Name: ______________________________ **PID:**__________

Circle your section: A01 (11am-12pm) or A02 (12pm-1pm)

MATH 10B: PRACTICE FINAL EXAM - SOLUTIONS

July 29th, 2012

Do not turn the page until instructed to begin.

Turn off and put away your cell phone.

No calculators or any other devices are allowed.
You may use one 8.5×11 page of handwritten notes, but no other assistance.
Read each question carefully, answer each question completely, & show all of your work.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
If any question is not clear, ask for clarification.
Good luck!

#	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Σ	80	

1. Take the following indefinite integrals.

(a) (5 points) $\displaystyle\int \frac{(2-x)^2}{x}\,dx.$ (hint: FOIL and separate terms)

$$\begin{aligned}\int \frac{(2-x)^2}{x}\,dx &= \int \frac{4-4x+x^2}{x}\,dx\\ &= \int \left(\frac{4}{x}-\frac{4x}{x}+\frac{x^2}{x}\right)dx\\ &= 4\int \frac{1}{x}\,dx - \int 4\,dx + \int x\,dx\\ &= \boxed{4\ln|x| - 4x + \tfrac{x^2}{2} + C}\end{aligned}$$

(b) (5 points) $\displaystyle\int \frac{x}{(2-x)^2}\,dx.$ (hint: u-substitution)

$$u = 2-x \quad\Longrightarrow\quad \frac{du}{dx} = -1 \quad\Longrightarrow\quad dx = -du.$$

Using substitutions, we now have

$$\int \frac{x}{(2-x)^2}\,dx = -\int \frac{x}{u^2}\,du.$$

We still need to replace the x in the last integral in order to put everything in terms of u so we can take the integral. Luckily we can solve $u = 2-x$ for x to get $x = 2-u$, and now we have

$$\begin{aligned}\int \frac{x}{(2-x)^2}\,dx &= -\int \frac{2-u}{u^2}\,du\\ &= -\int \left(\frac{2}{u^2}-\frac{u}{u^2}\right)du\\ &= -2\int \frac{1}{u^2}\,du + \int \frac{1}{u}\,du\\ &= -2\frac{u^{-1}}{-1} + \ln|u| + C\\ &= \frac{2}{u} + \ln|u| + C\\ &= \boxed{\frac{2}{2-x} + \ln|2-x| + C}\end{aligned}$$

2. (10 points) A company has a continuous income stream of $P(t) = 10 + t$ million dollars per year. Assuming this income is directly deposited in an account making 5% interest per year, how much money will the company have in 2 years?

The formula for the future value of a continuous income stream is

$$\text{Future value} = \int_0^M P(t)e^{r(M-t)}\,dt,$$

where $M = 2$ and $r = 0.05$ (the interest rate). Substituting in $P(t) = 10 + t$, we get

$$\text{Future value} = \int_0^2 (10+t)e^{r(2-t)}\,dt = \int_0^2 10e^{r(2-t)}\,dt + \int_0^2 te^{r(2-t)}\,dt.$$

Let's calculate these last two integrals separately and then add them to get the final answer. The first integral requires u-substitution with $u = r(2-t)$, which gives us $\frac{du}{dt} = -r$, or in other words $dt = \frac{-1}{r}\,du$, and substituting this in we get

$$\begin{aligned}
\int_0^2 10e^{r(2-t)}\,dt &= \int 10e^u\ \left(\tfrac{-1}{r}\right)\,du \\
&= 10\left(\tfrac{-1}{r}\right)\int e^u\ du \\
&= 10\left(\tfrac{-1}{r}\right)\left(e^{r(2-t)}\Big|_0^2\right) \\
&= 10\left(\tfrac{-1}{r}\right)\left(e^{r(2-2)} - e^{r(2-0)}\right) \\
&= 10\left(\tfrac{-1}{r}\right)\left(1 - e^{2r}\right) \\
&= -200\left(1 - e^{1/10}\right) \qquad (\approx 21.03).
\end{aligned}$$

The second integral involves integration by parts.

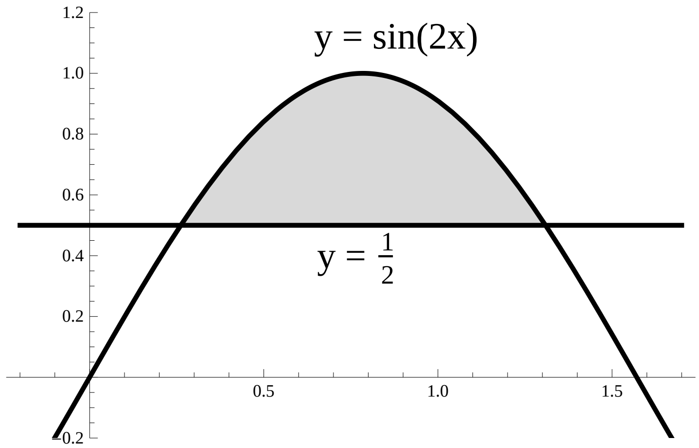
$$\begin{aligned}
u = t \quad &\Longrightarrow \quad du = dt \\
dv = e^{r(2-t)}\,dt \quad &\Longrightarrow \quad v = \tfrac{-1}{r}e^{r(2-t)}
\end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$

$$\begin{aligned}
\int \underbrace{t}_{u}\ \underbrace{e^{r(2-t)}\,dt}_{dv} &= \underbrace{t}_{u}\ \underbrace{\tfrac{-1}{r}e^{r(2-t)}}_{v} - \int \underbrace{\tfrac{-1}{r}e^{r(2-t)}}_{v}\ \underbrace{dt}_{du} \\
&= -\tfrac{t}{r}\,e^{r(2-t)} - \tfrac{1}{r^2}e^{r(2-t)} + C \\
\int_0^2 te^{r(2-t)}\,dt &= \left(-\tfrac{t}{r}\,e^{r(2-t)} - \tfrac{1}{r^2}e^{r(2-t)}\right)\Big|_0^2 \\
&= -\tfrac{2}{r} - \tfrac{1}{r^2} + \tfrac{1}{r^2}e^{2r} \\
&= -40 - 400 + 400e^{1/10} \qquad (\approx 2.07).
\end{aligned}$$

So the answer is about 23.10 million dollars.

3. (10 points) Find the shaded area between the curves depicted below.



The curves intersect when $\sin(2x) = \frac{1}{2}$. The two angles which have a sine of $\frac{1}{2}$ are $\pi/6$ and $5\pi/6$, so setting $2x$ equal to these angles and solving for x gives us $x = \pi/12$ and $x = 5\pi/12$. The area between the curves is

$$\begin{aligned}
\int_{\pi/12}^{5\pi/12} \left(\sin(2x) - \tfrac{1}{2}\right) dx &= \int_{\pi/12}^{5\pi/12} \sin(2x)\, dx - \int_{\pi/12}^{5\pi/12} \tfrac{1}{2}\, dx \\
&= -\tfrac{1}{2}\cos(2x)\Big|_{\pi/12}^{5\pi/12} - \tfrac{1}{2}x\Big|_{\pi/12}^{5\pi/12} \\
&= \left(-\tfrac{1}{2}\cos(5\pi/6) + \tfrac{1}{2}\cos(\pi/6)\right) - \left(\tfrac{5\pi}{24} - \tfrac{\pi}{24}\right) \\
&= \left(-\tfrac{1}{2}\left(-\tfrac{\sqrt{3}}{2}\right) + \tfrac{1}{2}\left(\tfrac{\sqrt{3}}{2}\right)\right) - \left(\tfrac{5\pi}{24} - \tfrac{\pi}{24}\right) \\
&= \boxed{\tfrac{\sqrt{3}}{2} - \tfrac{\pi}{6}}
\end{aligned}$$

4. Find more indefinite integrals.

(a) (5 points) $\displaystyle\int \frac{1}{1+9x^2}\,dx.$ (hint: trig. substitution)

Let $x = \frac{1}{3}\tan\theta$. Then $dx = \frac{1}{3}\sec^2\theta\,d\theta$. Using the trigonometric identity for the tangent: $1+\tan^2\theta = \sec^2\theta$, we have

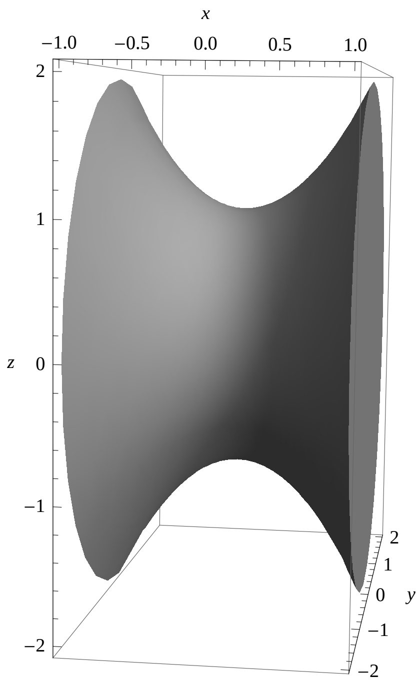
$$\begin{aligned}
\int \frac{1}{1+9x^2}\,dx &= \int \frac{1}{1+\tan^2\theta}\,dx \\
&= \int \frac{1}{\sec^2\theta}\,\tfrac{1}{3}\sec^2\theta\,d\theta \\
&= \int \tfrac{1}{3}\,d\theta \\
&= \tfrac{1}{3}\theta + C \\
&= \boxed{\tfrac{1}{3}\arctan(3x) + C}.
\end{aligned}$$

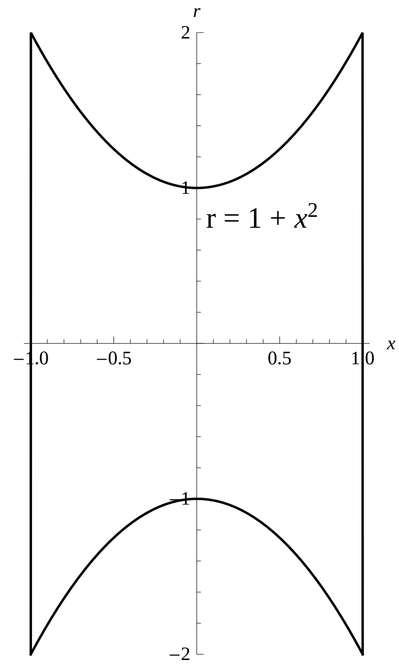
(b) (5 points) $\displaystyle\int \frac{18x}{1+9x^2}\,dx.$ (hint: u-substitution)

Let $u = 1+9x^2$. Then $du = 18x\,dx$, and

$$\begin{aligned}
\int \frac{18x}{1+9x^2}\,dx &= \int \frac{1}{u}\,du \\
&= \ln|u| + C \\
&= \boxed{\ln(1+9x^2) + C}
\end{aligned}$$

5. (10 points) Find the volume of the spool depicted below with radius given by the curve $r = 1 + x^2$ for $-1 \leq x \leq 1$.





The volume formula for a volume of revolution is

$$\text{Volume} = \pi \int (r(x))^2 \, dx,$$

where $r(x)$ is the radius function. So in this case we have

$$\begin{aligned}
\text{Volume} &= \pi \int_{-1}^{1} (1 + x^2)^2 \, dx \\
&= \pi \int_{-1}^{1} 1 + 2x^2 + x^4 \, dx \\
&= \pi \left(\left. x + \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5 \right|_{-1}^{1} \right) \\
&= \pi \left(\left(1 + \tfrac{2}{3} + \tfrac{1}{5}\right) - \left(-1 - \tfrac{2}{3} - \tfrac{1}{5}\right) \right) = \boxed{\frac{14}{15}\pi}.
\end{aligned}$$

6. Take the following indefinite integrals.

(a) (5 points) $\displaystyle\int \frac{\ln(x)}{x}\,dx.$ (hint: u-substitution)

Letting $u = \ln(x)$ gives us $\frac{du}{dx} = \frac{1}{x}$, or $du = \frac{1}{x}dx$.

$$\begin{aligned}\int \frac{\ln(x)}{x}\,dx &= \int u\,du \\ &= \frac{u^2}{2} + C \\ &= \boxed{\frac{(\ln(x))^2}{2} + C}\end{aligned}$$

(b) (5 points) $\displaystyle\int \frac{2}{x^2 - x}\,dx.$ (hint: partial fractions)

We factor $x^2 - x$ on the bottom into $x(x-1)$, so the general form for the partial fraction expansion should be

$$\frac{2}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}.$$

To add the fractions on the right hand side, we need to make the denominators the same

$$\begin{aligned}\frac{2}{x(x-1)} &= \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1)}{x(x-1)} + \frac{Bx}{x(x-1)} \\ &= \frac{Ax - A + Bx}{x(x-1)} = \frac{(A+B)x - A}{x(x-1)}.\end{aligned}$$

From here, we need the top to be equal to $(0)x + 2$, so we get the two equations

$$A + B = 0 \qquad \text{and} \qquad -A = 2,$$

so $A = -2$ and $B = 2$. Then the original integral can be calculated.

$$\int \frac{2}{x^2 - x}\,dx = \int \left(\frac{-2}{x} + \frac{2}{x-1}\right)dx = \boxed{-2\ln|x| + 2\ln|x-1| + C}$$

7. Solve the following initial value problem.

$$\frac{dy}{dx} = x\sin(x) \qquad\qquad y(0) = 1.$$

Integrate both sides of the differential equation to get $y(x) = \int x\sin(x)\,dx$. This integral can be solved with integration by parts.

$$\begin{aligned} u = x \quad &\Longrightarrow \quad du = dx \\ dv = \sin(x)\,dx \quad &\Longrightarrow \quad v = -\cos(x) \\ \int u\,dv &= uv - \int v\,du \\ y(x) = \int x\sin(x)\,dx &= -x\cos(x) + \int \cos(x)\,dx \\ &= -x\cos(x) + \sin(x) + C \end{aligned}$$

Therefore $y(x) = -x\cos(x) + \sin(x) + C$. Using the initial condition $y(0) = 1$, we have

$$1 = y(0) = -0\cos(0) + \sin(0) + C = 0 + 0 + C \qquad \Longrightarrow \qquad C = 1.$$

So the final answer is $\boxed{y(x) = -x\cos(x) + \sin(x) + 1}$.

8. (10 points) You're drinking coffee while working on a practice final, but you're so concentrated on the questions that you forget to continue drinking. Your coffee starts at $150°$F and the temperature of the room is $75°$F. According to Newton's Law of Cooling, the coffee's temperature $y(t)$ (in degrees Fahrenheit) as a function of time (in minutes) is given by the differential equation

$$\frac{dy}{dt} = \frac{75 - y}{50} \qquad\qquad y(0) = 150,$$

How long until the coffee reaches $100°$F?

This is a separable differential equation. It can be rewritten as

$$\frac{1}{(75 - y)}\, dy = \frac{1}{50}\, dt,$$

and integrating both sides gives us

$$-\ln(75 - y) = \frac{t}{50} + C.$$

We now solve for y in terms of t.

$$\begin{aligned}
\ln(75 - y) &= -\frac{t}{50} - C \\
75 - y &= e^{-\frac{t}{50} - C} \\
75 - y &= e^{-\frac{t}{50}} e^{-C} \qquad\qquad (\text{let } e^{-C} = C_0) \\
75 - y &= C_0 e^{-\frac{t}{50}} \\
y(t) &= 75 - C_0 e^{-\frac{t}{50}},
\end{aligned}$$

The initial condition is $y(0) = 150$. Plugging this in we get

$$150 = y(0) = 75 - C_0 e^{-\frac{0}{50}} = 75 - C_0 \qquad \Longrightarrow \qquad C_0 = -75.$$

Therefore the temperature of the coffee is $y(t) = 75 + 75e^{-t/50}$. To answer the question of how long it takes for the coffee to reach $100°$F, we write the equation $y(t) = 100$ and solve for t.

$$\begin{aligned}
75 + 75e^{-t/50} &= 100 \\
75e^{-t/50} &= 25 \\
e^{-t/50} &= \tfrac{1}{3} \\
-\tfrac{t}{50} &= \ln(\tfrac{1}{3}) \\
t &= \boxed{-50\ln(\tfrac{1}{3})} \qquad (\approx 23.86 \text{ minutes})
\end{aligned}$$

TABLE OF INTEGRALS

BASIC FUNCTIONS

1. $\displaystyle\int x^n\,dx = \frac{1}{n+1}\,x^{n+1} + C \qquad (\text{if } n \neq 1)$
2. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$
3. $\displaystyle\int a^x\,dx = \frac{1}{\ln(a)}\,a^x + C \qquad (\text{if } a > 0)$
4. $\displaystyle\int \ln(x)\,dx = x\,\ln(x) - x + C$
5. $\displaystyle\int \sin(x)\,dx = -\cos(x) + C$
6. $\displaystyle\int \cos(x)\,dx = \sin(x) + C$
7. $\displaystyle\int \tan(x)\,dx = -\ln|\cos(x)| + C$

PRODUCTS OF e^x, $\cos(x)$, $\sin(x)$

8. $\displaystyle\int e^{ax}\sin(bx)\,dx = \frac{1}{a^2+b^2}\,e^{ax}\left[a\sin(bx) - b\cos(bx)\right] + C$
9. $\displaystyle\int e^{ax}\cos(bx)\,dx = \frac{1}{a^2+b^2}\,e^{ax}\left[a\cos(bx) + b\sin(bx)\right] + C$
10. $\displaystyle\int \sin(ax)\sin(bx)\,dx = \frac{1}{b^2-a^2}\left[a\cos(ax)\sin(bx) - b\sin(ax)\cos(bx)\right] + C \qquad (\text{if } a \neq b)$
11. $\displaystyle\int \cos(ax)\cos(bx)\,dx = \frac{1}{b^2-a^2}\left[b\cos(ax)\sin(bx) - a\sin(ax)\cos(bx)\right] + C \qquad (\text{if } a \neq b)$
12. $\displaystyle\int \sin(ax)\cos(bx)\,dx = \frac{1}{b^2-a^2}\left[b\sin(ax)\sin(bx) + a\cos(ax)\cos(bx)\right] + C \qquad (\text{if } a \neq b)$

PRODUCT OF POLYNOMIAL $p(x)$ WITH $\ln(x)$, e^x, $\cos(x)$, $\sin(x)$

13. $\displaystyle\int x^n\ln(x)\,dx = \frac{1}{n+1}\,x^{n+1}\ln(x) - \frac{1}{(n+1)^2}x^{n+1} + C \qquad (\text{if } n \neq -1)$
14. $\displaystyle\int p(x)e^{ax}\,dx = \frac{1}{a}p(x)e^{ax} - \frac{1}{a}\int p'(x)\,e^{ax}\,dx$

$$= \frac{1}{a}p(x)e^{ax} - \frac{1}{a^2}p'(x)e^{ax} + \frac{1}{a^3}p''(x)\,e^{ax} - \cdots \qquad (+-+-+-\ldots)$$

15. $\displaystyle\int p(x)\sin(ax)\,dx = -\frac{1}{a}p(x)\cos(ax) + \frac{1}{a}\int p'(x)\cos(ax)\,dx$

$$= -\frac{1}{a}p(x)\cos(ax) + \frac{1}{a^2}p'(x)\sin(ax) + \frac{1}{a^3}p''(x)\cos(ax) - \cdots \qquad (-++--++\ldots)$$

16. $\displaystyle\int p(x)\cos(ax)\,dx = \frac{1}{a}p(x)\sin(ax) - \frac{1}{a}\int p'(x)\sin(ax)\,dx$

$$= \frac{1}{a}p(x)\sin(ax) + \frac{1}{a^2}p'(x)\cos(ax) - \frac{1}{a^3}p''(x)\sin(ax) - \cdots \qquad (++--++\ldots)$$

INTEGER POWERS OF $\sin(x)$, $\cos(x)$

17. $\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\,\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx + C \qquad (\text{if } n > 0)$

18. $\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\,\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx + C \qquad (\text{if } n > 0)$

19. $\int \frac{1}{\sin^m(x)}\,dx = \frac{-1}{m-1}\,\frac{\cos(x)}{\sin^{m-1}(x)} + \frac{m-2}{m-1}\int \frac{1}{\sin^{m-2}(x)}\,dx \qquad (\text{if } m > 1)$

20. $\int \frac{1}{\sin(x)}\,dx = \frac{1}{2}\ln\left|\frac{\cos(x)-1}{\cos(x)+1}\right| + C$

21. $\int \frac{1}{\cos^m(x)}\,dx = \frac{1}{m-1}\,\frac{\sin(x)}{\cos^{m-1}(x)} + \frac{m-2}{m-1}\int \frac{1}{\cos^{m-2}(x)}\,dx \qquad (\text{if } m > 1)$

22. $\int \frac{1}{\cos(x)}\,dx = \frac{1}{2}\ln\left|\frac{\sin(x)+1}{\cos(x)-1}\right| + C$

23. $\int \sin^m(x)\cos^n(x)\,dx$

If m is odd, let $w = \cos(x)$. If n is odd, let $w = \sin(x)$. If both m and n are even and nonnegative, convert all to $\sin(x)$ or all to $\cos(x)$ (using $\cos^2(x) + \sin^2(x) = 1$), and use 17 or 18. If m and n are even and one of them is negative, convert to whichever function is in the denominator and use 19 or 21. If both m and n are even and negative, substitute $w = \tan(x)$.

QUADRATIC IN THE DENOMINATOR

24. $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

25. $\int \frac{bx+c}{x^2+a^2}\,dx = \frac{b}{2}\ln|x^2+a^2| + \frac{c}{a}\arctan\left(\frac{x}{a}\right) + C \qquad (\text{if } a \neq 0)$

26. $\int \frac{1}{(x-a)(x-b)}\,dx = \frac{1}{a-b}(\ln|x-a| - \ln|x-b|) + C \qquad (\text{if } a \neq b)$

27. $\int \frac{cx+d}{(x-a)(x-b)}\,dx = \frac{1}{a-b}[(ac+d)\ln|x-a| - (bc+d)\ln|x-b|] + C \qquad (\text{if } a \neq b)$

INTEGRANDS INVOLVING $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$, $\quad a > 0$

28. $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$

29. $\int \frac{1}{\sqrt{x^2 \pm a^2}}\,dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$

30. $\int \sqrt{a^2 \pm x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2 \pm x^2} + a^2\int \frac{1}{\sqrt{a^2 \pm x^2}}\,dx\right) + C$

31. $\int \sqrt{x^2 - a^2}\,dx = \frac{1}{2}\left(x\sqrt{x^2 - a^2} - a^2\int \frac{1}{\sqrt{x^2 - a^2}}\,dx\right) + C$