Math 301: Homework 7

Due Wednesday November 1 at noon on Canvas

1. Show that any graph on n vertices that has at least nd edges contains a subgraph of minimum degree $d+1$.

2. (a) Modify the proof of the upper bound for $\mathrm{ex}(n, K_{2,t})$ that we did in class to prove the Kővari-Sós-Turán Theorem. For $2 \leq s \leq t$ there exists a constant c such that $\mathrm{ex}(n, K_{s,t}) \leq cn^{2-1/s}$.

   (b) Give the best lower bound you can for $\mathrm{ex}(n, K_{s,t})$.

3. (a) Let G be a graph where $V(G)$ consists of n points in the Euclidean plane and two points are adjacent if and only if they are at distance 1 from each other. Show that no matter how the points are placed, the number of edges in the graph is $O(n^{3/2})$.

   (b) Make a construction of n points in the plane that has as many pairs at unit distance as you can. How many edges are in the graph?

4. Let T be a tree on $t+1$ vertices.

   (a) Assume that n is divisible by t. Show that $\mathrm{ex}(n, T) \geq \frac{t-1}{2}n$. (Hint: K_t cannot contain a copy of T).

   (b) Use Problem 1 to show that $\mathrm{ex}(n, T) \leq (t-1)n$.

5. Let k be fixed. Show that there is a constant c so that $\mathrm{ex}(n, \{C_3, C_4, \cdots, C_{2k}\}) \leq cn^{1+1/k}$. (Hint: you need to show that if G has more than $cn^{1+1/k}$ edges then it must contain a cycle of length at most $2k$. Assume G has this many edges and use Problem 1 to start with a graph of minimum degree $c'n^{1/k}$. Do a breadth first search and show that you must find your cycle).