaces are in a 2-dimensional vector space over $\mathbb{F}_q$?)

   (b) Show that every pair of vertices has exactly one path of length 2 between them (Hint: this is *much* easier to do geometrically than algebraically).

   (c) Show that there are loops in the graph (you may allow q to be of any form that is convenient for you).

   (d) It is known that there are $q+1$ loops in this graph. Let G_π be the graph with the loops removed. Then G_π is a graph on q^2+q+1 vertices with q^2 vertices of degree $q+1$ and $q+1$ vertices of degree q.

   (e) Use part (b) and (d) to count the number of triangles in G_π.

   (f) It is known that for any $\epsilon > 0$, there is an M such that for $m \geq M$, there is a prime number in the interval $[m, (1+\epsilon)m]$. Use this to show that $\mathrm{ex}(n, C_4) \sim \frac{1}{2}n^{3/2}$.

2. The *multicolor Ramsey number* of a graph H, denoted $r_k(H)$ is the minimum n such that any k-coloring of the edges of K_n contains a monochromatic copy of H. We think of k as going to infinity. Assume that $\mathrm{ex}(n, H) = \Theta(n^\alpha)$ for some $1 < \alpha \leq 2$.

   (a) Use the pigeonhole principle to show that $r_k(H) = O(k^{1/(2-\alpha)})$.

   (b) Use the probabilistic method to show that $r_k(H) = \Omega\left(k^{1/(2-\alpha)}/\mathrm{polylog}(k)\right)$.

3. (**) Let G be a graph. A hypergraph H is said to be *Berge-G*, if there is a bijection $\phi : E(G) \to E(H)$ such that for each edge $e \in E(G)$, $e \subset \phi(e)$. We say that a hypergraph is Berge-G free if it does not contain any subhypergraph which is Berge-G.

We denote by $\mathrm{ex}_r(n, \mathrm{Berge}{-}G)$ the maximum number of edges in an n-vertex r-uniform hypergraph which is Berge-G free. It is known that $\mathrm{ex}_r(n, \mathrm{Berge} - C_4) = O(n^{3/2})$.

(a) (***) Show that $\mathrm{ex}_3(n, \mathrm{Berge} - C_4) = \Omega(n^{3/2})$. What can you say for general r?

(b) (****) For a family of hypergraphs $\mathcal{F}$, define the multicolor Ramsey number $r_k^{(3)}(\mathcal{F})$ to be the minimum n such that for any k coloring of the edge set of the complete 3-uniform hypergraph, there is a monochromatic copy of some graph in $\mathcal{F}$. Show that $r_k^{(3)}(\mathrm{Berge} - C_4) = \Theta(k^{2/3})$. (The upper bound is the same as in problem 2, using the Turán number and the pigeonhole principle. For the lower bound, it is equivalent to showing that the complete 3-uniform hypergraph on n vertices can be edge-partitioned into $O(n^{3/2})$ subgraphs, each of which has no Berge-C_4).