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1. Let G be a graph. We motivate a solution by picturing the edges in a maxcut as being the edges in a bipartite graph (removing all other edges) $A\dot{\cup}B$. If we want to maximize the number of edges, then $|A| = |B|$ or as close as possible- if there is some vertex in A that has more edges in the original G, ie this vertex would be adjacent to more vertices in A than in B, we can swap it.
   Next we split the problem into two cases- $n = 2k$, $n = 2k + 1$.

   - $n = 2k$: Let A have k elements, B other k where both sets can have $\binom{2k}{k}$ possibilities, and let X represent the number of edges e such that $|e \cap A| = 1$, which we will denote as event C. Then

     $$\mathbb{E}(X) = \mathbb{E}(\sum_{e \in E(G)} \chi_C) = \sum_e \mathbb{E}(\chi_C) = \sum_e \mathbb{P}(C)$$

     Suppose $e = \{u, v\}$ and we fix $u \in A$, $v \notin A$. The rest of A is $\binom{2k-2}{k-1}$. This idea is symmetric, so the probability of an edge being half in A is

     $$\frac{2\binom{2k-2}{k-1}}{\binom{2k}{k}} = \frac{k}{2k-1} = \frac{n}{2n-2}$$

     Thus the maxcut is at least $|E(G)|\frac{n}{2n-2}$ (which is greater than $\frac{n}{2n-1}$).
   - $n = 2k+1$: we use the same reasoning in this case where $|A| = k+1$, $|B| = k$. Everything else is identical to the prior proof except that A has one more vertex. If $u \in A$, $v \notin A$, the rest of A can be made in $\binom{2k-1}{k}$ (same for u, v swapped). Note that $\binom{2k-1}{k} = \binom{2k-1}{k-1}$ and $\binom{2k+1}{k+1} = \binom{2k+1}{k}$, so it does not matter whether we look at a vertex being half in A or B. The probability of such an occurrence is

     $$\frac{2\binom{2k-1}{k}}{\binom{2k+1}{k+1}} = \frac{(k+1)(k+1)}{k(2k+1)} = (\frac{n+1}{n})(\frac{n+1}{2n-2})$$

     Since $1 \leq \frac{n+1}{n}$, $|E(G)|\frac{n}{2n-1} \leq |E(G)|(\frac{n+1}{n})(\frac{n+1}{2n-2}) \leq maxcut(G)$.

2. First, we state that it is sufficient to consider complete bipartite graphs. If we can assign a valid list coloring to a $K_{s,t}$, $s+t = n$, then such a coloring

Problem 2

Let G be a bipartite graph on n vertices. For each vertex $v \in V(G)$, let $L(v)$ be a list of colors associated to v of size $\lfloor \log_2 n \rfloor + 1$. Show that it is possible to choose for each vertex v a color from $L(v)$ such that no edge has two endpoints that are the same color.

Let G be partioned into A and B, so that $V(G) = A \sqcup B$ and $E(G) \subseteq A \times B$. In the worst case, $E = A \times B$, which suggests we must avoid choosing the same color for a vertex in A and a vertex in B. Define $L(G) = \bigcup_{v \in V(G)} L(v)$ to be the list of all possible colors. We'll partition $L(G)$ into $L(A)$ and $L(B)$ by independently putting each color in $L(A)$ with probability p. For every $a \in A$, let X_a be an indicator for the event that $L(a) \cap L(A) = \emptyset$. Likewise for every $b \in B$, let X_b be an indicator for the event that $L(b) \cap L(B) = \emptyset$. Define $X = \sum_{a \in A} X_a + \sum_{b \in B} X_b$. In words, X is the number of vertices which cannot be assigned a color based on the partitioning of $L(G)$. Then,

$$\begin{aligned} \mathbb{E}(X) &= \sum_{a \in A} \mathbb{E}(X_a) + \sum_{b \in B} \mathbb{E}(X_b) \\ &= \sum_{a \in A} (1-p)^{\lfloor \log_2 n \rfloor + 1} + \sum_{b \in B} p^{\lfloor \log_2 n \rfloor + 1} \end{aligned}$$

We set $p = \frac{1}{2}$ and write $\lfloor \log_2 n \rfloor + 1$ as $\log_2 n + \epsilon$ for some $\epsilon > 0$. Then we can simplify to

$$\mathbb{E}(X) = \sum_{v \in V(G)} \left(\frac{1}{2}\right)^{\log_2 n + \epsilon} = n \left(\frac{1}{n}\right) \left(\frac{1}{2}\right)^{\epsilon} = \left(\frac{1}{2}\right)^{\epsilon}$$

Thus $\mathbb{E}(X) < 1$, which implies that there exists a partition of the colors $L(G)$ into $L(A)$ and $L(B)$ such that every vertex $v \in V(G)$ can be colored such that no edge has two endpoints of the same color.

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3. (a) Note that $\binom{4}{2}\frac{1}{2} = 3$
Given a fully connected graph on 4 vertices, observe the following 3 cycles:

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A---B   A---- B   A   B
|   |     \ /     | \/|
C---D     / \     |/ \|
         /   \    C   D
       C -----D
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In a fully connected graph, we have $\binom{n}{4}$ fully connected graphs on 4 vertices, each with 3 4-cycles. So the total amount of 4-cycles is $3\binom{n}{4} = \binom{n}{4}\binom{4}{2}\frac{1}{2}$

(b) Instead of going for a probabilistic proof, we'll go for an explicit construction here. Consider the tree on n vertices where each node has at most 1 child (it forms just a path). Connect the end of the path to the node two before it to create a 3-cycle at the end of the graph. This has n edges, the the max must be bounded below by n. Thus,we know that $\text{ex}(n, C_4) \in \Omega(n)$

(c) Suppose that we construct our graph by connecting two vertices with an edge with probability p.
Let P be a random variable representing the amount of edges in the graph.
Let Q be a random variable representing the amount of 4-cycles in the graph.
To break all the 4-cycles, we can remove one edge from every 4-cycle we create in this process.
Thus, the amount of edges in the graph can be expressed as $P - Q$.
We are interested in the expected value of $P - Q$.
$E[P - Q] = E[P] - E[Q]$ by linearity of expectation.
$E[P]$ is just the amount of possible edges times the probability of an edge existing, or $\binom{n}{2}p$.
The expected value of the amount of four cycles is the total amount of 4-cycles times the probability of all four edges existing. This is just $\binom{n}{4}\binom{4}{2}\frac{1}{2}p^4$

So the expected value ends up being $\binom{n}{2}p - \binom{n}{4}\binom{4}{2}\frac{1}{2}p^4$
We want to find the value of p that maximizes this formula.
To do this, we take the derivative with respect to p, yielding:

$$\binom{n}{2} - 2\binom{n}{4}\binom{4}{2}p^3$$

And we find the zero is at $p = \frac{1}{((n-3)(n-2))^{\frac{1}{3}}}$

Plugging p back into our original formula, we get:

$$\frac{n(n-1)}{2\sqrt[3]{(n-3)(n-2)}} - \frac{(n-3)(n-2)(n-1)n}{8((n-3)(n-2))^{\frac{4}{3}}}$$

This value is always positive, and the first term is approximately $\frac{n^2}{n^{\frac{2}{3}}} = n^{\frac{4}{3}}$ and the second term is similarly a factor of $n^{\frac{4}{3}}$, but is dominated by the first term. Thus, the expected value of edges in the graph created with our chosen p is in $\Omega(n^{\frac{4}{3}})$. Since our expected value is in $\Omega(n^{\frac{4}{3}})$, we know the true value must be as well.