Sylver Coinage

How to play

There are two people who are in charge of the national mint.

They alternate naming new denominations of coins to produce

They can't name a denomination if it is the sum of existing denominations.

Example: After 5 and 7 have been named, some examples of illegal plays are

12 = 5+7   10 = 5+5   19 = 5+7+7   24 = 5+5+7+7

If you name 1, then no more types of coins can be minted, the mint workers lose their jobs, and these newly unemployed people will throw you off the high tower.

Length

Question: Can we play cooperatively and protect each other from the angry mob by making the game go on forever?

Length

Unbounded :

Length

Unbounded :

1000, 999, 998, 997, ... , 3, 2, 1

Length

Unboundeded :

1000, 999, 998, 997, ... , 3, 2, 1

Unboundedly unbounded:

Length

Unbounded :

1000, 999, 998, 997, ... , 3, 2, 1

Unboundedly unbounded:

2^{1000}, 2^{999}, 2^{998}, ... , 8, 4, 2,

Length

Unbounded :

1000, 999, 998, 997, ... , 3, 2, 1

Unboundedly unbounded:

2^{1000}, 2^{999}, 2^{998}, ... , 8, 4, 2,

100001, 99999, 99997, ... , 7, 5, 3, 1

Length

Unbounded :

1000, 999, 998, 997, ... , 3, 2, 1

Unboundedly unbounded:

2^{1000}, 2^{999}, 2^{998}, ... , 8, 4, 2,

100001, 99999, 99997, ... , 7, 5, 3, 1

Unboundedly unboundedly unbouded:

Length

Unbounded :

$$1000, 999, 998, 997, \ldots , 3, 2, 1$$

Unboundedly unbounded:

$$2^{1000}, 2^{999}, 2^{998}, \ldots , 8, 4, 2,$$
$$100001, 99999, 99997, \ldots , 7, 5, 3, 1$$

Unboundedly unboundedly unbouded:

$$6^{1000}, 6^{999}, 6^{998}, \ldots , 216, 36, 6,$$
$$2^{10000}, 2^{9999}, 2^{9998}, \ldots , 8, 4, 2,$$
$$100001, 99999, 99997, \ldots , 7, 5, 3, 1$$

Length

Unbounded :

$$1000, 999, 998, 997, \ldots , 3, 2, 1$$

Unboundedly unbounded:

$$2^{1000}, 2^{999}, 2^{998}, \ldots , 8, 4, 2,$$
$$100001, 99999, 99997, \ldots , 7, 5, 3, 1$$

Unboundedly unboundedly unbouded:

$$6^{1000}, 6^{999}, 6^{998}, \ldots , 216, 36, 6,$$
$$2^{10000}, 2^{9999}, 2^{9998}, \ldots , 8, 4, 2,$$
$$100001, 99999, 99997, \ldots , 7, 5, 3, 1$$

Et cetera

Length

Theorem (Sylvester): The game will end after finitely many plays

Length

Theorem (Sylvester): The game will end after finitely many plays

Proof: At any point in time, consider the GCD of all played numbers. This quantity will never increase, and there are only finitely many plays that will keep the GCD the same. After that the GCD must decrease, and once the GCD is 1, there are only finitely many plays left.

Length

Theorem: (Sylvester) The game will end after finitely many plays

Proof: At any point in time, consider the GCD of all played numbers. This quantity will never increase, and there are only finitely many plays that will keep the GCD the same. After that the GCD must decrease, and once the GCD is 1, there are only finitely many plays left.

GCD = 42

Example:
Played numbers: 42

Length

Theorem: (Sylvester) The game will end after finitely many plays

Proof: At any point in time, consider the GCD of all played numbers. This quantity will never increase, and there are only finitely many plays that will keep the GCD the same. After that the GCD must decrease, and once the GCD is 1, there are only finitely many plays left.

GCD = 6

Example:
Played numbers: 42 78

Length

Theorem: (Sylvester) The game will end after finitely many plays

Proof: At any point in time, consider the GCD of all played numbers. This quantity will never increase, and there are only finitely many plays that will keep the GCD the same. After that the GCD must decrease, and once the GCD is 1, there are only finitely many plays left.

GCD = 6

Example:
Played numbers: 42 78 24

Length

Theorem: (Sylvester) The game will end after finitely many plays

Proof: At any point in time, consider the GCD of all played numbers. This quantity will never increase, and there are only finitely many plays that will keep the GCD the same. After that the GCD must decrease, and once the GCD is 1, there are only finitely many plays left.

Example:

GCD = 6

Played numbers: 42 78 24

Playable numbers: 6, 12, 18, 30, 36, 54, 60,
1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, ...

Length

Theorem: (Sylvester) The game will end after finitely many plays

Proof: At any point in time, consider the GCD of all played numbers. This quantity will never increase, and there are only finitely many plays that will keep the GCD the same. After that the GCD must decrease, and once the GCD is 1, there are only finitely many plays left.

GCD = 6

Example:
Played numbers: 42 78 24 18

Playable numbers: 6, 12, 30, 60,
1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, ...

Length

Theorem: (Sylvester) The game will end after finitely many plays

Proof: At any point in time, consider the GCD of all played numbers. This quantity will never increase, and there are only finitely many plays that will keep the GCD the same. After that the GCD must decrease, and once the GCD is 1, there are only finitely many plays left.

GCD = 6

Example:
Played numbers: 42 78 24 18 6

Playable numbers:
1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, ...

Length

Theorem: (Sylvester) The game will end after finitely many plays

Proof: At any point in time, consider the GCD of all played numbers. This quantity will never increase, and there are only finitely many plays that will keep the GCD the same. After that the GCD must decrease, and once the GCD is 1, there are only finitely many plays left.

Example:

GCD = 3

Played numbers: 42 78 24 18 6 15

Playable numbers: 3, 9

1, 2, 4, 5, 7, 8, 10, 11, 13, 14, ...

Length

Theorem: (Sylvester) The game will end after finitely many plays

Proof: At any point in time, consider the GCD of all played numbers. This quantity will never increase, and there are only finitely many plays that will keep the GCD the same. After that the GCD must decrease, and once the GCD is 1, there are only finitely many plays left.

Example: GCD = 3

Played numbers: 42 78 24 18 6 15 3

Playable numbers:

1, 2, 4, 5, 7, 8, 10, 11, 13, 14, ...

Length

Theorem: (Sylvester) The game will end after finitely many plays

Proof: At any point in time, consider the GCD of all played numbers. This quantity will never increase, and there are only finitely many plays that will keep the GCD the same. After that the GCD must decrease, and once the GCD is 1, there are only finitely many plays left.

GCD = 1

Example:
Played numbers: 42 78 24 18 6 15
3 10
Playable numbers: 1, 2, 4, 5, 7, 8, 11, 14, 17

Length

Theorem: (Sylvester) The game will end after finitely many plays

Proof: At any point in time, consider the GCD of all played numbers. This quantity will never increase, and there are only finitely many plays that will keep the GCD the same. After that the GCD must decrease, and once the GCD is 1, there are only finitely many plays left.

GCD = 1

Example:

Played numbers: 42 78 24 18 6 15
3 10

Playable numbers: 1, 2, 4, 5, 7, 8, 11, 14, 17

Game will end within the next 9 moves

Mating Strategies

What is the correct response to 4?

Mating Strategies

What is the correct response to 4?
Guess: Is it 5?

Mating Strategies

What is the correct response to 4?
Guess: Is it 5?

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23

Mating Strategies

What is the correct response to 4?
Guess: Is it 5?

Available plays:

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23

Mating Strategies

What is the correct response to 4?
Guess: Is it 5?

Available plays:

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23

If I play 2, you'll respond with 3 and vice versa.

Mating Strategies

What is the correct response to 4?
Guess: Is it 5?

Available plays:

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23

If I play 2, you'll respond with 3 and vice versa.

If I play 6, you'll respond with 7 and vice versa.

Mating Strategies

What is the correct response to 4?
Guess: Is it 5?

Available plays:

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23

If I play 2, you’ll respond with 3 and vice versa.

If I play 6, you’ll respond with 7 and vice versa.

We say 2 and 3 are “Mated” and 6 and 7 are “Mated”.

Mating Strategies

What is the correct response to 4?
Guess: Is it 5?

Available plays:

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23

If I play 2, you'll respond with 3 and vice versa.

If I play 6, you'll respond with 7 and vice versa.

We say 2 and 3 are "Mated" and 6 and 7 are "Mated".
Actually, I'll play 11, and force you to take 2,3,6, or 7.

Mating Strategies

What is the correct response to 4?
Guess: Is it 5?

Available plays:

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23

If I play 2, you'll respond with 3 and vice versa.

If I play 6, you'll respond with 7 and vice versa.

We say 2 and 3 are "Mated" and 6 and 7 are "Mated".
Actually, I'll play 11, and force *you* to take 2,3,6, or 7.
So 5 was not a good idea.

Mating Strategies

What is the correct response to 4?
Guess: Is it 5?

Mating Strategies

What is the correct response to 4?
Guess: Is it 5?

No.

Mating Strategies

What is the correct response to 4?
Guess: Is it 6?

Mating Strategies

What is the correct response to 4?
Guess: Is it 6?

Available plays:

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23

Mating Strategies

What is the correct response to 4?
Guess: Is it 6?

Available plays:

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23

2 and 3 are mated

Mating Strategies

What is the correct response to 4?
Guess: Is it 6?

Available plays:

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23

2 and 3 are mated
5 and 7 are mated

Mating Strategies

What is the correct response to 4?
Guess: Is it 6?

Available plays:

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23

2 and 3 are mated
5 and 7 are mated
9 and 11 are mated

Mating Strategies

What is the correct response to 4?
Guess: Is it 6?

Available plays:

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23

2 and 3 are mated
5 and 7 are mated
9 and 11 are mated
13 and 15 are mated

Mating Strategies

What is the correct response to 4?
Guess: Is it 6?

Available plays:

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23

2 and 3 are mated
5 and 7 are mated
9 and 11 are mated
13 and 15 are mated
Etc.

Mating Strategies

What is the correct response to 4?
Guess: Is it 6?

Available plays:

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23

2 and 3 are mated
5 and 7 are mated
9 and 11 are mated
13 and 15 are mated
Etc.

Any play I make, you have a good response, so 6 was a good play for you to make.

Mating Strategies

What is the correct response to 4?
Guess: Is it 6?

Mating Strategies

What is the correct response to 4?
Guess: Is it 6?

Yes.

P-positions so far

P-positions so far

{ 2, 3 }

	1
2	3
4	5
6	7
8	9
10	11

P-positions so far

{ 2, 3 }
{ 4, 5, 11 }

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23

P-positions so far

{ 2, 3 }
{ 4, 5, 11 }
{ 4, 6 }

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23

P-positions so far

{ 2, 3 }
{ 4, 5, 11 }
{ 4, 6 }

So far we know that 1, 2, 3, 4, and 6 are all bad starting plays.
Are there any good starting plays?

Good first plays

Theorem (Hutchings):

If a and b are co-prime and $\{ a, b \} \neq \{ 2, 3 \}$ then $\{ a, b \}$ is an N-position

Good first plays

Theorem (Hutchings):

If a and b are co-prime and $\{ a, b \} \neq \{ 2, 3 \}$ then $\{ a, b \}$ is an N-position

Corollary 1: If $p \geq 5$ is prime, then $\{ p \}$ is a P-position

Good first plays

Theorem (Hutchings):

If a and b are co-prime and $\{ a, b \} \neq \{ 2, 3 \}$ then $\{ a, b \}$ is an N-position

Corollary 1: If $p \geq 5$ is prime, then $\{ p \}$ is a P-position

"*p*-positions are P-positions"

Good first plays

Theorem (Hutchings):

If a and b are co-prime and $\{ a, b \} \neq \{ 2, 3 \}$ then $\{ a, b \}$ is an N-position

Corollary 1: If $p \geq 5$ is prime, then $\{ p \}$ is a P-position

Good first plays

Theorem (Hutchings):

If a and b are co-prime and $\{ a, b \} \neq \{ 2, 3 \}$ then $\{ a, b \}$ is an N-position

Corollary 1: If $p \geq 5$ is prime, then $\{ p \}$ is a P-position

Proof: After the first player plays p, the second player cannot play a multiple of p so he must play something co-prime to p, resulting in an N-position

Good first plays

Theorem (Hutchings):

If a and b are co-prime and $\{ a, b \} \neq \{ 2, 3 \}$ then $\{ a, b \}$ is an N-position

Corollary 1: If $p \geq 5$ is prime, then $\{ p \}$ is a P-position

Good first plays

Theorem (Hutchings):

If a and b are co-prime and $\{ a, b \} \neq \{ 2, 3 \}$ then $\{ a, b \}$ is an N-position

Corollary 1: If $p \geq 5$ is prime, then $\{ p \}$ is a P-position

Corollary 2: If n is a composite number, $n \neq 2^a 3^b$ then $\{ n \}$ is an N-position

Good first plays

Theorem (Hutchings):

If a and b are co-prime and $\{ a, b \} \neq \{ 2, 3 \}$ then $\{ a, b \}$ is an N-position

Corollary 1: If $p \geq 5$ is prime, then $\{ p \}$ is a P-position

Corollary 2: If n is a composite number, $n \neq 2^a 3^b$ then $\{ n \}$ is an N-position

"*n*-positions are N-positions"

Good first plays

Theorem (Hutchings):

If a and b are co-prime and $\{ a, b \} \neq \{ 2, 3 \}$ then $\{ a, b \}$ is an N-position

Corollary 1: If $p \geq 5$ is prime, then $\{ p \}$ is a P-position

Corollary 2: If n is a composite number, $n \neq 2^a 3^b$ then $\{ n \}$ is an N-position

Good first plays

Theorem (Hutchings):

If a and b are co-prime and $\{ a, b \} \neq \{ 2, 3 \}$ then $\{ a, b \}$ is an N-position

Corollary 1: If $p \geq 5$ is prime, then $\{ p \}$ is a P-position

Corollary 2: If n is a composite number, $n \neq 2^a 3^b$ then $\{ n \}$ is an N-position

Proof: If the first player plays n, then the second player should play p, a prime factor of n. Now the game is in the same position as if player 2 had started by playing p which is a P-position.

Good first plays

Theorem (Hutchings):

If a and b are co-prime and $\{ a, b \} \neq \{ 2, 3 \}$ then $\{ a, b \}$ is an N-position

Corollary 1: If $p \geq 5$ is prime, then $\{ p \}$ is a P-position

Corollary 2: If n is a composite number, $n \neq 2^a 3^b$ then $\{ n \}$ is an N-position

These three results are basically all there is to know, so we'll write them on the board before we prove Hutchings' Theorem.

Proof of Hutchings' Theorem

Proof by strategy stealing.

Proof of Hutchings' Theorem

Proof by strategy stealing.

Let a and b be coprime and t be the largest playable number.

Proof of Hutchings' Theorem

Proof by strategy stealing.

Let a and b be coprime and t be the largest playable number.
$t = ab - a - b$

Proof of Hutchings' Theorem

Proof by strategy stealing.

Let a and b be coprime and t be the largest playable number.
$t = ab - a - b$

Case 1: $\{ a,b,t \}$ is a P-position
    Then $\{ a,b \}$ is an N-position because t is a good next move.

Proof of Hutchings' Theorem

Proof by strategy stealing.

Let a and b be coprime and t be the largest playable number.
$t = ab - a - b$

Case 1: { a,b,t } is a P-position
Then { a,b } is an N-position because t is a good next move.

Case 2: { a,b,t } is an N-position
Then there is some s such that { a,b,t,s } is a P-position. However, if we play s, it will exclude t from being played, (not obvious) so { a, b, s } is a P-position and { a,b } is an N-position.

Proof of Hutchings' Theorem

Example: 9 and 11 have been played.

	1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16	17
18	19	20	21	22	23	24	25	26
27	28	29	30	31	32	33	34	35
36	37	38	39	40	41	42	43	44
45	46	47	48	49	50	51	52	53
54	55	56	57	58	59	60	61	62
63	64	65	66	67	68	69	70	71
72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98

Proof of Hutchings' Theorem

Example: 9 and 11 have been played.

	1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16	17
18	19	20	21	22	23	24	25	26
27	28	29	30	31	32	33	34	35
36	37	38	39	40	41	42	43	44
45	46	47	48	49	50	51	52	53
54	55	56	57	58	59	60	61	62
63	64	65	66	67	68	69	70	71
72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98

Note that any play will exclude 79

Proof of Hutchings' Theorem

Example: 9 and 11 have been played.

	1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16	17
18	19	20	21	22	23	24	25	26
27	28	29	30	31	32	33	34	35
36	37	38	39	40	41	42	43	44
45	46	47	48	49	50	51	52	53
54	55	56	57	58	59	60	61	62
63	64	65	66	67	68	69	70	71
72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98

Note that any play will exclude 79

Eg. 26

Proof of Hutchings' Theorem

Example: 9 and 11 have been played.

	1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16	17
18	19	20	21	22	23	24	25	26
27	28	29	30	31	32	33	34	35
36	37	38	39	40	41	42	43	44
45	46	47	48	49	50	51	52	53
54	55	56	57	58	59	60	61	62
63	64	65	66	67	68	69	70	71
72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98

Note that any play will exclude 79

Eg. 26
By adding 11, we exclude 37, 48, 59, and 70.

Proof of Hutchings' Theorem

Example: 9 and 11 have been played.

	1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16	17
18	19	20	21	22	23	24	25	26
27	28	29	30	31	32	33	34	35
36	37	38	39	40	41	42	43	44
45	46	47	48	49	50	51	52	53
54	55	56	57	58	59	60	61	62
63	64	65	66	67	68	69	70	71
72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98

Note that any play will exclude 79

Eg. 26
By adding 11, we exclude 37, 48, 59, and 70.
By adding 9, we exclude 79.

Proof of Hutchings' Theorem

Formally, let *s* be any playable number.

Proof of Hutchings' Theorem

Formally, let *s* be any playable number.

Since *a* and *b* are co-prime, we can write

$ax + by = s$ $\qquad$ $x = x_0 + nb$

$y = y_0 - na$

Proof of Hutchings' Theorem

Formally, let s be any playable number.

Since a and b are co-prime, we can write

$ax + by = s$ $\quad\quad x = x_0 + nb$

$y = y_0 - na$

Choose n so that $0 \le x < b$

Proof of Hutchings' Theorem

Formally, let s be any playable number.

Since a and b are co-prime, we can write

$ax + by = s$ $\qquad$ $x = x_0 + nb$

$y = y_0 - na$

Choose n so that $0 \leq x < b$

Since $x \neq 0$ and since x and y can’t both be positive, we have

$0 < x < b$ *and* $y < 0$.

Proof of Hutchings' Theorem

Formally, let *s* be any playable number.

Since *a* and *b* are co-prime, we can write

$ax + by = s$ $\quad$ $x = x_0 + nb$

$y = y_0 - na$

Choose *n* so that $0 \leq x < b$

Since $x \neq 0$ and since *x* and *y* can't both be positive, we have

$0 < x < b$ *and* $y < 0.$

$$t - (ax+by) = t - s$$

$$(ab-a-b) - (ax+by) = t - s$$

$$a(b-x-1) + b(-y-1) = t - s$$

$$s + a(b-x-1) + b(-y-1) = t$$

.

Proof of Hutchings' Theorem

Formally, let *s* be any playable number.

Since *a* and *b* are co-prime, we can write

$ax + by = s$ $\qquad$ $x = x_0 + nb$

$$y = y_0 - na$$

Choose *n* so that $0 \leq x < b$

Since $x \neq 0$ and since *x* and *y* can't both be positive, we have

$0 < x < b$ *and* $y < 0$.

$$t - (ax+by) = t - s$$

$$(ab-a-b) - (ax+by) = t - s$$

$$a(b-x-1) + b(-y-1) = t - s$$

$$s + a(b-x-1) + b(-y-1) = t$$

So *t* is a positive linear combination of *s, a,* and *b* as desired.

Proof of Hutchings' Theorem

Proof by strategy stealing.

Let a and b be coprime and t be the largest playable number.
$t = ab - a - b$

Case 1: { a,b,t } is a P-position
    Then { a,b } is an N-position because t is a good next move.

Case 2: { a,b,t } is an N-position
    Then there is some s such that { a,b,t,s } is a P-position. However, if we play s, it will exclude t from being played, (not obvious) so { a, b, s } is a P-position and { a,b } is an N-position.

Now What?

Now What?

	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	29
30	31	32	33	34
35	36	37	38	39
40	41	42	43	44
45	46	47	48	49
50	51	52	53	54
55	56	57	58	59
60	61	62	63	64

Now What?

We still want to know more about N- and P- positions.

Other N- and P- Positions

Definition: Let $A = \{ a_1 , a_2 , a_3 , \ldots , a_k \}$ be the set of numbers played, and let t be the topmost number not yet played. A is a <u>quiet end position</u> if for all playable s,

$$t = s + \sum x_i a_i$$

i.e., we can exclude t using only one copy of s.

Other N- and P- Positions

Definition: Let $A = \{ a_1 , a_2 , a_3 , \ldots , a_k \}$ be the set of numbers played, and let t be the topmost number not yet played. A is a quiet end position if for all playable s,

$$t = s + \sum x_i a_i$$

i.e., we can exclude t using only one copy of s.

Example: if a and b are co-prime, $\{ a, b \}$ is a quiet end position.

Other N- and P- Positions

Definition: Let $A = \{ a_1 , a_2 , a_3 , \ldots , a_k \}$ be the set of numbers played, and let t be the topmost number not yet played. A is a quiet end position if for all playable s,

$$t = s + \sum x_i a_i$$

i.e., we can exclude t using only one copy of s.

Example: if a and b are co-prime, $\{ a, b \}$ is a quiet end position.

We can use the same strategy stealing argument to show that quiet end positions are N-positions.

Other N- and P- Positions

Theorem: Let a be co-prime both to b_1 and b_2. Then

$\{ a, b_1 c^1 , b_1 c^2 , b_1 c^3 , \ldots, b_1 c^i \}$ is a quiet end position

if and only if

$\{ a, b_2 c^1 , b_2 c^2 , b_2 c^3 , \ldots, b_2 c^i \}$ is a quiet end position

Other N- and P- Positions

Theorem: Let a be co-prime both to b_1 and b_2. Then

$$\{ a, b_1 c^1 , b_1 c^2 , b_1 c^3 , \ldots, b_1 c^i \} \text{ is a quiet end position}$$

if and only if

$$\{ a, b_2 c^1 , b_2 c^2 , b_2 c^3 , \ldots, b_2 c^i \} \text{ is a quiet end position}$$

Why is this useful?

Other N- and P- Positions

Theorem: Let a be co-prime both to b_1 and b_2. Then

$\{ a, b_1 c^1 , b_1 c^2 , b_1 c^3 , \ldots, b_1 c^i \}$ is a quiet end position

if and only if

$\{ a, b_2 c^1 , b_2 c^2 , b_2 c^3 , \ldots, b_2 c^i \}$ is a quiet end position

Why is this useful?

We know that { 3, 4 } is a quiet end position

Other N- and P- Positions

Theorem: Let a be co-prime both to b_1 and b_2. Then

$\{ a, b_1 c^1 , b_1 c^2 , b_1 c^3 , \ldots, b_1 c^i \}$ is a quiet end position

if and only if

$\{ a, b_2 c^1 , b_2 c^2 , b_2 c^3 , \ldots, b_2 c^i \}$ is a quiet end position

Why is this useful?

We know that { 3, 4 } is a quiet end position
{ 3, 4, 7 } is equivalent to { 3, 4 } in this game.

Other N- and P- Positions

Theorem: Let a be co-prime both to b_1 and b_2. Then

$\{ a, b_1 c^1 , b_1 c^2 , b_1 c^3 , \ldots, b_1 c^i \}$ is a quiet end position

if and only if

$\{ a, b_2 c^1 , b_2 c^2 , b_2 c^3 , \ldots, b_2 c^i \}$ is a quiet end position

Why is this useful?

We know that { 3, 4 } is a quiet end position
{ 3, 4, 7 } is equivalent to { 3, 4 } in this game.
Since {7, 1*3, 1*4 } is a quiet end position,
so are {7, 3*3, 3*4} and {7, 5*3, 5*4}

Other N- and P- Positions

Theorem: Let a be co-prime both to b_1 and b_2. Then

$\{ a, b_1 c^1 , b_1 c^2 , b_1 c^3 , \ldots, b_1 c^i \}$ is a quiet end position

if and only if

$\{ a, b_2 c^1 , b_2 c^2 , b_2 c^3 , \ldots, b_2 c^i \}$ is a quiet end position

Why is this useful?

We know that { 3, 4 } is a quiet end position
{ 3, 4, 7 } is equivalent to { 3, 4 } in this game.
Since {7, 1*3, 1*4 } is a quiet end position,
so are {7, 3*3, 3*4} and {7, 5*3, 5*4}

So {7, 9, 12} and {7, 15, 20} are N-positions.

Other N- and P- Positions

Theorem: Let a be co-prime both to b_1 and b_2. Then

$\{ a, b_1 c^1, b_1 c^2, b_1 c^3, \ldots, b_1 c^i \}$ is a quiet end position

if and only if

$\{ a, b_2 c^1, b_2 c^2, b_2 c^3, \ldots, b_2 c^i \}$ is a quiet end position

Why is this useful?

We know that { 3, 4 } is a quiet end position
{ 3, 4, 7 } is equivalent to { 3, 4 } in this game.
Since {7, 1*3, 1*4 } is a quiet end position,
so are {7, 3*3, 3*4} and {7, 5*3, 5*4}

So {7, 9, 12} and {7, 15, 20} are N-positions.

Proof of the Quiet End Theorem

Proof by example: {7, 9, 12} is a QEP iff {7, 15, 20} is a QEP

Proof of the Quiet End Theorem

Proof by example: {7, 9, 12} is a QEP iff {7, 15, 20} is a QEP

	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69
70	71	72	73	74	75	76
77	78	79	80	81	82	83

	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69
70	71	72	73	74	75	76
77	78	79	80	81	82	83

Proof of the Quiet End Theorem

Proof by example: {7, 9, 12} is a QEP iff {7, 15, 20} is a QEP

	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69
70	71	72	73	74	75	76
77	78	79	80	81	82	83

	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69
70	71	72	73	74	75	76
77	78	79	80	81	82	83

Note that the numbers at the tops of the columns are proportional.

Proof of the Quiet End Theorem

Proof by example: {7, 9, 12} is a QEP iff {7, 15, 20} is a QEP

	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69
70	71	72	73	74	75	76
77	78	79	80	81	82	83

	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69
70	71	72	73	74	75	76
77	78	79	80	81	82	83

Note that the numbers at the tops of the columns are proportional.

Proof of the Quiet End Theorem

Proof by example: {7, 9, 12} is a QEP iff {7, 15, 20} is a QEP

	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69
70	71	72	73	74	75	76
77	78	79	80	81	82	83

	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69
70	71	72	73	74	75	76
77	78	79	80	81	82	83

Note that the numbers at the tops of the columns are proportional.

Proof of the Quiet End Theorem

Proof by example: {7, 9, 12} is a QEP iff {7, 15, 20} is a QEP

	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69
70	71	72	73	74	75	76
77	78	79	80	81	82	83

	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69
70	71	72	73	74	75	76
77	78	79	80	81	82	83

Note that the numbers at the tops of the columns are proportional.

Proof of the Quiet End Theorem

Proof by example: {7, 9, 12} is a QEP iff {7, 15, 20} is a QEP

	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69
70	71	72	73	74	75	76
77	78	79	80	81	82	83

	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69
70	71	72	73	74	75	76
77	78	79	80	81	82	83

Note that the numbers at the tops of the columns are proportional.

Proof of the Quiet End Theorem

Proof by example: {7, 9, 12} is a QEP iff {7, 15, 20} is a QEP

	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69
70	71	72	73	74	75	76
77	78	79	80	81	82	83

	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69
70	71	72	73	74	75	76
77	78	79	80	81	82	83

Note that the numbers at the tops of the columns are proportional.

Proof of the Quiet End Theorem

Proof by example: {7, 9, 12} is a QEP iff {7, 15, 20} is a QEP

	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69
70	71	72	73	74	75	76
77	78	79	80	81	82	83

	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69
70	71	72	73	74	75	76
77	78	79	80	81	82	83

Note that the numbers at the tops of the columns are proportional.

Proof of the Quiet End Theorem

Proof by example: {7, 9, 12} is a QEP iff {7, 15, 20} is a QEP

	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69
70	71	72	73	74	75	76
77	78	79	80	81	82	83

	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69
70	71	72	73	74	75	76
77	78	79	80	81	82	83

Note that the numbers at the tops of the columns are proportional.

(rest of proof on the board)

Conclusion

Good first moves:

primes of at least 5

Good second moves to bad first moves:

prime factors of at least 5

After that:

Use quiet end theorem to guide you, but the full set of N- and P- positions is hard to calculate.