Name: ________________________________ **PID:**____________

Circle your section: A01 (11am-12pm) or A02 (12pm-1pm)

MATH 10B: MIDTERM EXAM 2

July 26th, 2012

Do not turn the page until instructed to begin.

Turn off and put away your cell phone.

No calculators or any other devices are allowed.
You may use one 8.5×11 page of handwritten notes, but no other assistance.
Read each question carefully, answer each question completely, & show all of your work.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
If any question is not clear, ask for clarification.
Good luck!

#	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
E.C.	3	
Σ	50	

1. Take the following indefinite integrals.

   (a) (5 points) $\displaystyle\int \frac{1}{(2-x)^2}\,dx.$

   Doing integration by substitution with $u = 2 - x$, we get $du = -dx$, which can be rewritten as $dx = (-1)du$. Then doing the substitution,

$$\begin{aligned}\int \frac{1}{(2-x)^2}\,dx &= \int \frac{1}{u^2}\,(-1)\,du \\ &= -\int u^{-2}\,du \\ &= -\frac{u^{-1}}{-1} + C \\ &= \frac{1}{u} + C \\ &= \boxed{\frac{1}{2-x} + C}\end{aligned}$$

   (b) (5 points) $\displaystyle\int \frac{\ln(x)}{x^2}\,dx.$

   We either use #13 from the table of integrals, or we do integration by parts with $u = \ln(x)$ and $dv = \frac{1}{x^2}\,dx$.

$$\begin{aligned} u &= \ln(x) &\implies&& du &= \frac{1}{x}\,dx \\ dv &= \frac{1}{x^2}\,dx &\implies&& v &= \frac{-1}{x}\end{aligned}$$

$$\begin{aligned}\int u\,dv &= uv - \int v\,du \\ \int \frac{\ln(x)}{x^2}\,dx &= \frac{-\ln(x)}{x} + \int \frac{1}{x^2}\,dx \\ &= \boxed{\frac{-\ln(x)}{x} + \frac{-1}{x} + C}\end{aligned}$$

2. Also take these indefinite integrals.

(a) (5 points) $$\int \frac{2}{\sqrt{4-x^2}}\,dx.$$

Using #28 from the table of integrals with $a = 2$, we have

$$\int \frac{2}{\sqrt{4-x^2}}\,dx = 2\int \frac{1}{\sqrt{4-x^2}}\,dx = \boxed{2\arcsin(\tfrac{x}{2}) + C}.$$

(b) (5 points) $$\int \frac{2x}{\sqrt{4-x^2}}\,dx.$$

Letting $u = 4 - x^2$ and doing integration by substitution, we get

$$u = 4 - x^2 \qquad \Longrightarrow \qquad \frac{du}{dx} = -2x\,dx \qquad \Longrightarrow \qquad (-1)du = 2x\,dx,$$

so that using substitution we get

$$\begin{aligned}\int \frac{2x}{\sqrt{4-x^2}}\,dx &= \int \frac{1}{\sqrt{u}}(-1)\,du \\ &= -\int u^{-1/2}\,du \\ &= -\frac{u^{1/2}}{1/2} + C \\ &= -2\sqrt{u} + C \\ &= \boxed{-2\sqrt{4-x^2} + C}\end{aligned}$$

3. Here are more indefinite integrals. Take them as well.

(a) (5 points) $$\int \frac{x-2}{x^2(x+2)}\,dx.$$

We first split into partial fractions using $\frac{x-2}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$. Adding these fractions back together gives us

$$\begin{aligned}\frac{x-2}{x^2(x+2)} &= \frac{Ax(x+2)}{x^2(x+2)} + \frac{B(x+2)}{x^2(x+2)} + \frac{Cx^2}{x^2(x+2)}\\ &= \frac{Ax^2+2Ax+Bx+2B+Cx^2}{x^2(x+2)}\\ &= \frac{(A+C)x^2+(2A+B)x+(2B)}{x^2(x+2)},\end{aligned}$$

and since $(x-2)$ in the numerator of the original fraction can be rewritten as $(0)x^2+(1)x+(-2)$, we get three equations:

$$A+C=0 \qquad 2A+B=1 \qquad 2B=-2.$$

The third equation gives us $B=-1$. Plugging this into the second equation gives us $A=1$. And plugging that into the first equation gives us $C=-1$.

$$\int \frac{x-2}{x^2(x+2)}\,dx = \int \left(\frac{1}{x} + \frac{-1}{x^2} + \frac{-1}{x+2}\right)dx = \boxed{\ln|x| + \tfrac{1}{x} - \ln|x+2| + C}$$

(b) (5 points) $$\int \sin^2(x)\,\cos(x)\,dx.$$

Letting $u=\sin(x)$ gives us $du=\cos(x)\,dx$, so with integration by substitution we get

$$\begin{aligned}\int \sin^2(x)\,\cos(x)\,dx &= \int u^2\,du\\ &= \frac{u^3}{3}+C\\ &= \boxed{\frac{\sin^3(x)}{3}+C}\end{aligned}$$

4. (10 points) You work for NASA and the current mission is to send a rover to explore a recently discovered planet in another galaxy. Your job is to determine the acceleration due to gravity for this planet. Once on the planet's surface, the rover drops a weight from a height of 2 meters and it measures how long it takes the weight to hit the ground. The rover determines this time is 1 second. What is the acceleration due to gravity on this planet?

   (Your answer is assumed to be in units of m/s^2)

The formula for the height of anything that is falling is

$$h(t) = -\frac{g}{2}t^2 + v_0 t + h_0,$$

where g is the acceleration due to gravity, v_0 is the initial velocity, and h_0 is the initial height. We know that $v_0 = 0$ in this problem because the object is dropped (not thrown), and we know that $h_0 = 2$ because the object is dropped from 2 meters off the ground. Therefore the equation that describes the height is

$$h(t) = -\frac{g}{2}t^2 + 2.$$

We also know that $h(1) = 0$, because the object hits the ground after 1 second.

$$0 = h(1) = -\frac{g}{2}(1)^2 + 2 \qquad \Longrightarrow \qquad \boxed{g = 4}.$$

5. For each of the following integrals, if it converges, write the number it converges to. Otherwise, you may simply write "diverges" as your answer.

   (a) (5 points) $\displaystyle\int_1^{\infty} \frac{1}{x\sqrt{x}}\,dx.$

$$\begin{aligned}\int_1^{\infty} \frac{1}{x\sqrt{x}}\,dx &= \lim_{b\to\infty} \int_1^{b} \frac{1}{x\sqrt{x}}\,dx \\ &= \lim_{b\to\infty} \int_1^{b} x^{-3/2}\,dx \\ &= \lim_{b\to\infty} \left(\frac{x^{-1/2}}{-1/2}\Bigg|_1^b \right) \\ &= \lim_{b\to\infty} \left(-\frac{2}{\sqrt{x}}\Bigg|_1^b \right) \\ &= \lim_{b\to\infty} \left(-\frac{2}{\sqrt{b}} + \frac{2}{\sqrt{1}} \right) \\ &= 0 + 2 \\ &= \boxed{2}\end{aligned}$$

   (b) (5 points) $\displaystyle\int_0^{5} \frac{1}{(x-1)^2}\,dx.$

$\frac{1}{(x-1)^2}$ is discontinuous at $x = 1$, so the integral has to be broken up into two separate integrals.

$$\int_0^5 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^5 \frac{1}{(x-1)^2}\,dx.$$

If either of these integrals diverge, then the original one does too. Let's calculate $\int \frac{1}{(x-1)^2}\,dx$ first and then use it to evaluate the definite integrals. We do this by using the substitution $u = x - 1$, which implies $du = dx$.

$$\int \frac{1}{(x-1)^2}\,dx = \int \frac{1}{u^2}\,du = \int u^{-2}\,du = \frac{u^{-1}}{-1} + C = \frac{-1}{u} + C = \frac{-1}{x-1} + C.$$

We can now evaluate the definite integrals.

$$\int_0^1 \frac{1}{(x-1)^2}\,dx = \lim_{b\to 1} \frac{-1}{x-1}\Bigg|_0^b = \lim_{b\to 1} \frac{-1}{b-1} + \frac{-1}{0-1} = \lim_{b\to 1} \frac{-1}{b-1} + 1 = \infty,$$

so the integral $\boxed{\text{diverges}}$.

Extra Credit: (3 points) Take one more indefinite integral:

$$\int e^{\sqrt{x}}\,dx$$

If the problem were $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$, then we could do a substitution with $w = \sqrt{x}$ (which implies $dw = \frac{1}{2\sqrt{x}}\,dx$, or $2dw = \frac{1}{\sqrt{x}}\,dx$), and we would then get

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^w\,dw = 2e^w + C = 2e^{\sqrt{x}} + C.$$

But this problem is different. We can still make use of this information though, if we do integration by parts with $u = \sqrt{x}$ and $dv = \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

$$u = \sqrt{x} \qquad \Longrightarrow \qquad du = \frac{1}{2\sqrt{x}}\,dx$$

$$dv = \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx \qquad \Longrightarrow \qquad v = 2e^{\sqrt{x}} \qquad \text{(from above)}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int e^{\sqrt{x}} = \int \underbrace{\sqrt{x}}_{u}\,\underbrace{\frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx}_{dv} = \underbrace{\sqrt{x}}_{u}\,\underbrace{2e^{\sqrt{x}}}_{v} - \int \underbrace{2e^{\sqrt{x}}}_{v}\,\underbrace{\frac{1}{2\sqrt{x}}\,dx}_{du}$$

$$= 2\sqrt{x}e^{\sqrt{x}} - \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$$

$$= \boxed{2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C}.$$

TABLE OF INTEGRALS

BASIC FUNCTIONS

1. $\displaystyle\int x^n\,dx = \frac{1}{n+1}\,x^{n+1} + C \qquad (\text{if } n \neq 1)$
2. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$
3. $\displaystyle\int a^x\,dx = \frac{1}{\ln(a)}\,a^x + C \qquad (\text{if } a > 0)$
4. $\displaystyle\int \ln(x)\,dx = x\,\ln(x) - x + C$
5. $\displaystyle\int \sin(x)\,dx = -\cos(x) + C$
6. $\displaystyle\int \cos(x)\,dx = \sin(x) + C$
7. $\displaystyle\int \tan(x)\,dx = -\ln|\cos(x)| + C$

PRODUCTS OF e^x, $\cos(x)$, $\sin(x)$

8. $\displaystyle\int e^{ax}\sin(bx)\,dx = \frac{1}{a^2+b^2}\,e^{ax}\,[a\sin(bx) - b\cos(bx)] + C$
9. $\displaystyle\int e^{ax}\cos(bx)\,dx = \frac{1}{a^2+b^2}\,e^{ax}\,[a\cos(bx) + b\sin(bx)] + C$
10. $\displaystyle\int \sin(ax)\sin(bx)\,dx = \frac{1}{b^2-a^2}\,[a\cos(ax)\sin(bx) - b\sin(ax)\cos(bx)] + C \qquad (\text{if } a \neq b)$
11. $\displaystyle\int \cos(ax)\cos(bx)\,dx = \frac{1}{b^2-a^2}\,[b\cos(ax)\sin(bx) - a\sin(ax)\cos(bx)] + C \qquad (\text{if } a \neq b)$
12. $\displaystyle\int \sin(ax)\cos(bx)\,dx = \frac{1}{b^2-a^2}\,[b\sin(ax)\sin(bx) + a\cos(ax)\cos(bx)] + C \qquad (\text{if } a \neq b)$

PRODUCT OF POLYNOMIAL $p(x)$ WITH $\ln(x)$, e^x, $\cos(x)$, $\sin(x)$

13. $\displaystyle\int x^n\ln(x)\,dx = \frac{1}{n+1}\,x^{n+1}\ln(x) - \frac{1}{(n+1)^2}x^{n+1} + C \qquad (\text{if } n \neq -1)$
14. $\displaystyle\int p(x)e^{ax}\,dx = \frac{1}{a}p(x)e^{ax} - \frac{1}{a}\int p'(x)\,e^{ax}\,dx$

    $\displaystyle= \frac{1}{a}p(x)e^{ax} - \frac{1}{a^2}p'(x)e^{ax} + \frac{1}{a^3}p''(x)\,e^{ax} - \cdots \qquad (+-+-+-\ldots)$
15. $\displaystyle\int p(x)\sin(ax)\,dx = -\frac{1}{a}p(x)\cos(ax) + \frac{1}{a}\int p'(x)\cos(ax)\,dx$

    $\displaystyle= -\frac{1}{a}p(x)\cos(ax) + \frac{1}{a^2}p'(x)\sin(ax) + \frac{1}{a^3}p''(x)\cos(ax) - \cdots \qquad (-++--++\ldots)$
16. $\displaystyle\int p(x)\cos(ax)\,dx = \frac{1}{a}p(x)\sin(ax) - \frac{1}{a}\int p'(x)\sin(ax)\,dx$

    $\displaystyle= \frac{1}{a}p(x)\sin(ax) + \frac{1}{a^2}p'(x)\cos(ax) - \frac{1}{a^3}p''(x)\sin(ax) - \cdots \qquad (++--++\ldots)$

Integer Powers of $\sin(x)$, $\cos(x)$

17. $\displaystyle\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\,\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx + C \qquad (\text{if } n > 0)$

18. $\displaystyle\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\,\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx + C \qquad (\text{if } n > 0)$

19. $\displaystyle\int \frac{1}{\sin^m(x)}\,dx = \frac{-1}{m-1}\,\frac{\cos(x)}{\sin^{m-1}(x)} + \frac{m-2}{m-1}\int \frac{1}{\sin^{m-2}(x)}\,dx \qquad (\text{if } m > 1)$

20. $\displaystyle\int \frac{1}{\sin(x)}\,dx = \frac{1}{2}\ln\left|\frac{\cos(x)-1}{\cos(x)+1}\right| + C$

21. $\displaystyle\int \frac{1}{\cos^m(x)}\,dx = \frac{1}{m-1}\,\frac{\sin(x)}{\cos^{m-1}(x)} + \frac{m-2}{m-1}\int \frac{1}{\cos^{m-2}(x)}\,dx \qquad (\text{if } m > 1)$

22. $\displaystyle\int \frac{1}{\cos(x)}\,dx = \frac{1}{2}\ln\left|\frac{\sin(x)+1}{\cos(x)-1}\right| + C$

23. $\displaystyle\int \sin^m(x)\cos^n(x)\,dx$

If m is odd, let $w = \cos(x)$. If n is odd, let $w = \sin(x)$. If both m and n are even and nonnegative, convert all to $\sin(x)$ or all to $\cos(x)$ (using $\cos^2(x) + \sin^2(x) = 1$), and use 17 or 18. If m and n are even and one of them is negative, convert to whichever function is in the denominator and use 19 or 21. If both m and n are even and negative, substitute $w = \tan(x)$.

Quadratic in the Denominator

24. $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

25. $\displaystyle\int \frac{bx+c}{x^2+a^2}\,dx = \frac{b}{2}\ln|x^2+a^2| + \frac{c}{a}\arctan\left(\frac{x}{a}\right) + C \qquad (\text{if } a \neq 0)$

26. $\displaystyle\int \frac{1}{(x-a)(x-b)}\,dx = \frac{1}{a-b}(\ln|x-a| - \ln|x-b|) + C \qquad (\text{if } a \neq b)$

27. $\displaystyle\int \frac{cx+d}{(x-a)(x-b)}\,dx = \frac{1}{a-b}[(ac+d)\ln|x-a| - (bc+d)\ln|x-b|] + C \qquad (\text{if } a \neq b)$

Integrands involving $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$, $\quad a > 0$

28. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$

29. $\displaystyle\int \frac{1}{\sqrt{x^2\pm a^2}}\,dx = \ln\left|x + \sqrt{x^2\pm a^2}\right| + C$

30. $\displaystyle\int \sqrt{a^2\pm x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2\pm x^2} + a^2\int \frac{1}{\sqrt{a^2\pm x^2}}\,dx\right) + C$

31. $\displaystyle\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}\left(x\sqrt{x^2-a^2} - a^2\int \frac{1}{\sqrt{x^2-a^2}}\,dx\right) + C$