ANSWERS 3/15

1. $\dfrac{\partial z}{\partial x} = \sin y + 2x = \dfrac{\sqrt{2}}{2}$

   $\dfrac{\partial z}{\partial y} = x \cdot \cos y = 0$

   $D_{\hat{v}} f = \left(-\dfrac{8}{17}\right)\left(\dfrac{\sqrt{2}}{2}\right) + \left(\dfrac{15}{17}\right)(0) = \boxed{-\dfrac{4\sqrt{2}}{17}}$

2. $F(x,y,z) = x^2 - y^3 - z^2 = 0$

   $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{2x}{-2z} = \dfrac{x}{z} = 3$

   $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{-3y^2}{-2z} = -\dfrac{12}{2} = -6$

   $D_{\hat{v}} f = \left(-\dfrac{1}{\sqrt{2}}\right)(3) + \left(-\dfrac{1}{\sqrt{2}}\right)(-6) = \dfrac{-3+6}{\sqrt{2}} = \boxed{\dfrac{3}{\sqrt{2}}}$

3. $\hat{v} = \dfrac{1}{\sqrt{10}}\langle 1, -3 \rangle$

   $\dfrac{\partial z}{\partial x} = \dfrac{y^2}{xy^2} = \dfrac{1}{x} = 1$

   $\dfrac{\partial z}{\partial y} = \dfrac{2xy}{xy^2} = \dfrac{2}{y} = \dfrac{2}{e}$

   $D_{\hat{v}} f = \dfrac{1}{\sqrt{10}}\left[1 \cdot 1 + (-3)\dfrac{2}{e}\right] = \boxed{\dfrac{e-6}{e\sqrt{10}}}$

4. $\hat{v} = \dfrac{1}{13}\langle -5, 12 \rangle$

   $F(x,y,z) = e^{yz} - zx + \cos(xy)$

   $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{-z - y\sin(xy)}{ye^{yz} - x} = \dfrac{1+0}{0-1} = -1$

   $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{ze^{yz} - x\sin(xy)}{ye^{yz} - x} = \dfrac{-1+0}{0-1} = 1$

   $D_{\hat{v}} f = \dfrac{1}{13}\big((-5)(-1) + (12)(1)\big)$

   $= \boxed{\dfrac{17}{13}}$

⑤

$\nabla f = \langle -2-2x, 4-8y \rangle =$

$\nabla f = 0$ at $(-1, \frac{1}{2})$.

$f(-1, \frac{1}{2}) = 9 + 2 + 2 - 1 - 1 = 11$

$\Rightarrow$ critical point at $\boxed{(-1, \frac{1}{2}, 11)}$

$f_{xx} = -2, \quad f_{xy} = 0, \quad f_{yy} = -8$

$D = (-2)(-8) - (0)^2 = 16$

$D > 0 \Rightarrow$ local max/min.

$f_{xx} < 2 \Rightarrow$ $\boxed{\text{local max}}$

⑥ $z = x + y + x^2y + xy^2$

$\nabla z = \langle 1 + 2xy + y^2, 1 + 2xy + x^2 \rangle$

$$\begin{aligned} &1 + 2xy + y^2 = 0 \\ -&(1 + 2xy + x^2 = 0) \\ \hline &y^2 - x^2 = 0 \end{aligned}$$

$y = x$ or $y = -x$.

if $y = x$

$1 + 2x^2 + x^2 = 0$

$3x^2 = -1$ can't happen.

if $y = -x$,

$1 - 2x^2 + x^2 = 0$

$1 - x^2 = 0$

$x = \pm 1$.

so $(1, -1, 0)$

$(-1, 1, 0)$ are the critical points.

$f_{xx} = 2y$

$f_{xy} = 2x + 2y$

$f_{yy} = 2x$

$D = (2y)(2x) - (2x+2y)^2$

$= 4xy - 4x^2 - 8xy - 4y^2$

$= -4(x^2 + xy + y^2)$.

$(1, -1, 0)$ $D = -4(1 - 1 + 1) < 0$

SADDLE POINT

$(-1, 1, 0)$ $D = -4(1 - 1 + 1) < 0$

SADDLE POINT

(7) $z = e^x \cos y$

$\nabla f = \langle e^x \cos y, -e^x \sin y \rangle$

e^x is never 0, so we need $\cos y = 0$ and $\sin y = 0$ simultaneously.

This can never happen, so there are no critical points.

(8) $z = x \sin y$

$\nabla f = \langle \sin y, x \cos y \rangle$

$\sin y = 0$, so $y = n\pi$ for some integer n.

Then $\cos y = \pm 1$, so we need $x = 0$.

Thus $(0, n\pi, 0)$ is a critical point for all integers n,

$f_{xx} = 0$, $f_{xy} = \cos y$, $f_{yy} = -x \sin y$

$D = (0)(-x\sin y) - (\cos y)^2 = -(\cos y)^2 = -(\cos(n\pi))^2 = -(\pm 1)^2 = -1.$

Thus $\boxed{(0, n\pi, 0) \text{ is a saddle point}}$ for all integers n.

(9) $z = (x^2+y^2)e^{y^2-x^2}$

$\nabla z = \langle (2x + (x^2+y^2)(-2x))e^{y^2-x^2}, (2y + (x^2+y^2)(2y))e^{y^2-x^2} \rangle$

$2x - 2x^3 - 2xy^2 = 0 \longrightarrow 2x(1-x^2-y^2) = 0$

$2y + 2x^2y + 2y^3 = 0 \longrightarrow 2y(1+x^2+y^2) = 0$

cant have $1+x^2+y^2 = 0$,

So $y = 0$.

Then $2x(1-x^2) = 0$

$x = 0, 1, -1.$

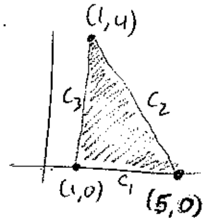
$f_{xx} = [(2-6x^2-2y^2) + (2x-2x^3-2xy^2)(-2x)]e^{y^2-x^2}$

$f_{xy} = [(-4xy) + (2x-2x^3-2xy^2)(2y)]e^{y^2-x^2}$

$f_{yy} = [(2+2x^2+6y^2) + (2y+2x^2y+2y^3)]e^{y^2-x^2}$

	$x=0$, $y=0$, $z=0$	$x=1$, $y=0$, $z=1/e$	$x=-1$, $y=0$, $z=-\frac{1}{e}$
f_{xx}	2	$-\frac{4}{e}$	$-\frac{4}{e}$
f_{xy}	0	0	0
f_{yy}	2	$\frac{4}{e}$	$\frac{4}{e}$
	$D = 4$	$D = -\frac{16}{e^2}$	$D = -\frac{16}{e^2}$
	MINIMUM	SADDLE POINTS	

(10) $\nabla f = \langle 4, -5 \rangle \Rightarrow$ no critical points.



C_1: $y=0$, $1 \le x \le 5$ $\quad f(x,0) = 1+4x \quad$ max at $x=5$ $(5,0,21)$; min at $x=1$ $(1,0,5)$

C_2: $y = -x+5$, $1 \le x \le 5$ $\quad f(x,-x+5) = 1+4x+5x-25 = -24+9x \quad$ max at $x=5$ $(5,0,21)$; min at $x=1$ $(1,4,-15)$

C_3: $x=1$, $0 \le y \le 4$ $\quad f(1,y) = 5-5y \quad$ min at $y=4$ $(1,4,-15)$; max at $y=0$ $(1,0,5)$

OF the three candidates

$(5,0,21) \leftarrow$ Abs max

$(1,0,5)$

$(1,4,-15) \leftarrow$ Abs min

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(11) $\nabla f = \langle 2x+2xy, 2y+x^2 \rangle$

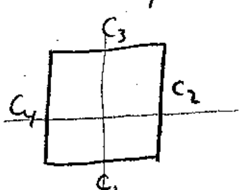
$2x(1+y)=0$

$x=0$ or $y=-1$

$y=0$ $\quad x=\pm\sqrt{2}$

Critical points: $(0,0,4)$

$(\sqrt{2},-1,5)$, $(-\sqrt{2},-1,5)$ } not in domain



C_1: $y=-1$, $-1 \le x \le 1$ $\quad f(x,-1) = x^2+1-x^2+4 = 5 \Rightarrow$ all pts on this line are candidates: $(x,-1,5)$.

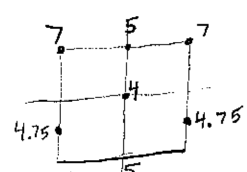
C_2: $x=1$, $-1 \le y \le 1$ $\quad f(1,y) = y^2+y+5 \quad$ max at $y=1$ $(1,1,7)$; min at $y=-\frac{1}{2}$ $(1,-\frac{1}{2},4.75)$

C_3: $y=1$, $-1 \le x \le 1$ $\quad f(x,1) = 2x^2+5 \quad$ max at $x=1$ $(1,1,7)$; min at $x=0$ $(0,1,5)$

C_4: $x=-1$, $-1 \le y \le 1$ $\quad f(-1,y) = y^2+y+5 \quad$ max at $y=1$ $(-1,1,7)$; min at $y=-\frac{1}{2}$ $(-1,-\frac{1}{2},4.75)$

$\Rightarrow$ Abs max at $(1,1,7)$

Abs min at $(0,0,4)$



⑫ I

$\nabla f = \langle 4x^3 - 4y,\ 4y^3 - 4x \rangle$

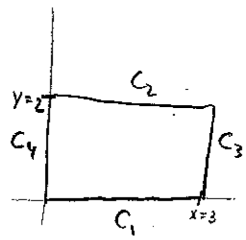
$y = x^3,\ x = y^3$

$x = x^9$

$x = -1, 0, 1$

critical pts: $(0,0,2)$, $(1,1,0)$, $(-1,-1,0)$ ← not in domain.

II



C_1: $y=0$, $0 \le x \le 3$ $\quad f(x,0) = x^4 + 2$ $\quad$ max at $x=3$ $(3,0,83)$; min at $x=0$ $(0,0,2)$

C_2: $y = 2$, $0 \le x \le 3$ $\quad f(x,2) = x^4 - 8x + 18$

where is this max/min?

$4x^3 - 8 = 0$

$x = \sqrt[3]{2}$

$(\sqrt[3]{2},\ 2,\ 18 - 6\sqrt[3]{2})$

$(0, 2, 18)$

$(3, 2, 75)$

$(3,2,75)$, $(\sqrt[3]{2}, 2, 18 - 6\sqrt[3]{2})$

C_3: $x = 3$, $0 \le y \le 2$ $\quad f(3,y) = y^4 - 12y + 83$

where is this max/min?

$4y^3 - 12 = 0$

$y = \sqrt[3]{3}$

$(3,\ \sqrt[3]{3},\ 83 - 9\sqrt[3]{3})$ min on C_3

$(3, 0, 83)$ max on C_3

$(3, 2, 75)$

$(3,0,83)$, $(3, 2, 83 - 9\sqrt[3]{3})$

C_4: $x = 0$, $0 \le y \le 2$ $\quad f(0,y) = y^4 + 2$

max at $y=2$ $(0,2,18)$

min at $y=0$ $(0,0,2)$

III possibilities for z are

$2, 0, 83, 75, 18 - 6\sqrt[3]{2}, 83 - 9\sqrt[3]{3}, 18$

abs max at $(3,0,83)$

abs min at $(1,1,0)$

(13) point (x,y,z) minimizes the distance.

$$x+y-z=1 \Rightarrow x+y=z+1$$

$$\text{dist}=\sqrt{(x-2)^2+(y-1)^2+(z+1)^2}$$

$$\text{dist}^2=(x-2)^2+(y-1)^2+(z+1)^2=(x-2)^2+(y-1)^2+(x+y)^2$$

$$\frac{\partial}{\partial x}\text{dist}^2=2(x-2)+2(x+y)$$

$$\frac{\partial}{\partial y}\text{dist}^2=2(y-1)+2(x+y).$$

$$\begin{aligned} x-2+x+y&=0 \\ y-1+x+y&=0 \end{aligned} \quad \begin{aligned} 2x+y-2&=0 \\ x+2y-1&=0 \end{aligned} \Rightarrow \quad x=1,\ y=0. \Rightarrow z=0.$$

$$\text{dist}=\sqrt{1^2+1^2+1^2}=\boxed{\sqrt{3}}$$

(14) point (x,y,z) minimizes the distance

$$z^2=x^2+y^2$$

$$\begin{aligned}\text{dist}^2&=(x-4)^2+(y-2)^2+z^2 \\ &=(x-4)^2+(y-2)^2+x^2+y^2\end{aligned}$$

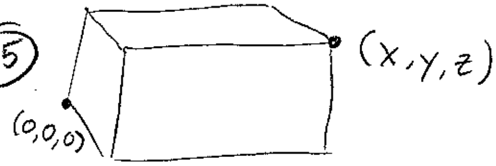
$$\frac{\partial}{\partial x}\text{dist}^2=2(x-4)+2x=4x-8=0$$

$$\frac{\partial}{\partial y}\text{dist}^2=2(y-2)+2y=4y-4=0$$

$$x=2,\ y=0,\ z=\pm 2.$$

$$\text{dist}=\sqrt{2^2+2^2+2^2}=\boxed{\sqrt{12}}$$

(15)



$$x+2y+3z=6$$

maximize $V = xyz = (6-2y-3z)yz$

$$= 6yz - 2y^2z - 3yz^2$$

$$\frac{\partial V}{\partial y} = 6z - 4yz - 3z^2 = 0$$

$$z(6-4y-3z)=0$$

$$\frac{\partial V}{\partial z} = 6y - 2y^2 - 6yz = 0$$

$$2y(3-y-3z)=0$$

if $z=0$, then $2y(3-y)=0$

$y=0$ or $y=3$

if $z = \frac{6-4y}{3}$, $2y(3-y-6+4y)=0$

$$2y(-3+3y)=0$$

$y=0$, $z=2$ or $y=1$, $z=\frac{2}{3}$

Critical pts: $(6,0,0)$ vol $=0$

$(0,3,0)$ vol $=0$

$(0,0,2)$ vol $=0$

$(2,1,\frac{2}{3})$ vol $=\frac{4}{3}$

Max at $(2,1,\frac{2}{3})$

volume $= \frac{4}{3}$

Question: we get these 0 volume points because they are minima and therefore critical points. why dont we get $(4,1,0)$?

(16)

maximize $V_i = xyz$

Subject to $x+y+z=c$

$$V = xy(c-x-y)$$
$$= cxy - x^2y - xy^2$$
$$\frac{\partial V}{\partial x} = cy - 2xy - y^2$$
$$y(c-2x-y)=0$$
$$\frac{\partial V}{\partial y} = cx - x^2 - 2xy$$
$$x(c-x-2y)=0.$$

if $y=0$, then $x(c-x)=0$

$x=0$ or $x=c$.

if $y=c-2x$ then $x(c-x-2c+4x)=0$

$x(-c+3x)=0$

$x=0$ or $x=\frac{c}{3}$

$y=c$ $\quad$ $y=\frac{c}{3}$

Critical points: $(0,0,c)$ vol $=0$

$(c,0,0)$ vol $=0$

$(0,c,0)$ vol $=0$

$(\frac{c}{3}, \frac{c}{3}, \frac{c}{3})$ vol $= \frac{c^3}{27}$

dimensions are $\frac{c}{3}, \frac{c}{3}, \frac{c}{3}$