Math 301: Homework 7

Due Wednesday November 1 at noon on Canvas

1. Show that any graph on n vertices that has at least nd edges contains a subgraph of minimum degree $d+1$.

   Solution: Let G have n vertices and nd edges. If G has minimum degree at least $d+1$ we are done. Otherwise, let v be a vertex of minimum degree at most d and let $G_1 = G \setminus \{v\}$. Continue this process recursively: if G_i has minimum degree at least $d+1$ we are done, otherwise there is a vertex v with degree in G_i at most d and let $G_{i+1} = G_i \setminus \{v\}$. We claim that this process stops, and then you are left with your graph of minimum degree i. If not, then we remove vertices until we are left with just one vertex remaining and no edges. In this case we have removed at most $(n-1)d$ edges. But the original graph contained nd edges, a contradiction.

2. (a) Modify the proof of the upper bound for $\text{ex}(n, K_{2,t})$ that we did in class to prove the Kővari-Sós-Turán Theorem. For $2 \le s \le t$ there exists a constant c such that $\text{ex}(n, K_{s,t}) \le cn^{2-1/s}$.

   (b) Give the best lower bound you can for $\text{ex}(n, K_{s,t})$.

   Solution: See bottom.

3. (a) Let G be a graph where $V(G)$ consists of n points in the Euclidean plane and two points are adjacent if and only if they are at distance 1 from each other. Show that no matter how the points are placed, the number of edges in the graph is $O(n^{3/2})$.

   Solution: In a unit distance graph, the vertices adjacent to a fixed vertex v must be on the circle with radius 1 which is centered at v. Therefore, in order for a vertex to be adjacent to both v and u, it must be on the intersection two unit circles, one centered at u and one at v. Since two circles can intersect in at most two points, u and v may have at most 2 neighbors in common. Since u and v are arbitrary, this means that any unit distance graph is $K_{2,3}$ free. By the KST theorem it contains at most $O(n^{3/2})$ edges.

   (b) Make a construction of n points in the plane that has as many pairs at unit distance as you can. How many edges are in the graph?

   Solution: Here is an example of a unit distance graph with $n \log n$ edges. This is not best possible (no one knows what is best possible). Let $n = 2^k$ and write the vertices of a graph on n vertices as binary vectors of length k. Let $\mathbf{v}_1, \cdots, \mathbf{v}_k$

be k distinct unit vectors (it doesn't matter what they are). Given a vertex $(x_1, \cdots, x_k) \in \{0,1\}^k$, place the vertex in the Euclidean plane at point

$$\sum_{i=1}^{k} x_i \mathbf{v}_i.$$

Then any pair of vertices with Hamming distance 1 will be placed at unit distance in the Euclidean plane. This gives a graph on 2^k vertices with minimum degree k.

4. Let T be a tree on $t+1$ vertices.

   (a) Assume that n is divisible by t. Show that $\mathrm{ex}(n, T) \geq \frac{t-1}{2}n$. (Hint: K_t cannot contain a copy of T).

   Solution: If n is divisible by t we may place down $\frac{n}{t}$ disjoint copies of K_t. Since T has $t+1$ vertices, this graph does not contain a copy of T, and it has $\frac{t-1}{2}n$ edges.

   (b) Use Problem 1 to show that $\mathrm{ex}(n, T) \leq (t-1)n$.

   Solution: Let G be a graph on n vertices with $(t-1)n$ edges. We will show that G contains a copy of T. By problem 1, G has a subgraph G' which has minimum degree t. We embed T greedily in G' as a breadth first search. Since T has only $t+1$ vertices, and G' has minimum degree t, we may always choose a vertex which has not yet been used in continue on our breadth first search.

5. Let k be fixed. Show that there is a constant c so that $\mathrm{ex}(n, \{C_3, C_4, \cdots, C_{2k}\}) \leq cn^{1+1/k}$. (Hint: you need to show that if G has more than $cn^{1+1/k}$ edges then it must contain a cycle of length at most $2k$. Assume G has this many edges and use Problem 1 to start with a graph of minimum degree $c'n^{1/k}$. Do a breadth first search and show that you must find your cycle).

   Solution: Let $c = 2$ and by way of contradiction assume that there is a graph on n vertices with $2n^{1+1/k}$ edges and no cycle of length at most $2k$. By problem 1, there is a subgraph with minimum degree at least $n^{1/k} + 1$. Do a breadth first search starting from an arbitrary vertex. Once the search is k levels from the root, the graph induced by those vertices must be a tree, otherwise there is a cycle of length at most $2k$ in the graph. But a tree with $k+1$ levels (including the root) and minimum degree $n^{1/k} + 1$ must contain more than n vertices, a contradiction.

Proof. Fix some positive natural numbers n, s, and t with $s \le t$, and choose an arbitrary edge-maximal $K_{s,t}$-free graph G on n vertices and m edges. We first claim that $\delta(G) \ge s-1$. If this were not the case, then there would be some vertex v of degree at most $s-2$ in G. Form a graph G' by adding an edge between v and an arbitrary other vertex.[2] Then we cannot have added any $K_{s,t}$ to this graph: clearly any new $K_{s,t}$ would need to use this new edge, so v would need to be in this $K_{s,t}$, but

$$\deg_{G'}(v) = \deg_G(v) + 1 \le s-1 < s \le t;$$

if v were part of a $K_{s,t}$ then its degree would need to be either s or t, and this is not possible. Therefore, any $K_{s,t}$ in G' exists in G, so the fact that G is $K_{s,t}$-free implies that G' is, too. But this is a contradiction: we selected G as an edge-maximal $K_{s,t}$-free graph, yet G' witnesses that it is not.

Now, we proceed with the main proof. For all $S \subseteq V(G)$, let

$$\deg(S) = \Big|\bigcap_{v\in S} \Gamma(v)\Big|$$

be the mutual degree of the vertices in S. Then for all $S \subseteq V(G)$ with $|S| = s$, we must have

$$\deg(S) \le t-1$$

because otherwise S and its mutual neighborhood would contain a $K_{s,t}$. We therefore note that

$$\sum_{\substack{S\subseteq V(G)\\ |S|=s}} \deg(S) \le \binom{n}{s}(t-1).$$

Note that this summation is one way to count the number of $K_{1,s}$s in G: we can form a $K_{1,s}$ by first picking a set S to form the satellites, and then picking any of their $\deg(S)$-many mutual neighbors to be the center. Alternatively, we can first pick any $v \in G$ as the center, and then choose any s-subset of its neighbors. Therefore, we have

$$\sum_{v\in V(G)} \binom{\deg(v)}{s}. = \sum_{\substack{S\subseteq V(G)\\ |S|=s}} \deg(S) \le \binom{n}{s}(t-1).$$

Now, let X be a random variable whose value is the degree of a vertex chosen uniformly at random from $V(G)$. Let $\varphi : \{z \in \mathbb{R} : z \ge s-1\} \to \mathbb{R}$ be given by $\varphi(z) = \binom{z}{s}$; we know from lemma 1 that φ is convex. Note that $\varphi(X)$ is a well-defined random variable, because each value that X takes is at least $\delta(G)$, which is at least $s-1$ as previously justified. Similarly, $\varphi(\mathbb{E}(X))$ is well-defined because $\mathbb{E}(X) \ge \min X \ge \delta(G)$. Therefore, Jensen's inequality dictates that

$$\varphi(\mathbb{E}(X)) \le \mathbb{E}(\varphi(X))$$
$$\binom{\frac{1}{n}\sum_{v\in V}\deg(v)}{s} \le \frac{1}{n}\sum_{v\in V}\binom{\deg(v)}{s}.$$

For the left-hand side, we have

$$\binom{\frac{1}{n}\sum_{v\in V}\deg(v)}{s} = \binom{2m/n}{s} \ge \left(\frac{2m}{ns}\right)^s = (2/s)^s m^s n^{-s}.$$

On the right-hand side, we have

$$\frac{1}{n}\sum_{v\in V}\binom{\deg(v)}{s} \le \frac{1}{n}\binom{n}{s}(t-1) \le \frac{1}{n}n^s(t-1) = n^{s-1}(t-1).$$

[2] This is not possible only if G is complete, but if G is complete and contains no $K_{s,t}$ then n must be smaller than $s+t$. There are only finitely many (isomorphism classes of) such graphs, and so we may simply adjust our final constant C_0 by making sure that it is large enough to handle all of these "special cases." For instance, choosing $C = \max\{C_0, (s+t)^2\}$ suffices.

Combining these yields

$$\begin{aligned}(2/s)^s m^s n^{-s} &\le n^{s-1}(t-1)\\ m^s &\le (t-1)(s/2)^s \cdot n^{2s-1}\\ m &\le (t-1)^{1/s}(s/2) \cdot n^{2-1/s}.\end{aligned}$$

Because $(t-1)^{1/s}(s/2)$ is a constant depending only on s and t, the proof is now complete. □

(b) We use the method of alterations. Fix some s and t. Suppose that we construct a random graph on n vertices by choosing each vertex independently with some probability $p \in (0,1)$. Let X be the number of edges in this graph, and let Y be the number of $K_{s,t}$s that appear in this graph. Then by removing at most one arbitrary edge from each $K_{s,t}$ in G, we form a $K_{s,t}$-free graph with $X - Y$ edges. Note that

$$\mathbb{E}(X) = p\binom{n}{2} \approx pn^2$$

and

$$\mathbb{E}(Y) \le p^{st}\binom{n}{s}\binom{n-s}{t} \approx p^{st}n^{s+t},$$

because there are at most $\binom{n}{s}\binom{n-s}{t}$ possible copies of $K_{s,t}$ in G. (Interestingly, this count is exact for $s \neq t$, but when $s = t$ we are counting each $K_{s,t}$ twice and so must divide this count by 2.) It then follows that

$$\mathbb{E}(X-Y) = \mathbb{E}(X) - \mathbb{E}(Y) \ge p\binom{n}{2} - p^{st}\binom{n}{s}\binom{n-s}{t}.$$

To maximize this value, we should expect to choose p such that $pn^2 \approx p^{st}n^{s+t}$, so choose

$$p = \varepsilon \cdot n^{(s+t-2)/(1-st)}$$

for some small $\varepsilon > 0$. Then, writing $x^y = x \uparrow y$ for legibility, we find a $K_{s,t}$-free graph with edge count at least

$$\begin{aligned}\mathbb{E}(X-Y) &\ge p\binom{n}{2} - p^{st}\binom{n}{s}\binom{n-s}{t}\\ &\ge p(n/2)^2 - p^{st}n^{s+t}\\ &= \frac{\varepsilon}{4}\cdot n \uparrow \left(2 + \frac{s+t-2}{1-st}\right) - \varepsilon^{st}\cdot n \uparrow \left(\frac{st(s+t-2)}{1-st} + s + t\right)\\ &= \frac{\varepsilon}{4}\cdot n \uparrow \frac{(2-2st)+(s+t-2)}{1-st} - \varepsilon^{st}\cdot n \uparrow \frac{(s^2t+t^2s-2st)+(s-s^2t)+(t-st^2)}{1-st}\\ &= \frac{\varepsilon}{4}\cdot n \uparrow \frac{s+t-2st}{1-st} - \varepsilon^{st}\cdot n \uparrow \frac{s+t-2st}{1-st}\\ &= (\varepsilon/4 - \varepsilon^{st})\cdot n \uparrow \frac{s+t-2st}{1-st},\end{aligned}$$

and by choosing a small enough ε we can make $\varepsilon/4 - \varepsilon^{st} > 0$ to guarantee that this expression is always positive.

This seems like a pretty good lower bound, which we can see by considering the case when $s \approx t$. Indeed, if we let $s = t$ then the exponent on n is

$$\frac{s+t-2st}{1-st} = \frac{2(s-s^2)}{1-s^2} = \frac{2s(1-s)}{(1+s)(1-s)} = \frac{2s}{1+s} = 2 - \frac{2}{1+s}.$$

So we have a lower bound of $2 - 2/(1+s)$, which is approximately $2 - 2/s$ and is thus quite close to our upper bound of $2 - 1/s$.