[redacted]

Problem 1:

We want to show G has a dominating set of size at most

$$\frac{n}{\delta+1}(\log(\delta+1)+1).$$

Choose a set X randomly by putting each vertex in X independently with probability p. We will choose p later. Let Y be the set of all vertices not in X that do not have a neighbor in X. Then the probability a vertex is in X is p, so

$$\mathbb{E}(X) = np.$$

Since each vertex has minimum degree δ, the probability a vertex v is in Y is the probability that v and none of its neighbors are in Y which is

$$\leq (1-p)^{\delta+1}.$$

Let Z be an indicator with value 1 is a vertex is in Y and 0 otherwise. Then

$$\mathbb{E}(X+Y) = E(X) + E(Y) \leq np + n(1-p)^{\delta+1}.$$

Note that the $S = X \cup Y$, which gives precisely those vertices which are either in G or has a neighbor in S. Then $S \leq np + n(1-p)^{\delta+1}$. From homework 1, we know that $(1-p) \leq e^{-p}$, so

$$S \leq np + ne^{-p(\delta+1)}.$$

Now if we pick $p = \frac{\log(\delta+1)}{\delta+1}$ we have

$$S \leq \frac{n\log(\delta+1)}{\delta+1} + ne^{-(\log(\delta+1))} = \frac{n}{\delta+1}(\log(\delta+1)+1),$$

as desired.

Problem 2:

(a) If $e < 4n$, then

$$\begin{aligned}\frac{e^3}{64n^2} - n &< \frac{(4n)^3}{64n^2} - n \\ &= \frac{64n^3}{64n^2} - n \\ &= n - n \\ &= 0 \\ &\leq \operatorname{cr}(G),\end{aligned}$$

so the lemma is indeed trivially true: surely any drawing of G must have at least zero crossings.

(d) It is easy to see that $\operatorname{cr}(F) \geq e(F) - e(F')$ for any edge-maximal planar F': as we add the edges back to form F, each edge adds at least one crossing. And, by Euler's formula,

$$-e(F') \geq -3v(F) + 6,$$

so

$$\begin{aligned}\operatorname{cr}(F) &\geq e(F) - e(F') \\ &\geq e(F) - 3v(F) + 6 \\ &\geq e(F) - 3v(F),\end{aligned}$$

as desired.

(e) As $Y = e(H)$ and $X = v(H)$, the previous step immediately implies that

$$Y - 3X \leq \operatorname{cr}(H).$$

Furthermore, in our drawing of G we have a drawing of H with exactly c_S crossings, so surely it is possible to draw H with at least this few; that is,

$$\operatorname{cr}(H) \leq c_S.$$

(f) Each of the n vertices of G is included in H with probability p, so

$$\mathbb{E}(X) = pn.$$

Each of the e edges of G is included in H with probability p^2—each of its endpoints must be selected—so

$$\mathbb{E}(Y) = p^2 e.$$

And each crossing corresponds to two disjoint edges, so each of the $\operatorname{cr}(G)$-many crossings in G is included in H (i.e., contributes 1 to c_S) with probability p^4, so

$$\mathbb{E}(c_S) = p^4 \operatorname{cr}(G).$$

Combining these values with our result from the previous step shows that

$$p^2 e - 3pn \leq p^4 \operatorname{cr}(G).$$

(g) We showed in the first step that it suffices to assume that $e < 4n$; under this assumption, it follows that $4n/e \in [0,1]$, so it is a valid probability. Choosing this for p yields

$$\begin{aligned}
\frac{16n^2}{e} - \frac{12n^2}{e} &\le \frac{256n^4}{e^4}\,\mathrm{cr}(G) \\
\mathrm{cr}(G) &\ge \frac{e^4}{256n^4}\Big(\frac{16n^2}{e} - \frac{12n^2}{e}\Big) \\
&= \frac{e^3}{16n^2} - \frac{3e^3}{64n^2} \\
&= \frac{4e^3}{64n^2} - \frac{3e^3}{64n^2} \\
&= \frac{e^3}{64n^2} \\
\mathrm{cr}(G) &\ge \frac{e^3}{64n^2} - n;
\end{aligned}$$

this completes the proof.

For the theorem...

(b) Suppose that we produce a drawing of G by placing the points at their actual positions in the plane, and connecting points with straight line segments. Each crossing in this drawing corresponds to an intersection of two distinct lines in L, and each pair of lines can contribute at most one such crossing. So

$$\mathrm{cr}(G) \le \binom{m}{2} < m^2.$$

(c) Let $\alpha(P', L')$ denote the number of incidences between a set of points P' and a set of lines L', so that $x = \alpha(P, L)$. Then note that

$$\begin{aligned}
\alpha(P, L) - |L| &= \sum_{\ell \in L} \big(\alpha(P, \{\ell\}) - 1\big) \\
&\le \sum_{\ell \in L} |\{\{u, v\} \in E(G) : u, v \in \ell\}| \\
&= e(G).
\end{aligned}$$

(The inequality in the second step is an equality unless $\ell \cap P = \emptyset$, in which case $\alpha(P, \{\ell\}) - 1 = -1$ but $|\{\{u, v\} \in E(G) : u, v \in \ell\}| = 0$.)

(d) It follows by our lemma that

$$\begin{aligned}
m^2 > \mathrm{cr}(G) &\ge \frac{e(G)^3}{64v(G)^2} - v(G) \\
m^2 &\ge \frac{(x-m)^3}{64n^2} - n \\
64n^2m^2 &\ge (x-m)^3 - 64n^3 \\
4(n^{2/3}m^{2/3}) &\ge (x-m) - 4n \\
4(n^{2/3}m^{2/3}) + 4n + m &\ge x.
\end{aligned}$$

Thus, x is $\mathcal{O}(n^{2/3}m^{2/3} + n + m)$, and we are done!

For the construction...

It's hard for me to think about coming up with a construction that depends explicitly on fractional powers. Instead, I'll choose an integer parameter k, and create a construction in terms of $\Theta(k^3)$ lines and $\Theta(k^3)$ points. So mn is $\Theta(k^6)$, and thus $m^{2/3}n^{2/3}$ is $\Theta(k^4)$.

Finding $\Theta(k^4)$ point–line incidences is thus sufficient for a construction. (As m and n are of the same order, the "$+m+n$" terms in the Szemeredi–Trotter theorem dwindle in comparison to $m^{2/3}n^{2/3}$, so we are justified in ignoring them.)

Let $\ell(a_0, a_1)$ denote the line in the plane that passes through $(0, a_0)$ and has slope a_1. Then consider the family of lines

$$L_i = \{\ell(i, m) : m \in [k]\}.$$

Of the points in $\{1, \ldots, k\} \times \{i+1, \ldots, i+k^2\}$, each line passes through exactly k, so this gives us k^2 incidences already—at the cost of k^3 points. But as we vary i, we will see that we only have to create a few new points. To be more specific: define

$$L = \bigcup_{i \in [k^2]} L_i.$$

For each $x \in [k]$, each line $\ell \in L$ has $(x, y) \in \ell$ for some $y \in [k^2 + k^2]$, because the slope of ℓ is at most k and the intercept is at most k^2. So choosing $P = [k] \times [2k^2]$ allows each of the k^3 lines to intersect exactly k points, while using only $\Theta(k^3)$ points in total. Therefore, we have $\Theta(k^4)$ intersections using $\Theta(k^3)$ points and $\Theta(k^3)$ lines, quod erat faciendum. $\square$

3. For both proofs, we will, for each $S \in \binom{V(G)}{4}$, let A_S be the event that $G[S] \cong K_4$, and let X_S be an indicator variable for A_S. Then also let $X = \sum_S X_S$ be a random variable that counts the number of K_4 in G. Note that each

$$\mathbb{E}(X_S) = \mathbb{P}(A_S) = p^6,$$

and so

$$\mathbb{E}(X) = \binom{n}{4} p^6;$$

by our standard inequalities, we thus have

$$\left(\frac{n}{4}\right)^4 p^6 \le \mathbb{E}(X) \le \left(\frac{en}{4}\right)^4 p^6.$$

(a) *Proof.* Suppose that $p \le 1/(\omega(n) n^{2/3})$. Then, as X is nonnegative, Markov's inequality yields that

$$\begin{aligned}
\mathbb{P}(G \text{ has a } K_4) &= \mathbb{P}(X \ge 1) \\
&\le \frac{\mathbb{E}(X)}{1} \\
&= \binom{n}{4} p^6 \\
&\le \frac{e^4}{256} n^4 p^6 \\
&\le \frac{e^4}{256} \cdot \frac{n^4}{\omega(n)^6 n^{12/3}} \\
&= \frac{e^4}{256} \cdot \frac{1}{\omega(n)^6}.
\end{aligned}$$

As $\omega(n) \to \infty$, certainly $\omega(n)^{-6} \to 0$. Therefore, the probability that G contains a K_4 tends to 0, so the probability that it does not contain a K_4 tends to 1, as desired. $\square$

(b) *Proof.* Suppose that $p \ge \omega(n)/n^{2/3}$. In fact, it suffices to suppose that exactly $p = \omega(n)/n^{2/3}$: if p is larger, then we can surely add in extra edges without decreasing the probability that there is a K_4 in the resulting graph.

We'd like to use Chebyshev's inequality, so we should compute the variance of X, and doing so requires computing the covariance of the X_S.

Choose $S, T \in \binom{V(G)}{4}$. We consider cases based on the size of $|S \cap T|$.

Case: $|S \cap T| \le 1$. In this case, there are no edges shared among $G[S]$ and $G[T]$, so A_S and A_T are independent: $\mathrm{Cov}(X_S, X_T) = 0$.

Case: $|S \cap T| = 2$. In this case, $G[S]$ and $G[T]$ share exactly one edge, and so $e(G[S \cup T]) = 11$. So

$$\mathbb{E}(X_S X_T) = \mathbb{P}(A_S \cap A_T) = p^{11},$$

and thus

$$\mathrm{Cov}(X_S, X_T) = \mathbb{E}(X_S X_T) - \mathbb{E}(X_S)\,\mathbb{E}(X_T) = p^{11} - p^{12}.$$

Case: $|S \cap T| = 3$. In this case, $G[S]$ and $G[T]$ share exactly three edges, and so $e(G[S \cup T]) = 9$. So

$$\mathbb{E}(X_S X_T) = \mathbb{P}(A_S \cap A_T) = p^{9},$$

and

$$\mathrm{Cov}(X_S, X_T) = \mathbb{E}(X_S X_T) - \mathbb{E}(X_S)\,\mathbb{E}(X_T) = p^{9} - p^{12}.$$

Case: $|S \cap T| = 4$. In this case, $G[S] = G[T]$, and we are just interested in computing the variance. We have

$$\mathrm{Cov}(X_S, X_T) = p^6 - p^{12}.$$

Then, we may write the variance of X as follows:

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{Var}\Big(\sum_{S \in \binom{V(G)}{4}} X_S\Big) \\
&= \sum_S \sum_T \mathrm{Cov}(X_S, X_T) \\
&= \sum_S \sum_{0 \le k \le 4} \sum_{\substack{T \\ |T \cap S| = k}} \mathrm{Cov}(X_S, X_T) \\
&= \sum_S \Big(0 + 0 + \binom{4}{2}\binom{n-4}{2}(p^{11} - p^{12}) + \binom{4}{3}\binom{n-4}{1}(p^9 - p^{12}) + 1 \cdot (p^6 - p^{12})\Big) \\
&= \binom{n}{4}\big(3(n-4)(n-3)(p^{11} - p^{12}) + 12(n-4)(p^9 - p^{12}) + (p^6 - p^{12})\big) \\
&\le n^4\big(3n^2(p^{11} - p^{12}) + 12n(p^9 - p^{12}) + (p^6 - p^{12})\big) \\
&\le 3n^6 p^{11} + 12 n^5 p^9 + n^4 p^6.
\end{aligned}$$

Recalling our bound from above that $\mathbb{E}(X) \ge \frac{1}{64} n^4 p^6$, we now apply Chebyshev's inequality:

$$\begin{aligned}
\mathbb{P}(X = 0) &\le \mathbb{P}(|X - \mathbb{E}(X)| \ge \mathbb{E}(X)) \\
&\le \frac{\mathrm{Var}(X)}{\mathbb{E}(X)^2} \\
&\le (64^2) \cdot \frac{3n^6 p^{11} + 12 n^5 p^9 + n^4 p^6}{n^8 p^{12}} \\
&= (64^2)\Big(\frac{3}{n^2 p} + \frac{12}{n^3 p^3} + \frac{1}{n^4 p^6}\Big) \\
&= (64^2)\Big(\frac{3}{n^2\,\omega(n)\,n^{-2/3}} + \frac{12}{n^3\,\omega(n)^3\,n^{-2}} + \frac{1}{n^4\,\omega(n)^6\,n^{-12/3}}\Big) \\
&= (64^2)\Big(\frac{3}{n^{4/3}\,\omega(n)} + \frac{12}{n\,\omega(n)^3} + \frac{1}{\omega(n)^6}\Big) \\
&\le \frac{(64)^2(3 + 12 + 1)}{\omega(n)} \\
&\in \mathcal{O}(1/\omega(n))
\end{aligned}$$

We know that $\omega(n) \to \infty$, so $1/\omega(n) \to 0$. Thus, the probability that G contains no K_4 tends to 0, and so the probability that G contains a K_4 tends to 1, as was to be shown. $\square$