Math 301: Homework 1

Due by email to `mtait@cmu.edu` Wednesday September 6 at noon

1. Determine the number of integral solutions there are to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 100$$

   which satisfy the constraints $x_1, x_2, x_3 \geq 0$, $x_4 \geq -2$ and $x_5 \geq 10$. Your answer should not contain a sum.

2. We proved in class that $n! \leq en\left(\frac{n}{e}\right)^n$.

   (a) Prove that for all real x, $1 + x \leq e^x$.

   (b) Use part (a) to give a second proof of the upper bound on $n!$ by induction on n.

   (c) Prove that $e\left(\frac{n}{e}\right)^n \leq n!$.

   (d) Prove that $n! \leq e\sqrt{n}\left(\frac{n}{e}\right)^n$.

3. (a) Let x and y be positive integers and let m be a positive integer. Prove the following identity:

$$\binom{x+y}{m} = \sum_{k=0}^{m} \binom{x}{k}\binom{y}{m-k}.$$

   Is it true for all real x and y? For a real number x we define $\binom{x}{k} := \frac{x(x-1)\cdots(x-k+1)}{k!}$.

   (b) Prove the same identity using a bijection with lattice paths.

   (c) Prove that the number of even subsets of $[n]$ is equal to the number of odd subsets of $[n]$ in two ways: using a bijection and using the binomial theorem.