A NOTE ON TORSION MODULES WITH PURE EMBEDDINGS

MARCOS MAZARI-ARMIDA

Abstract. We study Martsinkovsky-Russell torsion modules [MaRu20] with pure embeddings as an abstract elementary class. We give a model-theoretic characterization of the pure-injective and the Σ-pure-injective modules relative to the class of torsion modules assuming that the ring is right semihereditary. Our characterization of relative Σ-pure-injective modules extends the classical charactetization of [GrJe76] and [Zim77, 3.6].

We study the limit models of the class and determine when the class is superstable assuming that the ring is right semihereditary. As a corollary, we show that the class of torsion abelian groups with pure embeddings is strictly stable, i.e., stable not superstable.

1. Introduction

Martsinkovsky-Russell torsion modules were introduced in [MaRu20] as a natural generalization of torsion modules to rings that are not necessarily commutative domains (Definition 2.3). We will denote them by $\mathfrak{s}$-*torsion modules* throughout this paper. For a commutative domain, they are precisely the R-torsion modules, i.e., those modules such that every element of the module can be annihilate by a non-zero element of the ring.

For most rings the class of $\mathfrak{s}$-torsion modules is not first-order axiomatizable. For example, it is folklore that the class of torsion abelian groups is not first-order axiomatizable. For this reason, we use non-elementary model-theoretic methods to analyse the class. More precisely, we will study the class of $\mathfrak{s}$-torsion modules with pure embedding as an abstract elementary class (AEC for short).

An AEC $\mathbf{K}$ is a pair $(K, \leq_{\mathbf{K}})$ where K is a class of structures and $\leq_{\mathbf{K}}$ is a partial order on K. Additionally, the partial order on K extends the substructure relation, $\mathbf{K}$ is closed under unions of chains, and every set can be closed to a small structure in the class. The class of $\mathfrak{s}$-torsion modules with pure embedding is an abstract elementary class with amalgamation, joint embedding, and no maximal models. Moreover, it was shown in [Maz2, 4.16] that the class is stable. In this paper, assuming that the ring is right semihereditary, we study its class of limit models and use them to determine when the class is superstable. Recall that a *limit model* is a universal model with some level of homogeneity (Definition 2.9) and an AEC is *superstable* if there is a unique limit model up to isomorphims on a tail of cardinals (Definition 2.12).[2]

A difficulty when trying to understand the class of $\mathfrak{s}$-torsion modules is that the class might not be closed under pure-injective envelopes, see [Maz21a, 3.1] for the case of torsion abelian groups. Therefore, we begin by developing relative notions of pure-injectivity and Σ-pure-injectivity. The following result extends the classical result of [GrJe76] and [Zim77, 3.6] where they characterize Σ-pure-injective modules (see Remark 3.21).

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[2]A detailed account of the development of the notion of superstability in AECs can be consulted in the introductions of [GrVas17] and [Maz21b].

Lemma 3.19. *Assume R is right semihereditary and M is $\mathfrak{s}$-torsion. M is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective if and only if M has the low-pp descending chain condition.*

The study of limit models for the class of $\mathfrak{s}$-torsion modules and the characterization of superstability we obtain parallels that of previous results, [Maz21b], [Maz1] and [Maz2], with the added difficulty that we have to deal with relative pure-injective modules instead of with pure-injective or cotorsion modules. More precisely, we obtain the following result.

Theorem 4.14. *Assume R is right semihereditary and R_R is not absolutely pure. The following are equivalent.*

(1) *The class of $\mathfrak{s}$-torsion modules with pure embeddings is superstable.*
(2) *There exists a $\lambda \geq (|R| + \aleph_0)^+$ such that the class of $\mathfrak{s}$-torsion modules with pure embeddings has uniqueness of limit models of cardinality λ.*
(3) *Every limit model is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
(4) *Every $\mathfrak{s}$-torsion module is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
(5) *Every $\mathfrak{s}$-torsion module is $K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
(6) *For every $\lambda \geq |R| + \aleph_0$, the class of $\mathfrak{s}$-torsion modules with pure embeddings has uniqueness of limit models of cardinality λ.*
(7) *For every $\lambda \geq |R| + \aleph_0$, the class of $\mathfrak{s}$-torsion modules with pure embeddings is λ-stable.*

An important question that is left open is to determine if there is a ring satisfying any of the equivalent conditions of the above theorem (see Question 4.16 and Question 4.18). Nevertheless, the theorem is important as it allows us to show that certain classes are not superstable.

In particular, we use our results to show that the class of torsion abelian groups with pure embeddings is strictly stable, i.e., stable not superstable. Determining if the class was superstable was the original objective of this paper.

Theorem 5.6. *The class of torsion abelian groups with pure embeddings is λ-stable if and only if $\lambda^{\aleph_0} = \lambda$. Hence, it is strictly stable.*

This paper is part of a program to understand AECs of modules: [Maz20], [KuMa20], [Maz21b], [Maz1], [Maz21a], [Maz2]. Other papers that have studied AECs of modules include: [BCG+], [BET07], [ŠaTr12], [Sh17], [Bon20, §6] [LRV1, §6], [LRV2, §3].

The paper is divided into five sections. Section 2 has the preliminaries. Section 3 has new characterizations of relative pure-injective and Σ-pure-injective modules. Section 4 analyses the class of $\mathfrak{s}$-torsion modules with pure embeddings as an abstract elementary class. Section 5 shows how to use the previous results to show that the class of torsion abelian groups with pure embeddings is strictly stable.

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2. Preliminaries

In this section we briefly present the basic notions of module theory and abstract elementary classes that we will use in this paper. The module theoretic preliminaries include the definition of the class of $\mathfrak{s}$-torsion modules and assert some of its basic properties.

2.1. **Module theory.** All rings considered in this paper are associative with unity. We write $_RM$ to specify that M is a left R-module and M_R to specify that M is a right R-module. If we simply write M, we assume that M is a left R-module.

Recall that ϕ is a *positive primitive formula*, *pp*-formula for short, if ϕ is an existentially quantified system of linear equations. M is a *pure submodule* of N if *pp*-formulas are preserved between M and N and we denote it by $M \leq_p N$. The next family of *pp*-formulas was introduced in [Roth2].

Definition 2.1. *A pp-formula $\psi(x)$ is* low *if and only if $\psi[_R R] = 0$.*

Remark 2.2. *It is easy to show that if $\psi_1(x), \psi_2(x)$ are low formulas and $r \in R$, then $\psi_1 + \psi_2(x) := \exists y \exists z(\psi_1(y) \wedge \psi_2(z) \wedge x = y + z)$ and $r\psi_1(x) := \exists y(\psi(y) \wedge x = ry)$ are low formulas.*

Given $\mathbf{b} \in M^{<\omega}$ and $A \subseteq M$, the *pp*-type of $\mathbf{b}$ over A in M, denoted by $pp(\mathbf{b}/A, M)$, is the set of all *pp*-formulas that hold for $\mathbf{b}$ in M with parameters in A.

As mentioned in the introduction, in this paper we will study the class of $\mathfrak{s}$-torsion modules. These were introduced in [MaRu20] and studied from a model-theoretic perspective in [MaRo] and [Roth1]. Below we present their model-theoretic definition.

Definition 2.3. *We say that M is an $\mathfrak{s}$-*torsion module *if and only if for every $m \in M$ there is a low formula $\psi(x)$ such that $M \vDash \psi[m]$. We denote the class of $\mathfrak{s}$-torsion modules by $K^{\mathfrak{s}\text{-}Tor}$.*

Remark 2.4. *Let R be a commutative domain. Recall that a module M is an R-torsion module if for every $m \in M$ there is an $r \neq 0 \in R$ such that $rm = 0$. Denote the class of R-torsion modules by $K^{R\text{-}Tor}$. It was shown in [MaRu20, 2.2] that $K^{\mathfrak{s}\text{-}Tor} = K^{R\text{-}Tor}$. In particular, the class of of $\mathfrak{s}$-torsion abelian groups is precisely the class of torsion abelian groups.*

The following was introduced in [MaRu20, 2.1]. The description we present will appear in the forthcoming paper [MaRo].

Definition 2.5. *For a left R-module M, let*

$$\mathfrak{s}(M) = \{m \in M : M \vDash \psi[m] \text{ for some low formula } \psi\}.$$

Remark 2.6.

- *$M \in K^{\mathfrak{s}\text{-}Tor}$ if and only if $\mathfrak{s}(M) = M$.*
- *$K^{\mathfrak{s}\text{-}Tor}$ is closed under pure submodules and direct sums.*
- *([MaRu20, 2.19]) $\mathfrak{s}$ is a radical, i.e., for every M, N: $\mathfrak{s}(M)$ is a submodule of M, if $f : M \to N$ then $f(\mathfrak{s}(M)) \leq \mathfrak{s}(N)$, and $\mathfrak{s}(M/\mathfrak{s}(M)) = 0$.*

Remark 2.7. *It is important to notice that in general $\mathfrak{s}(\mathfrak{s}(M))$ might be different from $\mathfrak{s}(M)$, see [MaRu20, p. 69]. For this reason, for arbitrary rings it might be the case that $\mathfrak{s}(M)$ is not an $\mathfrak{s}$-torsion module.*

2.2. **Abstract elementary classes.** We summarize the notions of abstract elementary classes that are used in this paper. A more detailed introduction to abstract elementary classes from an algebraic point of view is given in [Maz21a, §2]. Abstract elementary classes were introduced by Shelah in [She88] to study those classes of structures that are axiomatizable in infinitary logics. An *abstract elementary class* $\mathbf{K}$ is a pair $(K, \leq_{\mathbf{K}})$ where K is a class of structures and $\leq_{\mathbf{K}}$ is a partial order on K. Additionally, the partial order on K extends the substructure relation, $\mathbf{K}$ is closed under unions of chains, and every set can be closed to a small structure in the class.

Given a model M, we write $|M|$ for its underlying set and $\|M\|$ for its cardinality. For an infinite cardinal λ, we denote by $\mathbf{K}_\lambda$ the models in $\mathbf{K}$ of cardinality λ. If we write $f : M \to N$ for $M, N \in K$, we assume that f is a $\mathbf{K}$-embedding unless specified otherwise. Recall that f is a $\mathbf{K}$-*embedding* if $f : M \cong f[M] \leq_{\mathbf{K}} N$. Finally, for $M, N \in K$ and $A \subseteq M$, we write $f : M \underset{A}{\to} N$ if f is a $\mathbf{K}$-embedding from M to N that fixes A point-wise.

We say that $\mathbf{K}$ has the *amalgamation property* if every span of models can be completed to a commutative square, $\mathbf{K}$ has the *joint embedding property* if every two models can be $\mathbf{K}$-embedded into a single model, and $\mathbf{K}$ has *no maximal models* if every model can be properly extended.

Shelah introduced a semantic notion of type in [Sh300], we call them *Galois-types* following [Gro02]. Intuitively, a Galois-type over a model M can be identified with an orbit of the group of automorphisms of the monster model which fixes M point-wise. The full definition can be consulted in [Maz21a, 2.6]. We denote by $\mathbf{gS}(M)$ the set of all Galois-types over M and we say that $\mathbf{K}$ is $(<\aleph_0)$-*tame* if for every $M \in K$ and $p \neq q \in \mathbf{gS}(M)$, there is a finite subset A of M such that $p\restriction_A \neq q\restriction_A$.

Definition 2.8. $\mathbf{K}$ *is* λ-stable *if* $|\mathbf{gS}(M)| \leq \lambda$ *for every* $M \in \mathbf{K}_\lambda$. *We say that* $\mathbf{K}$ *is* stable *if* $\mathbf{K}$ *is* λ*-stable for some* $\lambda \geq \mathrm{LS}(\mathbf{K})$.

A model M is *universal over* N if and only if $\|N\| = \|M\| = \lambda$ and for every $N^* \in \mathbf{K}_\lambda$ such that $N \leq_{\mathbf{K}} N^*$, there is $f : N^* \xrightarrow[N]{} M$.

Definition 2.9. *Let* λ *be an infinite cardinal and* $\alpha < \lambda^+$ *be a limit ordinal.* M *is a* (λ, α)-limit model over N *if and only if there is* $\{M_i : i < \alpha\} \subseteq \mathbf{K}_\lambda$ *an increasing continuous chain such that:*

(1) $M_0 = N$.
(2) $M = \bigcup_{i<\alpha} M_i$.
(3) M_{i+1} *is universal over* M_i *for each* $i < \alpha$.

M *is a* (λ, α)-limit model *if there is* $N \in \mathbf{K}_\lambda$ *such that* M *is a* (λ, α)*-limit model over* N. M *is a* λ-limit model *if there is a limit ordinal* $\alpha < \lambda^+$ *such that* M *is a* (λ, α)*-limit model.*

Fact 2.10 ([Sh:h, §II], [GrVan06, 2.9])**.** *Let* $\mathbf{K}$ *be an AEC with joint embedding, amalgamation, and no maximal models.* $\mathbf{K}$ *is* λ*-stable if an only if* $\mathbf{K}$ *has a* λ*-limit model.*

A model is *universal in* $\mathbf{K}_\lambda$ if it has cardinality λ and if every model in $\mathbf{K}$ of size λ can be $\mathbf{K}$-embedded into it. It is known that every λ-limit model is universal in $\mathbf{K}_\lambda$ if $\mathbf{K}$ has the joint embedding property [Maz20, 2.10].

We will also be interested in saturated models. Given $\lambda > \mathrm{LS}(\mathbf{K})$ we say that M is λ-*saturated* if every Galois-type over a $\mathbf{K}$-substructure of size strictly less than λ is realized in M. We have the following relation between saturated models and limit models.

Fact 2.11. *Let* $\mathbf{K}$ *be an AEC with joint embedding, amalgamation, and no maximal models. If* M *is a* (λ, α)*-limit model and* $\mathrm{cf}(\alpha) > \mathrm{LS}(\mathbf{K})$*, then* M *is* $\mathrm{cf}(\alpha)$*-saturated.*

We say that $\mathbf{K}$ has *uniqueness of limit models of cardinality* λ if $\mathbf{K}$ has λ-limit models and if any two λ-limit models are isomorphic.

Definition 2.12. $\mathbf{K}$ *is* superstable *if and only if* $\mathbf{K}$ *has uniqueness of limit models on a tail of cardinals.*

Superstability was first introduced in [She99] and further developed in [GrVas17] and [Vas18]. There it is shown that for AECs that have amalgamation, joint embedding, no maximal models, and $\mathrm{LS}(\mathbf{K})$-tameness, the definition above is equivalent to any other definition of superstability introduced for AECs. In particular, for a complete first-order theory T, $(Mod(T), \preceq)$ is superstable if and only if T is superstable as a first-order theory[3].

Finally, we say that $\mathbf{K}$ is *strictly stable* if $\mathbf{K}$ is stable and not superstable.

[3] T is superstable if and only if T is λ-stable for every $\lambda \geq 2^{|T|}$.

3. Relative pure-injective and Σ-pure-injective modules

In this section we extend classical results of pure-injective and Σ-pure-injective modules to our setting. The arguments are similar to the standard arguments, but we provide them to show that they come through in this non-first-order setting.

We assume the following hypothesis throughout the paper.

Hypothesis 3.1. *$K^{\mathfrak{s}\text{-}Tor}$ is non-trivial, i.e., there is a non-zero module in $K^{\mathfrak{s}\text{-}Tor}$.*

The following fact gives an algebraic condition that implies our hypothesis.

Fact 3.2 ([MaRu20, 2.32]). *Assume $\mathfrak{s}$ is idempotent, i.e., $\mathfrak{s}(\mathfrak{s}(M)) = \mathfrak{s}(M)$. R_R is absolutely pure*[4] *if and only if $K^{\mathfrak{s}\text{-}Tor}$ is trivial.*

Remark 3.3. *Since we will soon assume that R is right semihereditary (Hypothesis 3.4) and in that case $\mathfrak{s}$ is idempotent (Proposition 3.7) for our purposes we could have simply assumed that R_R is not absolutely pure.*

If R is a commutative domain, then $K^{\mathfrak{s}\text{-}Tor}$ is trivial if and only if R is a field.

We will assume the following hypothesis for the rest of this section.

Hypothesis 3.4. *R is right semihereditary, i.e., finitely generated right submodules of projective modules are projective.*

The only reason we assume that R is right semihereditary is because of the following fact.

Fact 3.5 ([MaRu20, 2.17]). *If R is right semihereditary, then $\mathfrak{s}(M) \leq_p M$ for every left R-module M.*

Remark 3.6. *Instead of assuming that R is right semihereditary our hypothesis could have been that $\mathfrak{s}(M) \leq_p M$ for every left R-module M as this is all we will use. We decided to assume that R is right semihereditary as it is a more natural hypothesis. An interesting question is to determine if both statements are equivalent. In the case of commutative domains, it is well-known that a commutative domain is semihereditary if and only if it is a a Prüfer domain. In that case, it is known that both assertions are equivalent [Lam07, p. 117].*

The next proposition follows directly from Fact 3.5, but we record it due to its importance.

Proposition 3.7.

(1) *$\mathfrak{s}$ is idempotent, i.e., $\mathfrak{s}(\mathfrak{s}(M)) = \mathfrak{s}(M)$ for every left R-module M.*
(2) *$\mathfrak{s}(M) \in K^{\mathfrak{s}\text{-}Tor}$ for every left R-module M.*

Recall that a module M is *pure-injective* if for every N_1, N_2, if $N_1 \leq_p N_2$ and $f : N_1 \to M$ is a homomorphism then there is a homomorphism $g : N_2 \to M$ extending f. Given a module M, *the pure-injective envelope of* M, denoted by $PE(M)$, is a pure-injective module with $M \leq_p PE(M)$ and it is minimum with respect to these properties, i.e., if N is pure-injective and there is $f : M \to N$ pure embedding then there is $g : PE(M) \to N$ pure embeddings extending f.

Let us recall the following notion and assertion.

Definition 3.8 ([Pre09, p. 145]). *Let $M \leq_p N$. M is a* pure-essential submodule *of N, denoted by $M \leq^e N$, if and only if for every homomorphism $f : N \to N'$, if $f \circ i$ is a pure embedding where $i : M \hookrightarrow N$ is the inclusion, then f is a pure embedding.*

Fact 3.9 ([Pre09, 4.3.15, 4.3.16]).

(1) *If $M \leq_p N_1 \leq_p N_2$ and $M \leq^e N_2$, then $M \leq^e N_1$.*

[4] M_R is absolutely pure if for every N_R, if $M_R \subseteq_R N_R$ then $M_R \leq_p N_R$

(2) $M \leq^e PE(M)$.

We now introduce a relative notion of pure injectivity and saturation.

Definition 3.10. *Let M be an $\mathfrak{s}$-torsion module.*

- *M is* $K^{\mathfrak{s}\text{-Tor}}$-pure-injective *if and only if for every $N_1, N_2 \in K^{\mathfrak{s}\text{-}Tor}$, if $N_1 \leq_p N_2$ and $f : N_1 \to M$ is a homomorphism then there is a homomorphism $g : N_2 \to M$ extending f.*
- *M is* relatively *pp*-saturated *in N if and only if $M \leq_p N$ and if every pp-type over M which is realized in N is realized in M.*

The following notion of partial homomorphism will also be useful.

Definition 3.11. *For two modules M, N, $A \subseteq |M|$ and $B \subseteq |N|$. A function $f : A \to B$ is a* pp-(M,N)-homomorphism *if and only if for every $\bar{a} \in A$ and $\phi(\bar{x})$ pp-formula:*

$$M \vDash \phi[\bar{a}] \Rightarrow N \vDash \phi[f(\bar{a})].$$

Observe that if $f : M \to N$ is a homomorphism then f is a *pp*-(M,N)-homomorphism as *pp*-formulas are preserved under homomorphism.

We now prove several equivalences of $K^{\mathfrak{s}\text{-Tor}}$-pure-injectivity. These extend classical characterizations of pure-injectivity, see Remark 3.21 and the detailed history presented right before Theorem 2.8 of [Pre88].

Lemma 3.12. *Assume $M \in K^{\mathfrak{s}\text{-}Tor}$. The following are equivalent.*

(1) *M is relatively pp-saturated in N for every $N \in K^{\mathfrak{s}\text{-}Tor}$.*
(2) *M is $K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
(3) *$M = \mathfrak{s}(PE(M))$*
(4) *$M = \mathfrak{s}(N)$ for some pure-injective module N.*
(5) *If $M \leq_p M^*$ and $M^* \in K^{\mathfrak{s}\text{-}Tor}$, then M is a direct summand of M^*.*

Proof. (1) $\Rightarrow$ (2): Let $N_1 \leq_p N_2$ and $f : N_1 \to M$ be a homomorphism. Let

$$\mathcal{P} = \{g : f \subseteq g,\ g \text{ is a } pp\text{-}(N_2, M)\text{-homomorphism, and } dom(g) = A\}.$$

It is clear that one can apply Zorn's lemma to $\mathcal{P}$, so let $g : A \to M$ be a maximal *pp*-(N_2, M)-homomorphism extending f. We show that $A = N_2$. Let $b \in N_2$ and $p = pp(b/A, N_2)$. Consider $q(x) = \{\phi(x, g(\bar{a})) : \phi(x, \bar{a}) \in p\}$. Clearly $q(x)$ is a $Th(M)$-type so there is M^* elementary extension of M and $m^* \in M^*$ such that $q(x) \subseteq pp(m^*/M, M^*)$.

Since $N_2 \in K^{\mathfrak{s}\text{-Tor}}$, there is ψ low such that $N_2 \vDash \psi(b)$. Hence $\psi \in q(x)$ and $m^* \in \mathfrak{s}(M^*)$. Let $q'(x) = pp(m^*/M, \mathfrak{s}(M^*))$. Then by (1), there is $m \in M$ realizing $q'(x)$ and it is clear that $g \cup \{(b, m)\}$ is a *pp*-(N_2, M)-homomorphism extending f. So by maximality of g we have that $b \in A$.

(2) $\Rightarrow$ (3): Let $N_1 = M$, $N_2 = \mathfrak{s}(PE(M))$ and $f = \mathrm{id}_M$. Then by (2) there is a $g : \mathfrak{s}(PE(M)) \to M$ extending f. Observe that by Fact 3.9 $M \leq^e \mathfrak{s}(PE(M))$ as $M \leq_p \mathfrak{s}(PE(M)) \leq_p PE(M)$ and $M \leq^e PE(M)$. Then it follows that g is a pure embedding, so $\mathfrak{s}(PE(M)) = M$.

(3) $\Rightarrow$ (4): Clear.

(4) $\Rightarrow$ (5): Let $M \leq_p M^*$ and $M^* \in K^{\mathfrak{s}\text{-Tor}}$. Then by (4) we have that $M = \mathfrak{s}(N) \leq_p N$ for N a pure-injective module. Since N is pure-injective, there is a homomorphism $g : M^* \to N$ with $g\restriction_M = \mathrm{id}_M$. One can check that $M^* = M \oplus ker(g)$ using that $g[M^*] \subseteq \mathfrak{s}(N) = M$.

(5) $\Rightarrow$ (1): Let $M \leq_p M^* \in K^{\mathfrak{s}\text{-Tor}}$ and $p = pp(a/M, M^*)$ for some $a \in M^*$. Then by (5) there is L such that $M^* = M \oplus L$. Let $\pi_1 : M^* = M \oplus L \to M$ be the projection onto the first coordinate. One can check that $\pi_1(a) \in M$ realizes p. $\square$

Recall the following notion introduced in [FuSa01, XIII.§6].

Definition 3.13. *Assume R is a Prüfer domain. M is torsion-ultracomplete if for every module N, if $M \leq_p N$ and $N/M \in K^{\mathfrak{s}\text{-}Tor}$, then M is a direct summand of N.*

The next lemma together with Lemma 3.12 show that torsion-ultracomplete modules can be described model theoretically for Prüfer domains.

Lemma 3.14. *Assume R is a Prüfer domain. Let M be an $\mathfrak{s}$-torsion module. The following are equivalent.*

- *M is $K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
- *M is torsion-ultracomplete.*

Proof. The forward direction follows from the fact that if $M, N/M \in K^{\mathfrak{s}\text{-Tor}}$, then $N \in K^{\mathfrak{s}\text{-Tor}}$. The backward direction is clear as quotient of torsion modules is torsion. □

The standard argument can be used to show the following proposition.

Proposition 3.15. *Assume M and N are $\mathfrak{s}$-torsion modules. If M and N are $K^{\mathfrak{s}\text{-}Tor}$-pure-injective, then $M \oplus N$ is $K^{\mathfrak{s}\text{-}Tor}$-pure-injective*

We turn our attention to Σ-$K^{\mathfrak{s}\text{-Tor}}$-pure-injective modules.

Definition 3.16. *Let M be an $\mathfrak{s}$-torsion module. M is* Σ-$K^{\mathfrak{s}\text{-Tor}}$-pure-injective *if and only if $M^{(I)}$ is $K^{\mathfrak{s}\text{-}Tor}$-pure-injective for every index set I where $M^{(I)}$ denotes the direct sum of M indexed by I.*

Let us now consider the following notion.

Definition 3.17. *Let M be an $\mathfrak{s}$-torsion module. M has* the low-pp descending chain condition *if and only if for every $\{\phi_n(x)\}_{n\in\omega}$ such that $\phi_0(x)$ is low and $\phi_n(x)$ is a pp-formula for every $n \in \omega$, if $\{\phi_n[M]\}_{n\in\omega}$ is a descending chain in M, then there exists $n_0 \in \omega$ such that $\phi_{n_0}[M] = \phi_k[M]$ for every $k \geq n_0$.*

We will soon see that the previous notion is actually equivalent to being Σ-$K^{\mathfrak{s}\text{-Tor}}$-pure-injective

The next result will be useful to characterize Σ-$K^{\mathfrak{s}\text{-Tor}}$-pure-injective modules.

Lemma 3.18. *Let M be an $\mathfrak{s}$-torsion module. If M has the low-pp descending chain condition, then M is $K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*

Proof. Let $p = pp(b/M, N)$ for some $N \in K^{\mathfrak{s}\text{-Tor}}$. It is enough to show that there is a $\phi \in p$ such that for every $\psi \in p$ and $c \in M$, $M \vDash \phi(c) \to \psi(c)$. Such a ϕ exists by the hypothesis on M and the fact that there is a low formula $\theta \in p$ as N is an $\mathfrak{s}$-torsion module. □

The next result extends a classic characterization of Σ-pure-injectivity, see [Pre88, 2.11] and Remark 3.21.

Lemma 3.19. *Let M be an $\mathfrak{s}$-torsion module. The following are equivalent.*

1. *M is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
2. *$M^{(\aleph_0)}$ is $K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
3. *M has the low-pp descending chain condition.*

Proof. (1) $\Rightarrow$ (2): Clear.

(2) $\Rightarrow$ (3): Assume for the sake of contradiction that there is a family of *pp*-formulas $\{\phi_n(x)\}_{n\in\omega}$ such that $\phi_0(x)$ is low and $\phi_n[M] \supset \phi_{n+1}[M]$ for every $n \in \omega$. For each $n \in \omega$ pick $a_n \in \phi_n[M] \backslash \phi_{n+1}[M]$ and set $b_n = (a_0, a_1, \cdots, a_{n-1}, 0, \cdots) \in M^{(\aleph_0)}$.

Let $p(x) = \{\phi_n(x - b_n) : n \geq 1\} \cup \{\phi_0(x)\}$. Realize that $p(x)$ is a $Th(M^{(\aleph_0)})$-type so there is $M^* \succeq M^{(\aleph_0)}$ and $c \in M^*$ realizing $p(x)$. Observe that $c \in \mathfrak{s}(M^*)$, then by hypothesis and Lemma 3.12.(1) there is $d \in M^{(\aleph_0)}$ realizing $pp(c/M^{(\aleph_0)}, \mathfrak{s}(M^*))$. Then one can show that $M \vDash \phi_{m+1}[a_m]$ for some $m \in \omega$, contradicting the choice of a_m.

(3) $\Rightarrow$ (1): It is known, see for example [Pre88, 2.10], that $\phi[N^{(I)}] = \phi[N]^{(I)}$ for every *pp*-formula ϕ. Therefore, it follows from (3) that $M^{(I)}$ has the low-pp descending chain condition and $M^{(I)} \in K^{\mathfrak{s}\text{-Tor}}$ by Remark 2.6. Hence $M^{(I)}$ is $K^{\mathfrak{s}\text{-Tor}}$-pure-injective by Lemma 3.18. □

The next corollary will be very useful.

Corollary 3.20. *Let M and N be $\mathfrak{s}$-torsion modules .*

- *If N is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective, then N is $K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
- *If $M \leq_p N$ and N is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective, then M is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
- *If M is elementarily equivalent to N and N is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective, then M is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*

Remark 3.21. *Let Φ be a collection of pp-formulas. We say that M is a $\mathfrak{s}_\Phi$-torsion module if and only if M satisfies:*

$$\forall x(\bigvee_{\phi \in \Phi} \phi),$$

and given a module M, let $\mathfrak{s}_\Phi(M) = \{m \in M : M \vDash \psi[m] \text{ for some } \psi \in \Phi\}$.

If Φ is such that for every M we have that $\mathfrak{s}_\Phi(M) \leq_p M$, then $\mathfrak{s}_\Phi$(-) is an idempotent radical and all the results we have proven so far hold for $\mathfrak{s}_\Phi$-torsion modules. In particular, by taking $\Phi := \{x = x\}$ it follows that the results in this section extend the classical characterizations of pure-injective and Σ-pure-injective modules of [Ste67], [Kie67], [War69], [GrJe76], and [Zim77]. Another example is given by taking the ring of integers and letting $\Phi = \{p^n x = 0 : n < \omega\}$ for a fixed prime number p, it is clear that the $\mathfrak{s}_\Phi$-torsion modules are precisely the abelian p-groups. As we do not know of any other interesting choice for Φ, we do not explore this idea any further.

4. $\mathfrak{s}$-torsion modules as an AEC

In this section we study the class of $\mathfrak{s}$-torsion modules with pure embeddings as an abstract elementary class. There are three reasons why we decided to study $\mathfrak{s}$-torsion modules with respect to pure embeddings instead than with respect to embeddings. Firstly, the class of $\mathfrak{s}$-torsion modules is defined with respect to all low *pp*-formulas and not only those low quantifier-free formulas. Secondly, the class of $\mathfrak{s}$-torsion modules is closed under pure submodules, but it is not necessarily closed under submodules. Finally, the original objective of this paper was to understand the class of torsion abelian groups with pure embeddings.

As in the previous section we are assuming *Hypothesis 3.1*, i.e., there is a non-zero module in $K^{\mathfrak{s}\text{-Tor}}$.

4.1. **Basic properties.** We begin by recalling some basic properties of the AEC of $\mathfrak{s}$-torsion modules with pure embeddings.

Fact 4.1. *Let R be a ring and $\mathbf{K}^{\mathfrak{s}\text{-}Tor} = (K^{\mathfrak{s}\text{-}Tor}, \leq_p)$.*

1. *$\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ is an AEC with $\mathrm{LS}(\mathbf{K}^{\mathfrak{s}\text{-}Tor}) = |R| + \aleph_0$.*
2. *$\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ has amalgamation, joint embedding, and no maximal models.*
3. *If $\lambda^{|R|+\aleph_0} = \lambda$, then $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ is λ-stable.*

Proof. (1) and (2) follow from [Maz2, 4.2.(4), 4.8] and (3) from [Maz2, 4.16]. □

We show next that $\mathbf{K}^{\mathfrak{s}\text{-Tor}} = (K^{\mathfrak{s}\text{-Tor}}, \leq_p)$ is nicely generated in $(R\text{-Mod}, \leq_p)$ in the sense of [Maz21a, 4.1], i.e., if $N_1, N_2 \in K^{\mathfrak{s}\text{-Tor}}$ and $N_1, N_2 \leq_p N$ for some module N, then there is $L \in K^{\mathfrak{s}\text{-Tor}}$ such that $N_1, N_2 \leq_p L \subseteq N$.

Lemma 4.2. $\mathbf{K}^{\mathfrak{s}\text{-}Tor} = (K^{\mathfrak{s}\text{-}Tor}, \leq_p)$ *is nicely generated in* $(R\text{-}Mod, \leq_p)$.

Proof. If $N_1, N_2 \in K^{\mathfrak{s}\text{-Tor}}$ and $N_1, N_2 \leq_p N$ for some module N, then $L = N_1 + N_2 \in K^{\mathfrak{s}\text{-Tor}}$ and $N_1, N_2 \leq_p L \subseteq N$. □

The next result follows directly from the previous lemma, [Maz21a, 4.5], and [KuMa20, 3.7].

Corollary 4.3. *Let* $N_1, N_2 \in K^{\mathfrak{s}\text{-}Tor}$, $M \leq_p N_1, N_2$, $\bar{b}_1 \in N_1^{<\omega}$ *and* $\bar{b}_2 \in N_2^{<\omega}$, *then:*

$$\mathbf{gtp}_{K^{\mathfrak{s}\text{-}Tor}}(\bar{b}_1/M; N_1) = \mathbf{gtp}_{K^{\mathfrak{s}\text{-}Tor}}(\bar{b}_2/M; N_2) \text{ if and only if } pp(\bar{b}_1/M, N_1) = pp(\bar{b}_2/M, N_2).$$

In particular, $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ *is* $(< \aleph_0)$*-tame.*

The next result follows from the previous lemma and [Maz21a, 4.6].

Corollary 4.4. *Let* $\lambda \geq |R| + \aleph_0$. *If* $(R\text{-}Mod, \leq_p)$ *is* λ*-stable, then* $(K^{\mathfrak{s}\text{-}Tor}, \leq_p)$ *is* λ*-stable.*

We can not prove anything else without extra assumptions on the ring.

4.2. **Limit models and superstability.** We characterize limit models algebraically and use them to characterize superstability.

We assume *Hypothesis 3.4* for the rest of this section, i.e., we assume that R is right semi-hereditary.

We begin by showing that saturated models are $K^{\mathfrak{s}\text{-Tor}}$-pure-injective.

Lemma 4.5. *If* M *is* $(|R| + \aleph_0)^+$*-saturated in* $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$*, then* M *is* $K^{\mathfrak{s}\text{-}Tor}$*-pure-injective.*

Proof. We use Lemma 3.12. Let $M \leq_p N \in K^{\mathfrak{s}\text{-Tor}}$ and $p = pp(b/M, N)$ for some $b \in N$. Given $\phi(x, \bar{y})$ a pp-formula, let $A_\phi = \{\bar{m} \in M : \phi(x, \bar{m}) \in p\}$ and let $\bar{m}_\phi$ be an element A_ϕ if $A_\phi \neq \emptyset$ and $\bar{m}_\phi = \bar{0}$ otherwise. Let $B = \bigcup_{\phi \in pp\text{-formula}} \bar{m}_\phi$ and M^* be the structure obtained by applying downward Löwenheim-Skolem to B in M. Observe that $\|M^*\| = |R| + \aleph_0$.

Let $q = \mathbf{gtp}(b/M^*; N)$. Since M is $(|R| + \aleph_0)^+$-saturated there is $c \in M$ such that $q = \mathbf{gtp}(c/M^*; M)$. Then $pp(c/M^*, M) = pp(b/M^*, N)$ by Lemma 4.3. Using that pp-formulas determine cosets [Pre88, 2.2] and the choices of the $\bar{m}_\phi$'s, it follows that c realizes p. □

It follows directly from the above result and Fact 2.11 that long limit models are $K^{\mathfrak{s}\text{-Tor}}$-pure-injective.

Corollary 4.6. *If* M *is a* (λ, α)*-limit model in* $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ *and* $cf(\alpha) \geq (|R| + \aleph_0)^+$*, then* M *is* $K^{\mathfrak{s}\text{-}Tor}$*-pure-injective.*

We would like to show that limit models with *long chains* are isomorphic. In order to do that, we obtain a couple of algebraic results regarding pure-injective modules. We begin by recalling the following fact.

Fact 4.7 ([GKS18, 2.5])**.** *Let* M, N *be pure-injective modules. If there are* $f : M \to N$ *a pure embedding and* $g : N \to M$ *a pure embedding, then* M *and* N *are isomorphic.*

Lemma 4.8. *Let* M, N *be any two modules. If there are* $f : M \to N$ *a pure embedding and* $g : N \to M$ *a pure embedding, then* $PE(M)$ *and* $PE(N)$ *are isomorphic.*

Proof. It is enough to show that there are $f' : PE(M) \to PE(N)$ and $g' : PE(N) \to PE(M)$ pure embeddings, as then the result follows directly from Fact 4.7. The existence of f' and g' follow from the minimality of $PE(M)$ and $PE(N)$ respectively. □

The next corollary follows directly from the previous result and Lemma 3.12.

Corollary 4.9. *Let M, N be $\mathfrak{s}$-torsion and $K^{\mathfrak{s}\text{-}Tor}$-pure-injective modules. If there are $f : M \to N$ a pure embedding and $g : N \to M$ a pure embedding, then M and N are isomorphic.*

Since λ-limit models are universal in $(\mathbf{K}^{\mathfrak{s}\text{-Tor}})_\lambda$, we obtain the following result.

Corollary 4.10. *Assume $\lambda \geq |R| + \aleph_0$. If M, N are λ-limit models in $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ and $K^{\mathfrak{s}\text{-}Tor}$-pure-injective, then M and N are isomorphic.*

Putting together the above assertion with Lemma 4.6, we obtain the promised result that limit models with long chains are isomorphic.

Corollary 4.11. *Assume $\lambda \geq (|R| + \aleph_0)^+$. If M is a (λ, α)-limit model in $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ and N is a (λ, β)-limit model in $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ such that $\mathrm{cf}(\alpha), \mathrm{cf}(\beta) \geq (|R| + \aleph_0)^+$, then M and N are isomorphic.*

Regarding limit models with lengths of countable cofinality, the standard argument can be used to obtain the following assertion by Lemma 3.12.(5) and Proposition 3.15. See for example [KuMa20, 4.5, 4.6].

Lemma 4.12. *Assume $\lambda \geq (|R| + \aleph_0)^+$. If M is a (λ, ω)-limit model in $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ and N is a $(\lambda, (|R| + \aleph_0)^+)$-limit model in $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$, then M and $N^{(\aleph_0)}$ are isomorphic.*

We also have that limit models are elementarily equivalent. The argument of [KuMa20, 4.2] can be used in this setting as the class has the joint embedding property.

Lemma 4.13. *If M and N are limit model in $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$, then M and N are elementarily equivalent.*

This is all we need to characterize superstability in $\mathbf{K}^{\mathfrak{s}\text{-Tor}}$.

Theorem 4.14. *Assume R is right semihereditary and R_R is not absolutely pure. The following are equivalent.*

1. *$\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ is superstable.*
2. *There exists a $\lambda \geq (|R| + \aleph_0)^+$ such that $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ has uniqueness of limit models of cardinality λ.*
3. *Every limit model in $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
4. *Every $M \in K^{\mathfrak{s}\text{-}Tor}$ is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
5. *Every $M \in K^{\mathfrak{s}\text{-}Tor}$ is $K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
6. *For every $\lambda \geq |R| + \aleph_0$, $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ has uniqueness of limit models of cardinality λ.*
7. *For every $\lambda \geq |R| + \aleph_0$, $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ is λ-stable.*

Proof. (1) $\Rightarrow$ (2): Clear.

(2) $\Rightarrow$ (3): The proof is similar to that of (2) to (3) of [Maz2, 3.15]. The reason that argument goes through in $\mathbf{K}^{\mathfrak{s}\text{-Tor}}$ is because of Lemma 4.12, Lemma 4.6, Lemma 3.19, Lemma 4.13, and Corollary 3.20.

(3) $\Rightarrow$ (4): Follows from Corollary 3.20 and the fact that limit models are universal.

(4) $\Rightarrow$ (5): Follows from Corollary 3.20.

(5) $\Rightarrow$ (6): By Corollary 4.10 for every cardinal $\lambda \geq |R| + \aleph_0$ there is at most one λ-limit model up to isomorphisms, so we only need to show existence. We show that $\mathbf{K}^{\mathfrak{s}\text{-Tor}}$ is λ-stable for every $\lambda \geq |R| + \aleph_0$, this is enough by Fact 2.10. Let $\lambda \geq |R| + \aleph_0$ and $M \in \mathbf{K}^{\mathfrak{s}\text{-Tor}}_\lambda$.

Let $N \in \mathbf{K}^{\mathfrak{s}\text{-Tor}}$ and $\{a_i : i \leq \kappa\} \subseteq N$ such that $\{\mathbf{gtp}(a_i/M; N) : i \leq \kappa\}$ is an enumeration without repetitions of $\mathbf{gS}(M)$. Let $\Delta := \{pp(a_i/M, N) : i \leq \kappa\}$ and observe that $|\mathbf{gS}(M)| \leq |\Delta|$ since $\Phi : \mathbf{gS}(M) \to \Delta$ given by $\Phi(\mathbf{gtp}(a_i/M; N)) = pp(a_i/M, N)$ is injective by Lemma 4.3.

Since N is Σ-$K^{\mathfrak{s}\text{-Tor}}$-pure-injective by (5), it follows from Lemma 3.19 that N has the low-pp descending chain condition. Then it follows, as in Lemma 3.18, that for every $p \in \Delta$ there is $\psi_p \in p$ such that for every $\theta \in p$ and $c \in N$, $N \vDash \psi_p(c) \to \theta(c)$. Let $\Psi : \Delta \to \{\phi(x, \bar{m}) : \phi(x, \bar{y})$ is a *pp*-formula and $\bar{m} \in M\}$ be given by $\Psi(p) = \psi_p$. It is easy to show that Ψ is injective and as $|\{\phi(x, \bar{m}) : \phi(x, \bar{y})$ is a *pp*-formula and $\bar{m} \in M\}| = (|R| + \aleph_0)\lambda = \lambda$, we can conclude that $|\Delta| \leq \lambda$. Therefore, $|\mathbf{gS}(M)| \leq \lambda$.

(6) $\Rightarrow$ (1): Clear.

(6) $\Rightarrow$ (7): Clear.

(7) $\Rightarrow$ (4): Assume for the sake of contradiction that there is $M \in K^{\mathfrak{s}\text{-Tor}}$ which is not Σ-$K^{\mathfrak{s}\text{-Tor}}$-pure-injective. It follows from Lemma 3.19 that there is a set of formulas $\{\phi_n(x)\}_{n\in\omega}$ such that $\phi_0(x)$ is low and $\phi_n(x)$ is a *pp*-formula for every $n \in \omega$ such that $\phi_n[M] \supset \phi_{n+1}[M]$ for every $n \in \omega$.

Let $\lambda = \beth_\omega(|R| + \aleph_0)$. Observe that since $[\phi_n[M] : \phi_{n+1}[M]] \geq 2$, it follows that $[\phi_n[M^{(\lambda)}] : \phi_{n+1}[M^{(\lambda)}]] = \lambda$ for each $n \in \omega$ and $M^{(\lambda)} \in K^{\mathfrak{s}\text{-Tor}}$ by Remark 2.6. For every $n \in \omega$ pick $\{a_{n,\alpha} : \alpha < \lambda\} \subseteq M^{(\lambda)}$ a complete set of representatives of $\phi_n[M^{(\lambda)}]/\phi_{n+1}[M^{(\lambda)}]$.

Let $A = \bigcup_{n<\omega}\{a_{n,\alpha} : \alpha < \lambda\}$ and N be a structure obtained by applying downward Löwenheim-Skolem to A in $M^{(\lambda)}$. It is clear that $N \in \mathbf{K}^{\mathfrak{s}\text{-Tor}}_\lambda$. For every $\eta \in \lambda^\omega$, let $\Phi_\eta = \{\phi_{n+1}(x - \Sigma_{i=0}^n a_{i,\eta(i)}) : n < \omega\} \cup \{\phi_0(x)\}$. Φ_η is a $Th(M^{(\lambda)})$-type, so pick $M_\eta \succeq M^{(\lambda)}$ and $c_\eta \in M_\eta$ realizing Φ_η. It is clear that $c_\eta \in \mathfrak{s}(M_\eta)$ so consider $q_\eta = \mathbf{gtp}(c_\eta/N; \mathfrak{s}(M_\eta))$.

Using Lemma 4.3 and that $\mathfrak{s}(M_\eta) \leq_p M_\eta$ for every $\eta \in \lambda^\omega$, it can be shown that if $\eta_1 \neq \eta_2 \in \lambda^\omega$ then $q_{\eta_1} \neq q_{\eta_2}$. Hence $|\mathbf{gS}(N)| \geq \lambda^{\aleph_0} > \lambda$ by the choice of λ and König's lemma. This contradicts our assumption that $\mathbf{K}^{\mathfrak{s}\text{-Tor}}$ was λ-stable. $\square$

Remark 4.15. *The equivalence between* (4) *and* (7) *of the above theorem is a natural extension of a result of Garavaglia and Macintyre [Gar80, Theo 1].*

Previous results that characterised superstability in classes of modules always corresponded to classical rings [Maz21b], [Maz1], [Maz2]. In this case we do not know if that is the case. Moreover, we do not even know if there exists a ring such that the class of $\mathfrak{s}$-torsion modules is superstable. So we ask the following question.

Question 4.16. *Is there a right semihereditary ring R such that R_R is not absolutely pure and R satisfies any of the equivalent conditions given in Theorem 4.14?*

Remark 4.17. *If there is R left pure-semsimple ring such that R is right semihereditary and R_R is not absolutely pure, then the above question would have a positive solution by Theorem 4.14.(5). For this reason, we think of a ring satisfying any of the conditions given in Theorem 4.14 as a weak pure-semisimple ring.*

As foreshadow by the remark, we think that the above question has a positive solution. Nevertheless, even if the above question has a negative solution Theorem 4.14 is still interesting as it can be used to show that certain classes are not superstable. An example of this is given in the next section.

A finer question would be to determine if there is a commutative domain satisfying any of the equivalent conditions given in Theorem 4.14. We ask the question in algebraic terms.

Question 4.18. *Is there a Prüfer domain such that R is not a field, but every torsion module is torsion-ultracomplete?*

Finally, a natural question is if any of the results presented in this section can be extended to rings that are not necessarily right semihereditary. We think it is unlikely. However, we think that if one studies the class of $\mathfrak{s}$-torsion modules with respect to other embeddings it is possible to obtain analogous results to the ones presented here for rings that are not necessarily right semihereditary.

5. Torsion abelian groups

In this section we apply our general results to the class of torsion abelian groups with pure embeddings. We show it is strictly stable, characterize its stability cardinals, and describe its limit models. We will denote the class of torsion abelian groups with pure embeddings by $\mathbf{K}^{Tor}$.

Remark 5.1. *Recall that the class of $\mathfrak{s}$-torsion abelian groups is precisely the class of torsion abelian groups, i.e., those groups such that every element has finite order. Moreover, $\mathbb{Z}$ is semi-hereditary since it is a Prüfer domain. Therefore, we can use the results obtained in the previous section to study the class of torsion abelian groups.*

The following fact collects what is known of the class of torsion abelian groups with pure embeddings. They were first obtained in [Maz21a, §4], but they also follow from the results of the previous section.

Fact 5.2. *Let $\mathbf{K}^{Tor} = (K^{Tor}, \leq_p)$.*

- *$\mathbf{K}^{Tor}$ is an AEC with $\mathrm{LS}(\mathbf{K}^{Tor}) = \aleph_0$ that has amalgamation, joint embedding, and no maximal models.*
- *If $\lambda^{\aleph_0} = \lambda$, then $\mathbf{K}^{Tor}$ is λ-stable.*
- *$\mathbf{K}^{Tor}$ is $(< \aleph_0)$-tame.*

We will use the following algebraic result to show that the class is not superstable. Given an abelian group G, we will denote its torsion part by the standard $t(G)$ instead of $\mathfrak{s}(G)$.

Remark 5.3 ([Fuc15, §10.3])**.** *Let $B_n = \mathbb{Z}(p^n)^{(\lambda)}$ and $B = \bigoplus_n B_n$. The following holds:*

$$g = (b_n)_{n\in\omega} \in t(PE(B)) \leq \Pi_n B_n \text{ if and only if the orders of } \{b_n\}_{n\in\omega} \text{ are bounded.}$$

Using the above characterization of $t(PE(B))$, it is easy to show that $\|t(PE(B))\| = \lambda^{\aleph_0}$ as $|B_n[p]| = |\{b \in B_n : pb = 0\}| = \lambda$ for every $n \in \omega$.

Lemma 5.4. *$\mathbf{K}^{Tor}$ is not superstable. Hence, $\mathbf{K}^{Tor}$ is strictly stable.*

Proof. Assume for the sake of contradiction that $\mathbf{K}^{Tor}$ is superstable. Let $\lambda = \beth_\omega$ and $B = \bigoplus_n B_n$ where $B_n = \mathbb{Z}(p^n)^{(\lambda)}$ for every $n < \omega$ as in Remark 5.3. Then by Theorem 4.14.(5) and Lemma 3.12.(3), it follows that $B = t(PE(B))$. This is a contradiction as $\|t(PE(B))\| = \lambda^{\aleph_0} > \lambda$ by König's lemma. □

Remark 5.5. *The previous result contrasts with the fact that the class of torsion abelian groups with embedding is superstable [Maz21a, 4.8].*

We are actually able to obtain a complete characterization of the stability cardinals.

Theorem 5.6. *$\mathbf{K}^{Tor}$ is λ-stable if and only if $\lambda^{\aleph_0} = \lambda$.*

Proof. The backward direction follows from Fact 5.2 so we show the forward direction. We divide the proof into two cases:

Case 1: $\lambda > \aleph_0$. Assume that $\mathbf{K}^{Tor}$ is λ-stable. Let M be a (λ, ω_1)-limit model and $B = \bigoplus_n B_n$ where $B_n = \mathbb{Z}(p^n)^{(\lambda)}$ for every $n < \omega$ as in Remark 5.3. Since M is a λ-limit model and B has size λ there is a pure embedding $f : B \to M$. Then there is $g : PE(B) \to PE(M)$ pure embedding extending f by the minimality of $PE(B)$.

In particular, $g\restriction_{t(PE(B))} : t(PE(B)) \to t(PE(M))$ is injective. So $\|t(PE(B))\| \leq \|t(PE(M))\|$. Since $t(PE(M)) = M$ by Corollary 4.6 and Lemma 3.12 and $\|t(PE(B))\| = \lambda^{\aleph_0}$ by Remark 5.3, it follows that $\lambda = \lambda^{\aleph_0}$

Case 2: $\lambda = \aleph_0$. Assume for the sake of contradiction that $\mathbf{K}^{Tor}$ is ω-stable. Since $\mathbf{K}^{Tor}$ is $(< \aleph_0)$-tame by Fact 5.2, it follows from [BKV06, 3.6] that $\mathbf{K}^{Tor}$ is $\beth_\omega$-stable. This contradicts the previous case as $\beth_\omega^{\aleph_0} > \beth_\omega$ by König's lemma. □

From the above results we can precisely describe the spectrum function for limit models.

Corollary 5.7. *If $\lambda^{\aleph_0} = \lambda$, then $\mathbf{K}^{Tor}$ has two non-isomorphic λ-limit models. Moreover, for every other λ, $\mathbf{K}^{Tor}$ has no λ-limit models.*

Proof. The first part follows from Corollary 4.11 and Theorem 4.14.(2). The moreover part follows from Theorem 5.6 and Fact 2.10. ☐

We go one step further and give an algebraic description of the limit models. Recall that given $n \in \mathbb{N}$ and G an abelian group, $G[n]$ denotes the elements of order n in G and nG denotes the elements of the form ng for some g in G.

Lemma 5.8. *Let λ be an infinite cardinal such that $\lambda^{\aleph_0} = \lambda$ and $\alpha < \lambda^+$ be a limit ordinal. If M is a (λ, α)-limit model in $\mathbf{K}^{Tor}$, then:*

1. *If $cf(\alpha) \geq \omega_1$, then $M \cong t(\Pi_p PE(\bigoplus_n \mathbb{Z}(p^n)^{(\lambda)})) \oplus \bigoplus_p \mathbb{Z}(p^\infty)^{(\lambda)}$.*
2. *If $cf(\alpha) = \omega$, then $M \cong t(\Pi_p PE(\bigoplus_n \mathbb{Z}(p^n)^{(\lambda)}))^{(\aleph_0)} \oplus \bigoplus_p \mathbb{Z}(p^\infty)^{(\lambda)}$.*

Proof. (2) follows directly from (1) and Lemma 4.12, so we show (1). By Lemma 3.12 we have that $M = t(G)$ for some pure-injective group G. Since G is pure-injective, it follows from [EkFi72, §1], that:

$$G = \Pi_p PE(\bigoplus_n \mathbb{Z}(p^n)^{(\alpha_{p,n})} \oplus \mathbb{Z}_p^{(\beta_p)}) \oplus \mathbb{Q}^{(\delta)} \oplus (\bigoplus_p \mathbb{Z}(p^\infty)^{(\gamma_p)}),$$

for some specific $\alpha_{p,n}$, β_p, δ, γ_p described in [EkFi72, §1] for p a prime number and $n < \omega$. Since

$$t(\Pi_p PE(\bigoplus_n \mathbb{Z}(p^n)^{(\alpha_{p,n})} \oplus \mathbb{Z}_p^{(\beta_p)}) \oplus \mathbb{Q}^{(\delta)} \oplus (\bigoplus_p \mathbb{Z}(p^\infty)^{(\gamma_p)})) = t(\Pi_p PE(\bigoplus_n \mathbb{Z}(p^n)^{(\alpha_{p,n})})) \oplus (\bigoplus_p \mathbb{Z}(p^\infty)^{(\gamma_p)}),$$

we only need to determine $\alpha_{p,n}$ and γ_p for p a prime number and $n < \omega$.

By [EkFi72, 1.9] for every prime number p we have that $\gamma_p = dim_{\mathbb{F}_p}(D(G)[p])$ where $D(G)$ is the divisible part of G. Let p be a prime number. Since $\mathbb{Z}(p^\infty)^{(\lambda)}$ can be purely embedded in M, because M is universal in $\mathbf{K}^{Tor}_\lambda$, it can be purely embedded in G. Hence, $\gamma_p = \lambda$.

By [EkFi72, 1.5] for every prime number p and $n < \omega$ we have that $\alpha_{p,n} = dim_{\mathbb{F}_p}((p^{n-1}G)[p]/(p^n G)[p])$. Let p be a prime number and $n < \omega$. Since $\mathbb{Z}(p^n)^{(\lambda)}$ can be purely embedded in M, because M is universal in $\mathbf{K}^{Tor}_\lambda$, it can be purely embedded in G. Hence $\alpha_{p,n} = \lambda$.

Therefore, we can conclude that $M = t(\Pi_p PE(\bigoplus_n \mathbb{Z}(p^n)^{(\lambda)})) \oplus \bigoplus_p \mathbb{Z}(p^\infty)^{(\lambda)}$. ☐

We finish by recording the following results for the class of abelian p-groups with pure embeddings. The proofs are similar to those for torsion abelian groups so we omit them. Recall that G is an abelian p-group if every element of G has order p^n for some $n \in \mathbb{N}$.

Lemma 5.9. *Let p be a fixed prime number and denote by $\mathbf{K}^{p\text{-}grp}$ the class of abelian p-groups with pure embeddings.*

1. *$\mathbf{K}^{p\text{-}grp}$ is strictly stable.*
2. *$\mathbf{K}^{p\text{-}grp}$ is λ-stable if and only if $\lambda^{\aleph_0} = \lambda$.*
3. *Let λ be an infinite cardinal such that $\lambda^{\aleph_0} = \lambda$ and $\alpha < \lambda^+$ be a limit ordinal. If M is a (λ, α)-limit model in $\mathbf{K}^{p\text{-}grp}$, then:*
    - *If $cf(\alpha) \geq \omega_1$, then $M \cong t(PE(\bigoplus_n \mathbb{Z}(p^n)^{(\lambda)})) \oplus \mathbb{Z}(p^\infty)^{(\lambda)}$.*
    - *If $cf(\alpha) = \omega$, then $M \cong t(PE(\bigoplus_n \mathbb{Z}(p^n)^{(\lambda)}))^{(\aleph_0)} \oplus \mathbb{Z}(p^\infty)^{(\lambda)}$.*

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E-mail address: `mmazaria@andrew.cmu.edu`

URL: `http://www.math.cmu.edu/~mmazaria/`

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, Pennsylvania, USA