

# ON UNIVERSAL MODULES WITH PURE EMBEDDINGS

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ABSTRACT. We show that certain classes of modules have universal models with respect to pure embeddings.

**Theorem 0.1.** *Let  $R$  be a ring,  $T$  a first-order theory with an infinite model extending the theory of  $R$ -modules and  $\mathbf{K}^T = (\text{Mod}(T), \leq_{pp})$  (where  $\leq_{pp}$  stands for pure submodule). Assume  $\mathbf{K}^T$  has joint embedding and amalgamation.*

*If  $\lambda^{|\mathbf{K}^T|} = \lambda$  or  $\forall \mu < \lambda (\mu^{|\mathbf{K}^T|} < \lambda)$ , then  $\mathbf{K}^T$  has a universal model of cardinality  $\lambda$ .*

As a special case we get a recent result of Shelah [Sh17, 1.2] concerning the existence of universal reduced torsion-free abelian groups with respect to pure embeddings.

We begin the study of limit models for classes of  $R$ -modules with joint embedding and amalgamation. We show that limit models with chains of long cofinality are pure-injective and we characterize limit models with chains of countable cofinality. This can be used to answer Question 4.25 of [Maz20].

As this paper is aimed at model theorists and algebraists an effort was made to provide the background for both.

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## 1. INTRODUCTION

The first result concerning the existence of universal uncountable objects in classes of modules was [Ekl71]. In it, Eklof showed that there exists a homogeneous universal  $R$ -module of cardinality  $\lambda$  in the class of  $R$ -modules if and only if  $\lambda^{<\gamma} = \lambda$  (where  $\gamma$  is the least cardinal such that every ideal of  $R$  is generated by less than  $\gamma$  elements).

Grossberg and Shelah [GrSh83] used the weak continuum hypothesis to answer a question of Macintyre and Shelah [MaSh76] regarding the existence of universal locally finite groups in uncountable cardinalities. Kojman and Shelah [KojSh95] and Shelah [Sh96], [Sh97], [Sh01] and [Sh17] continued the study of universal groups for certain classes of abelian groups with respect to embeddings and pure embeddings. For further historical comments the reader can consult [Dža05, §6].

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*Date:* February 27, 2020.

AMS 2010 Subject Classification: Primary: 03C48. Secondary: 03C45, 03C60, 13L05, 16D10. Key words and phrases. Modules; Torsion-free groups; Abstract Elementary Classes; Universal models; Limit models.

In this paper, we will give a positive answer to the question of whether certain classes of modules with pure embeddings have universal models in specific cardinals. More precisely, we obtain:

**Theorem 3.19.** *Let  $R$  be a ring,  $T$  a first-order theory with an infinite model extending the theory of  $R$ -modules and  $\mathbf{K}^T = (\text{Mod}(T), \leq_{pp})$  (where  $\leq_{pp}$  stands for pure submodule). Assume  $\mathbf{K}^T$  has joint embedding and amalgamation.*

*If  $\lambda^{|T|} = \lambda$  or  $\forall \mu < \lambda (\mu^{|T|} < \lambda)$ , then  $\mathbf{K}^T$  has a universal model of cardinality  $\lambda$ .*

There are many examples of theories satisfying the hypothesis of Theorem 3.19 (see Example 3.10). One of them is the theory of torsion-free abelian groups. So as straightforward corollary we get:

**Corollary 3.22.** *If  $\lambda^{\aleph_0} = \lambda$  or  $\forall \mu < \lambda (\mu^{\aleph_0} < \lambda)$ , then the class of torsion-free abelian groups with pure embeddings has a universal group of cardinality  $\lambda$ .*

In [Sh17, 1.2] Shelah shows a result analogous to the above theorem, but instead of working with the class of torsion-free abelian groups he works with the class of reduced torsion-free abelian groups. The reason Corollary 3.22 transfers to Shelah's setting is because every abelian group can be written as a direct sum of a unique divisible subgroup and a unique up to isomorphism reduced subgroup (see [Fuc15, §4.2.5]). Shelah's statement is Corollary 3.26 in this paper.

The proof presented here is not a generalization of Shelah's original idea. We prove first that the class is  $\lambda$ -Galois-stable (for  $\lambda^{|T|} = \lambda$ ) and then using that the class is an abstract elementary class we construct universal extensions of size  $\lambda$  (for  $\lambda^{|T|} = \lambda$ ). By contrast, Shelah first constructs universal extensions of cardinality  $\lambda$  (for  $\lambda^{\aleph_0} = \lambda$ ) and from it he concludes that the class is  $\lambda$ -Galois-stable.

The methods used in both proofs are also quite different. We exploit the fact that any theory of  $R$ -modules has *pp*-quantifier elimination and that our class is an abstract elementary class with joint embedding and amalgamation. By contrast, Shelah's argument seems to only work in the restricted setting of torsion-free abelian groups. This is the case since the main device of his argument is the existence of a metric in reduced torsion-free abelian groups and the completions obtained from this metric.

In [Maz20], the second author began the study of limit models in classes of abelian groups. In this paper we go one step further and begin the study of limit models in classes of  $R$ -modules with joint embedding and amalgamation. *Limit models* were introduced in [KolSh96] as a substitute for saturation in the context of AECs. Intuitively the reader can think of them as universal models with some level of homogeneity (see Definition 2.10). They have proven to be an important concept in tackling Shelah's eventual categoricity conjecture. The key question has been the uniqueness of limit models of the same cardinality but of different length.<sup>2</sup>

We show that limit models in  $\mathbf{K}^T$  are elementary equivalent (see Lemma 4.3). We generalize [Maz20, 4.10] by showing that limit models with chains of cofinality greater than  $|T|$  are pure-injective (see Theorem 4.5). We characterize limit models with chains of countable cofinality for classes that are closed under direct sums (see Theorem 4.9). The main feature is that there is a natural way to construct universal models over pure-injective modules. More precisely, given  $M$  pure-injective and  $U$  a universal model of size  $\|M\|$ ,  $M \oplus U$  is universal over  $M$ . As a by-product of our study of limit models and [Maz20, 4.15] we answer Question 4.25 of [Maz20].

**Theorem 4.14.** *If  $G$  is a  $(\lambda, \omega)$ -limit model in the class of torsion-free abelian groups with pure embeddings, then  $G \cong \mathbb{Q}^{(\lambda)} \oplus \prod_p \overline{\mathbb{Z}}_{(p)}^{(\lambda)}{}^{\aleph_0}$ .*

Finally, combining Corollary 3.22 and Theorem 4.14, we are able to construct universal extensions of cardinality  $\lambda$  for some cardinals such that the class of torsion-free groups with pure embeddings is not  $\lambda$ -Galois-stable (an example for such a  $\lambda$  is  $\beth_\omega$ ). This is the first example of an

<sup>2</sup>A more detailed account of the importance of limit models is given in [Maz20, §1].

AEC with joint embedding, amalgamation and no maximal models in which one can construct universal extensions of cardinality  $\lambda$  without the hypothesis of  $\lambda$ -Galois-stability.

The paper is organized as follows. Section 2 presents necessary background. Section 3 studies classes of the form  $\mathbf{K}^T$ , studies universal models in these classes and shows how [Sh17, 1.2] is a special case of the theory developed in the section. Section 4 begins the study of limit models for classes of  $R$ -modules with joint embedding and amalgamation. It also answers Question 4.25 of [Maz20].

This paper was written while the second author was working on a Ph.D. under the direction of Rami Grossberg at Carnegie Mellon University and he would like to thank Professor Grossberg for his guidance and assistance in his research in general and in this work in particular. We would also like to thank Sebastien Vasey for several comments that helped improve the paper. We would also like to thank John T. Baldwin for introducing us to one another and for useful comments that improved the paper. We are grateful to the referees for their comments that significantly improved the paper.

## 2. PRELIMINARIES

We introduce the key concepts of abstract elementary classes and the model theory of modules that are used in this paper. Our primary references for the former are [Bal09, §4 - 8] and [Gro1X, §2, §4.4]. Our primary references for the latter is [Pre88].

**2.1. Abstract elementary classes.** Abstract elementary classes (AECs) were introduced by Shelah in [Sh87a]. Among the requirements we have that an AEC is closed under directed colimits and that every set is contained in a small model in the class. Given a model  $M$ , we will write  $|M|$  for its underlying set and  $\|M\|$  for its cardinality.

**Definition 2.1.** *An abstract elementary class is a pair  $\mathbf{K} = (K, \leq_{\mathbf{K}})$ , where:*

- (1)  $K$  is a class of  $\tau$ -structures, for some fixed language  $\tau = \tau(\mathbf{K})$ .
- (2)  $\leq_{\mathbf{K}}$  is a partial order on  $K$ .
- (3)  $(K, \leq_{\mathbf{K}})$  respects isomorphisms:  
*If  $M \leq_{\mathbf{K}} N$  are in  $K$  and  $f : N \cong N'$ , then  $f[M] \leq_{\mathbf{K}} N'$ .  
 In particular (taking  $M = N$ ),  $K$  is closed under isomorphisms.*
- (4) If  $M \leq_{\mathbf{K}} N$ , then  $M \subseteq N$ .
- (5) *Coherence:* If  $M_0, M_1, M_2 \in K$  satisfy  $M_0 \leq_{\mathbf{K}} M_2$ ,  $M_1 \leq_{\mathbf{K}} M_2$ , and  $|M_0| \subseteq |M_1|$ , then  $M_0 \leq_{\mathbf{K}} M_1$ .
- (6) *Tarski-Vaught axioms:* Suppose  $\delta$  is a limit ordinal and  $\{M_i \in K : i < \delta\}$  is an increasing chain. Then:
  - (a)  $M_\delta := \bigcup_{i < \delta} M_i \in K$  and  $M_i \leq_{\mathbf{K}} M_\delta$  for every  $i < \delta$ .
  - (b) *Smoothness:* If there is some  $N \in K$  so that for all  $i < \delta$  we have  $M_i \leq_{\mathbf{K}} N$ , then we also have  $M_\delta \leq_{\mathbf{K}} N$ .
- (7) *Löwenheim-Skolem-Tarski axiom:* There exists a cardinal  $\lambda \geq |\tau(\mathbf{K})| + \aleph_0$  such that for any  $M \in K$  and  $A \subseteq |M|$ , there is some  $M_0 \leq_{\mathbf{K}} M$  such that  $A \subseteq |M_0|$  and  $\|M_0\| \leq |A| + \lambda$ . We write  $\text{LS}(\mathbf{K})$  for the minimal such cardinal.

**Notation 2.2.**

- If  $\lambda$  is cardinal and  $\mathbf{K}$  is an AEC, then  $\mathbf{K}_\lambda = \{M \in \mathbf{K} : \|M\| = \lambda\}$ .
- Let  $M, N \in \mathbf{K}$ . If we write “ $f : M \rightarrow N$ ” we assume that  $f$  is a  $\mathbf{K}$ -embedding, i.e.,  $f : M \cong f[M]$  and  $f[M] \leq_{\mathbf{K}} N$ . In particular  $\mathbf{K}$ -embeddings are always monomorphisms.
- Let  $M, N \in \mathbf{K}$  and  $A \subseteq M$ . If we write “ $f : M \xrightarrow{A} N$ ” we assume that  $f$  is a  $\mathbf{K}$ -embedding and that  $f \upharpoonright_A = id_A$ .

Let us recall the following three properties. They are satisfied by all the classes considered in this paper, although not every AEC satisfies them.

**Definition 2.3.**

- (1)  $\mathbf{K}$  has the amalgamation property if for every  $M, N, R \in \mathbf{K}$  such that  $M \leq_{\mathbf{K}} N, R$ , there is  $R^* \in \mathbf{K}$  with  $R \leq_{\mathbf{K}} R^*$  and a  $\mathbf{K}$ -embedding  $f : N \xrightarrow{M} R^*$ .
- (2)  $\mathbf{K}$  has the joint embedding property if for every  $M, N \in \mathbf{K}$ , there is  $R^* \in \mathbf{K}$  with  $N \leq_{\mathbf{K}} R^*$  and a  $\mathbf{K}$ -embedding  $f : M \rightarrow R^*$ .
- (3)  $\mathbf{K}$  has no maximal models if for every  $M \in \mathbf{K}$ , there is  $M^* \in \mathbf{K}$  such that  $M <_{\mathbf{K}} M^*$ .

In [Sh87b] Shelah introduced a notion of semantic type. The original definition was refined and extended by many authors who following [Gro02] call these semantic types Galois-types (Shelah recently named them orbital types). We present here the modern definition and call them Galois-types throughout the text. We follow the notation of [MaVa18, 2.5].

**Definition 2.4.** Let  $\mathbf{K}$  be an AEC.

- (1) Let  $\mathbf{K}^3$  be the set of triples of the form  $(\mathbf{b}, A, N)$ , where  $N \in \mathbf{K}$ ,  $A \subseteq |N|$ , and  $\mathbf{b}$  is a sequence of elements from  $N$ .
- (2) For  $(\mathbf{b}_1, A_1, N_1), (\mathbf{b}_2, A_2, N_2) \in \mathbf{K}^3$ , we say  $(\mathbf{b}_1, A_1, N_1) E_{at}^{\mathbf{K}} (\mathbf{b}_2, A_2, N_2)$  if  $A := A_1 = A_2$ , and there exists  $f_\ell : N_\ell \xrightarrow{A} N$   $\mathbf{K}$ -embeddings such that  $f_1(\mathbf{b}_1) = f_2(\mathbf{b}_2)$  and  $N \in \mathbf{K}$ .
- (3) Note that  $E_{at}^{\mathbf{K}}$  is a symmetric and reflexive relation on  $\mathbf{K}^3$ . We let  $E^{\mathbf{K}}$  be the transitive closure of  $E_{at}^{\mathbf{K}}$ .
- (4) For  $(\mathbf{b}, A, N) \in \mathbf{K}^3$ , let  $\mathbf{gtp}_{\mathbf{K}}(\mathbf{b}/A; N) := [(\mathbf{b}, A, N)]_{E^{\mathbf{K}}}$ . We call such an equivalence class a Galois-type. Usually,  $\mathbf{K}$  will be clear from the context and we will omit it.
- (5) For  $M \in \mathbf{K}$ ,  $\mathbf{gS}_{\mathbf{K}}(M) = \{\mathbf{gtp}_{\mathbf{K}}(b/M; N) : M \leq_{\mathbf{K}} N \in \mathbf{K} \text{ and } b \in N\}$
- (6) For  $\mathbf{gtp}_{\mathbf{K}}(\mathbf{b}/A; N)$  and  $C \subseteq A$ ,  $\mathbf{gtp}_{\mathbf{K}}(\mathbf{b}/A; N) \upharpoonright_C := [(\mathbf{b}, C, N)]_E$ .

**Remark 2.5.** If  $\mathbf{K}$  has amalgamation, it is straightforward to show that  $E_{at}^{\mathbf{K}}$  is transitive.

**Definition 2.6.** An AEC is  $\lambda$ -Galois-stable if for any  $M \in \mathbf{K}_\lambda$ ,  $|\mathbf{gS}_{\mathbf{K}}(M)| \leq \lambda$ .

The following notion was isolated by Grossberg and VanDieren in [GrVan06].

**Definition 2.7.**  $\mathbf{K}$  is  $(< \kappa)$ -tame if for any  $M \in \mathbf{K}$  and  $p \neq q \in \mathbf{gS}(M)$ , there is  $A \subseteq M$  such that  $|A| < \kappa$  and  $p \upharpoonright_A \neq q \upharpoonright_A$ .  $\mathbf{K}$  is  $\kappa$ -tame if it is  $(< \kappa^+)$ -tame.

Let us recall the following concept that was introduced in [KolSh96].

**Definition 2.8.** Let  $\mathbf{K}$  be an AEC.  $M$  is  $\lambda$ -universal over  $N$  if and only if  $N \leq_{\mathbf{K}} M$  and for any  $N^* \in \mathbf{K}_{\leq \lambda}$  such that  $N \leq_{\mathbf{K}} N^*$ , there is  $f : N^* \xrightarrow{N} M$ .  $M$  is universal over  $N$  if and only if  $\|N\| = \|M\|$  and  $M$  is  $\|M\|$ -universal over  $N$ .

The next fact gives conditions for the existence of universal extensions.

**Fact 2.9** ([Sh:h, §II], [GrVan06, 2.9]). Let  $\mathbf{K}$  an AEC with joint embedding, amalgamation and no maximal models. If  $\mathbf{K}$  is  $\lambda$ -Galois-stable, then for every  $P \in \mathbf{K}_\lambda$ , there is  $M \in \mathbf{K}_\lambda$  such that  $M$  is universal over  $P$ .

The following notion was introduced in [KolSh96] and plays an important role in this paper.

**Definition 2.10.** Let  $\alpha < \lambda^+$  a limit ordinal.  $M$  is a  $(\lambda, \alpha)$ -limit model over  $N$  if and only if there is  $\{M_i : i < \alpha\} \subseteq \mathbf{K}_\lambda$  an increasing continuous chain such that  $M_0 := N$ ,  $M_{i+1}$  is universal over  $M_i$  for each  $i < \alpha$  and  $M = \bigcup_{i < \alpha} M_i$ . We say that  $M \in \mathbf{K}_\lambda$  is a  $(\lambda, \alpha)$ -limit model if there is  $N \in \mathbf{K}_\lambda$  such that  $M$  is a  $(\lambda, \alpha)$ -limit model over  $N$ . We say that  $M \in \mathbf{K}_\lambda$  is a limit model if there is  $\alpha < \lambda^+$  limit such that  $M$  is a  $(\lambda, \alpha)$ -limit model.

Observe that by iterating Fact 2.9 there exist limit models in Galois-stability cardinals for AECs with joint embedding, amalgamation and no maximal models.

In this paper, we deal with the classical global notion of universal model.

**Definition 2.11.** *Let  $\mathbf{K}$  an AEC and  $\lambda$  a cardinal.  $M \in \mathbf{K}$  is a universal model in  $\mathbf{K}_\lambda$  if  $M \in \mathbf{K}_\lambda$  and if given any  $N \in \mathbf{K}_\lambda$ , there is  $f : N \rightarrow M$ .*

**Remark 2.12.** *When an abstract elementary class has joint embedding, then  $M$  is universal over  $N$  or  $M$  is a limit model implies that  $M$  is a universal model in  $\mathbf{K}_{\|M\|}$ . A proof is given in [Maz20, 2.10].*

**2.2. Model theory of modules.** For most of the basic results of the model theory of modules, we use the comprehensive text [Pre88] of M. Prest as our primary source. The detailed history of these results can be found there.

The following definitions are fundamental and will be used throughout the text.

**Definition 2.13.** *Let  $R$  be a ring and  $L_R = \{0, +, -\} \cup \{r : r \in R\}$  be the language of  $R$ -modules.*

- $\phi(\bar{v})$  is a pp-formula if and only if

$$\phi(\bar{v}) = \exists w_1 \dots \exists w_l \left( \bigwedge_{j=1}^m \sum_{i=1}^n r_{i,j} v_i + \sum_{k=1}^l s_{k,j} w_k = 0 \right),$$

where  $r_{i,j}, s_{k,j} \in R$  for every  $i \in \{1, \dots, n\}, j \in \{1, \dots, m\}, k \in \{1, \dots, l\}$ .

- Given  $N$  an  $R$ -module,  $A \subseteq N$  and  $\bar{b} \in N^{<\omega}$  we define the pp-type of  $\bar{b}$  over  $A$  in  $N$  as

$$\text{pp}(\bar{b}/A, N) = \{\phi(\bar{v}, \bar{a}) : \phi(\bar{v}, \bar{w}) \text{ is a pp-formula, } \bar{a} \in A \text{ and } N \models \phi[\bar{b}, \bar{a}]\}.$$

- Given  $M, N$   $R$ -modules we say that  $M$  is a pure submodule of  $N$ , written as  $M \leq_{\text{pp}} N$ , if and only if  $M \subseteq N$  and  $\text{pp}(\bar{a}/\emptyset, M) = \text{pp}(\bar{a}/\emptyset, N)$  for every  $\bar{a} \in M^{<\omega}$ . Observe that in particular if  $M \leq_{\text{pp}} N$  then  $M$  is a submodule of  $N$ .

A key property of  $R$ -modules is that they have pp-quantifier elimination, i.e., every formula in the language of  $R$ -modules is equivalent to a boolean combination of pp-formulas.

**Fact 2.14** (Baur-Monk-Garavaglia, see e.g. [Pre88, §2.4]). *Let  $R$  be a ring and  $M$  a (left)  $R$ -module. Every formula in the language of  $R$ -modules is equivalent modulo  $\text{Th}(M)$  to a boolean combination of pp-formulas.*

The above theorem makes the model theory of modules algebraic in character, and we will use many of its consequences throughout the text. See for example Facts 3.2, 3.3, 3.13 and 4.2.

Recall that given  $T$  a complete first-order theory and  $A \subseteq M$  with  $M$  a model of  $T$ ,  $S^T(A)$  is the set of complete first-order types with parameters in  $A$ . A complete first-order theory  $T$  is  $\lambda$ -stable if  $|S^T(A)| \leq \lambda$  for every  $A \subseteq M$  with  $|A| = \lambda$  and  $M$  a model of  $T$ . For a complete first-order theory  $T$  this is equivalent to  $(\text{Mod}(T), \preceq)$  being  $\lambda$ -Galois-stable, where  $\preceq$  is the elementary substructure relation.

**Fact 2.15** (Fisher, Baur, see e.g. [Pre88, 3.1]). *If  $T$  is a complete first-order theory extending the theory of  $R$ -modules and  $\lambda^{|T|} = \lambda$ , then  $T$  is  $\lambda$ -stable.*

Pure-injective modules generalize the notion of injective module.

**Definition 2.16.** *A module  $M$  is pure-injective if and only if for every module  $N$ , if  $M \leq_{\text{pp}} N$  then  $M$  is a direct summand of  $N$ .*

There are many statements equivalent to the definition of pure-injectivity. The following equivalence will be used in the last section:

**Fact 2.17** ([Pre88, 2.8]). *Let  $M$  be an  $R$ -module. The following are equivalent:*

- (1)  $M$  is pure-injective.
- (2) Every  $M$ -consistent pp-type  $p(x)$  over  $A \subseteq M$  with  $|A| \leq |R| + \aleph_0$ , is realized in  $M$ .<sup>3</sup>

That is, pure-injective modules are saturated with respect to pp-types. They often suffice as a substitute for saturated models in the model theory of modules.

We will also use the pure hull of a module. The next fact has all the information the reader will need about them. They are studied extensively in [Pre88, §4] and [Zie84, §3].

**Fact 2.18.**

- (1) For  $M$  a module the pure hull of  $M$ , denoted by  $\overline{M}$ , is a pure-injective module such that  $M \leq_{pp} \overline{M}$  and it is minimum with respect to this. Its existence follows from [Zie84, 3.6] and the fact that every module can be embedded in a pure-injective module.
- (2) [Sab70] For  $M$  a module,  $M \preceq \overline{M}$ .

**2.3. Torsion-free groups.** The following class will be studied in detail.

**Definition 2.19.** Let  $\mathbf{K}^{tf} = (K^{tf}, \leq_{pp})$  where  $K^{tf}$  is the class of torsion-free abelian groups in the language  $L_{\mathbb{Z}} = \{0, +, -\} \cup \{z : z \in \mathbb{Z}\}$  (the usual language of  $\mathbb{Z}$ -modules) and  $\leq_{pp}$  is the pure subgroup relation. Recall that  $H$  is a pure subgroup of  $G$  if for every  $n \in \mathbb{N}$ ,  $nG \cap H = nH$ .

It is known that  $\mathbf{K}^{tf}$  is an AEC with  $\text{LS}(\mathbf{K}^{tf}) = \aleph_0$  that has joint embedding, amalgamation and no maximal models (see [BCG+], [BET07] or [Maz20, §4]). Furthermore limit models of uncountable cofinality were described in [Maz20].

**Fact 2.20** ([Maz20, 4.15]). *If  $G \in \mathbf{K}^{tf}$  is a  $(\lambda, \alpha)$ -limit model and  $\text{cf}(\alpha) \geq \omega_1$ , then*

$$G \cong \mathbb{Q}^{(\lambda)} \oplus \prod_p \overline{\mathbb{Z}}_{(p)}^{(\lambda)}.$$

### 3. UNIVERSAL MODELS IN CLASSES OF $R$ -MODULES

In this section we will construct universal models for certain classes of  $R$ -modules.

**Notation 3.1.** *Given  $R$  a ring, we denote by  $\mathbf{Th}_R$  the theory of left  $R$ -modules. Given  $T$  a first-order theory (not necessarily complete) extending the theory of (left)  $R$ -modules, let  $\mathbf{K}^T = (\text{Mod}(T), \leq_{pp})$  and  $|T| = |R| + \aleph_0$ .*

Our first assertion will be that  $\mathbf{K}^T$  is always an abstract elementary class. In order to prove this, we will use the following two corollaries of pp-quantifier elimination (Fact 2.14). Given  $n \in \mathbb{N}$  and  $\phi, \psi$  pp-formulas such that  $\mathbf{Th}_R \vdash \psi \rightarrow \phi$  we denote by  $\text{Inv}(-, \phi, \psi) \geq n$  the first-order sentence satisfying:  $M \models \text{Inv}(-, \phi, \psi) \geq n$  if and only if  $[\phi(M) : \psi(M)] \geq n$ . Such a formula is called an *invariant condition*.

**Fact 3.2** ([Pre88, 2.15]). *Every sentence in the language of  $R$ -modules is equivalent, modulo the theory of  $R$ -modules, to a boolean combination of invariant conditions.*

**Fact 3.3** ([Pre88, 2.23(a)(b)]). *Let  $M, N$  be  $R$ -modules and  $\phi, \psi$  pp-formulas such that  $\mathbf{Th}_R \vdash \psi \rightarrow \phi$ .*

- (1) If  $M \leq_{pp} N$ , then  $\text{Inv}(M, \phi, \psi) \leq \text{Inv}(N, \phi, \psi)$ .
- (2)  $\text{Inv}(M \oplus N, \phi, \psi) = \text{Inv}(M, \phi, \psi) \text{Inv}(N, \phi, \psi)$ .

**Lemma 3.4.** *If  $T$  is a first-order theory extending the theory of  $R$ -modules, then  $\mathbf{K}^T$  is an abstract elementary class with  $\text{LS}(\mathbf{K}^T) = |T|$ .*

<sup>3</sup>For an incomplete theory  $T$  we say that a pp-type  $p(x)$  over  $A \subseteq M$  is  $M$ -consistent if it is realized in an elementary extension of  $M$ .

*Proof.* It is easy to check that  $\mathbf{K}^T$  satisfies all the axioms of an AEC except possibly the Tarski-Vaught axiom. Moreover if  $\delta$  is a limit ordinal,  $\{M_i \in \mathbf{K}^T : i < \delta\}$  is an increasing chain (with respect to  $\leq_{pp}$ ) and  $N \in \mathbf{K}^T$  such that  $\forall i < \delta (M_i \leq_{pp} N)$ , then  $\forall i < \delta (M_i \leq_{pp} M_\delta = \bigcup_{i < \delta} M_i \leq_{pp} N)$ . Therefore, we only need to show that if  $\delta$  is a limit ordinal and  $\{M_i \in \mathbf{K}^T : i < \delta\}$  is an increasing chain, then  $M_\delta$  is a model of  $T$ .

First, by Fact 3.2, every  $\sigma \in T$  is equivalent modulo  $\mathbf{Th}_R$  to a boolean combination of invariant conditions. By putting that boolean combination in conjunctive normal form and separating the conjuncts we conclude that:

$$\text{Mod}(T) = \text{Mod}(\mathbf{Th}_R \cup \{\theta_\beta : \beta < \alpha\}),$$

where  $\alpha \leq |T|$  and each  $\theta_\beta$  is a finite disjunction of invariant statements of the form  $\text{Inv}(-, \phi, \psi) \geq k$  or of the form  $\text{Inv}(-, \phi, \psi) < k$  (for some  $pp$ -formulas  $\phi, \psi$  such that  $\mathbf{Th}_R \vdash \psi \rightarrow \phi$  and some positive integer  $k$ ).

Let  $\delta$  be a limit ordinal and  $\{M_i \in \mathbf{K}^T : i < \delta\}$  an increasing chain. It is clear that  $M_\delta \models \mathbf{Th}_R$  and that  $M_i \leq_{pp} M_\delta$  for all  $i < \delta$ . Take  $\beta < \alpha$  and consider  $\theta_\beta$ . There are two cases:

Case 1: Some disjunct of  $\theta_\beta$  is of the form  $\text{Inv}(-, \phi, \psi) \geq k$  and for some  $i < \delta$ ,  $M_i \models \text{Inv}(-, \phi, \psi) \geq k$ . Since  $M_i \leq_{pp} M_\delta$ , by Fact 3.3.(1) it follows that  $\text{Inv}(M_i, \phi, \psi) \leq \text{Inv}(M_\delta, \phi, \psi)$ , and so  $M_\delta \models \theta_\beta$ .

Case 2: Every disjunct of  $\theta_\beta$  satisfied by a  $M_i$ , for  $i < \delta$ , is of the form  $\text{Inv}(-, \phi, \psi) < k$  (for some  $\phi, \psi$ , and  $k$ ). Since  $\delta$  is a limit ordinal and  $\theta_\beta$  is a finite disjunction, there is some cofinal subchain  $\{M_{i'}\}$  of  $\{M_i : i < \delta\}$ , such that each  $M_{i'}$  satisfies the same disjunct of  $\theta_\beta$ . So without loss of generality we can assume that this is true of the entire chain, i.e, there are  $\phi, \psi$ , and  $k$  such that  $M_i \models \text{Inv}(-, \phi, \psi) < k$  for all  $i < \delta$  and  $\text{Inv}(-, \phi, \psi) < k$  is a disjunct of  $\theta_\beta$ . A counterexample to  $\text{Inv}(M_\delta, \phi, \psi) < k$  would be witnessed by finitely many tuples from  $M_\delta$ , hence by finitely many tuples from  $M_i$  for some  $i < \delta$ , a contradiction. Therefore,  $M_\delta \models \theta_\beta$ .  $\square$

**Remark 3.5.** *If  $T$  has an infinite model, then  $\mathbf{K}^T$  has no maximal models. An infinite model  $M$  of  $T$  has arbitrarily large elementary extensions, which are, ipso facto, models of  $T$  and pure extensions of  $M$ .*

The reader might wonder if  $\mathbf{K}^T$  satisfies any other of the structural properties of an AEC such as joint embedding or amalgamation. We show that if  $\mathbf{K}^T$  is closed under direct sums, then  $\mathbf{K}^T$  has both of these properties. This will be done in three steps.

**Fact 3.6** ([Pre88, Exercise 1, §2.6]). *Let  $M, N_1, N_2 \in \mathbf{K}^T$ . If  $M \leq_{pp} N_1$  and  $M \preceq N_2$ , then there are  $N \in \mathbf{K}^T$  and  $f : N_1 \xrightarrow[M]{} N$  with  $f$  elementary embedding and  $N_2 \leq_{pp} N$ .*

*Proof sketch.* Introduce new distinct constant symbols for the elements of  $N_1$  and  $N_2$ , agreeing on their common part  $M$ . Let  $\Delta(N_1)$  be the (complete) elementary diagram of  $N_1$ , let  $p^+(N_2) = \{\phi(\bar{a}) : \phi \text{ is a } pp\text{-formula, } \bar{a} \in N_2^{<\omega} \text{ and } N_2 \models \phi[\bar{a}]\}$ , and let  $p^-(N_2) = \{\neg\phi(\bar{a}) : \phi \text{ is a } pp\text{-formula, } \bar{a} \in N_2^{<\omega} \text{ and } N_2 \models \neg\phi[\bar{a}]\}$ . Then it is straightforward to verify that

$$\Sigma = \Delta(N_1) \cup p^+(N_2) \cup p^-(N_2)$$

is finitely satisfiable in  $N_1$  and any model  $N$  of  $\Sigma$  has the desired properties.  $\square$

**Proposition 3.7.** *If  $\mathbf{K}^T$  is closed under direct sums, then pure-injective modules are amalgamation bases<sup>4</sup>.*

<sup>4</sup>Recall that  $N \in \mathbf{K}$  is an *amalgamation base*, if given  $N \leq_{\mathbf{K}} N_1, N_2 \in \mathbf{K}$ , there are  $L \in \mathbf{K}$  and  $f : N_2 \xrightarrow[M]{} L$  such that  $N_1 \leq_{\mathbf{K}} L$ .

*Proof.* Let  $N \leq_{pp} N_1, N_2$  all in  $\mathbf{K}^T$  with  $N$  pure-injective. Since  $N$  is pure-injective there are submodules  $M_1, M_2$  of  $N_1, N_2$  respectively, such that for  $l \in \{1, 2\}$  we have that  $N_l = N \oplus M_l$ . Let  $L = N_1 \oplus N_2 = (N \oplus M_1) \oplus (N \oplus M_2)$ . Since  $\mathbf{K}^T$  is closed under direct sums  $L \in \mathbf{K}^T$ . Define  $f_1 : N_1 \rightarrow L$  by  $f_1(n, m_1) = (n, m_1, n, 0)$  and  $f_2 : N_2 \rightarrow L$  by  $f_2(n, m_2) = (n, 0, n, m_2)$ . Clearly  $f_1, f_2$  are pure embeddings with  $f_1 \upharpoonright_N = f_2 \upharpoonright_N$ .  $\square$

**Lemma 3.8.** *If  $\mathbf{K}^T$  is closed under direct sums, then:*

- (1)  $\mathbf{K}^T$  has joint embedding.
- (2)  $\mathbf{K}^T$  has amalgamation.

*Proof.* For the joint embedding property observe that given  $M, N \in \mathbf{K}^T$ , they embed purely in  $M \oplus N$  which is in  $\mathbf{K}^T$  by hypothesis.

Regarding the amalgamation property, let  $M \leq_{pp} N_1, N_2$  all in  $\mathbf{K}^T$ . For  $\ell \in \{1, 2\}$ ,  $M, N_\ell, \overline{M}$  satisfy the hypothesis of Fact 3.6, since  $M \preceq \overline{M}$  by Fact 2.18.(2). Then for  $\ell \in \{1, 2\}$ , there are  $N_\ell^* \in \mathbf{K}^T$  and  $f_\ell : N_\ell \xrightarrow[M]{} N_\ell^*$ , with  $f_\ell$  an elementary embedding and  $\overline{M} \leq_{pp} N_\ell^*$ .

Since  $\overline{M} \leq_{pp} N_1^*, N_2^*$  and  $\overline{M}$  is pure-injective by Fact 2.18.(1), it follows from Proposition 3.7 that there are  $N \in \mathbf{K}^T$ ,  $g_1 : N_1^* \rightarrow N$  and  $g_2 : N_2^* \rightarrow N$  with  $g_1 \upharpoonright_{\overline{M}} = g_2 \upharpoonright_{\overline{M}}$  and  $g_1, g_2$  both  $\mathbf{K}^T$ -embeddings. Finally, observe that  $g_1 \circ f_1 : N_1 \rightarrow N$  and  $g_2 \circ f_2 : N_2 \rightarrow N$  are  $\mathbf{K}^T$ -embeddings such that  $g_1 \circ f_1 \upharpoonright_M = g_2 \circ f_2 \upharpoonright_M$ .  $\square$

From the algebraic perspective the natural hypothesis is to assume that  $\mathbf{K}^T$  is closed under direct sums. On the other hand, from the model theoretic perspective it is more natural to assume that  $\mathbf{K}^T$  has joint embedding and amalgamation. This is always the case when  $T$  is a complete theory, which is precisely Example 3.10.(2) below.

Since we just showed that in  $\mathbf{K}^T$  closure under direct sums implies joint embedding and amalgamation, we will assume these throughout the paper.

**Hypothesis 3.9.** *Let  $R$  be a ring and  $T$  a first-order theory (not necessarily complete) with an infinite model extending the theory of  $R$ -modules such that:*

- (1)  $\mathbf{K}^T$  has joint embedding.
- (2)  $\mathbf{K}^T$  has amalgamation.

Even after this discussion the reader might wonder if there are any natural classes that satisfy the above hypothesis. We give some examples:

**Example 3.10.**

- (1)  $\mathbf{K}^{tf} = (K^{tf}, \leq_{pp})$  where  $K^{tf}$  is the class of torsion-free abelian groups. In this case  $T$  is a first-order axiomatization of torsion-free abelian groups. Since torsion-free abelian groups are closed under direct sums, by Lemma 3.8  $\mathbf{K}^{tf}$  has joint embedding and amalgamation.
- (2)  $\mathbf{K}^T = (\text{Mod}(T), \leq_{pp})$  where  $T$  is a complete theory extending  $\mathbf{Th}_R$ . This follows from the fact that if  $M, N \models T$ , then  $M \leq_{pp} N$  if and only if  $M \preceq N$  by pp-quantifier elimination.
- (3)  $\mathbf{K}^{\text{Th}_R} = (\text{Mod}(\mathbf{Th}_R), \leq_{pp})$ . It is clear that  $\mathbf{K}^{\text{Th}_R}$  is closed under direct sums, so by Lemma 3.8  $\mathbf{K}^{\text{Th}_R}$  has joint embedding and amalgamation.
- (4)  $\mathbf{K} = (\chi, \leq_{pp})$  where  $\chi$  is a definable category of modules in the sense of [Pre09, §3.4]. In this case  $T = \{\forall x(\phi(x) \rightarrow \psi(x)) : \mathbf{Th}_R \vdash \psi \rightarrow \phi \text{ and } \phi(M) = \psi(M) \text{ for every } M \in \chi\}$  and  $\mathbf{K}$  has joint embedding and amalgamation because  $\mathbf{K}$  is closed under direct sums (by [Pre09, 3.4.7]) and by Lemma 3.8.
- (5)  $\mathbf{K} = (\mathbf{C}, \leq_{pp})$  where  $\mathbf{C}$  is a universal Horn class. In this case  $T = T_{\mathbf{C}}$  (where  $T_{\mathbf{C}}$  is an axiomatization of  $\mathbf{C}$ ) and  $\mathbf{K}$  has joint embedding and amalgamation because  $\mathbf{K}$  is closed under direct sums (by [Pre88, 15.8]) and by Lemma 3.8.



- (6)  $\mathbf{K} = (\mathcal{F}_r, \leq_{pp})$  where  $r$  is a radical of finite type and  $\mathcal{F}_r$  is the class of  $r$ -torsion-free modules. In this case  $T$  exists by [Pre88, 15.9] and  $\mathbf{K}$  has joint embedding and amalgamation because  $\mathbf{K}$  is closed under direct sums (by [Pre88, 15.8]) and by Lemma 3.8.
- (7)  $\mathbf{K} = (\mathcal{T}_r, \leq_{pp})$  where  $r$  is a left exact radical,  $\mathcal{T}_r$  is the class of  $r$ -torsion modules and  $\mathcal{T}_r$  is closed under products. In this case  $T$  exists by [Pre88, 15.14] and  $\mathbf{K}$  has joint embedding and amalgamation by a similar reason to (5).
- (8)  $\mathbf{K} = (K_{\text{flat}}, \leq_{pp})$  where  $K_{\text{flat}}$  is the class of (left) flat  $R$ -modules over a right coherent ring. In this case  $T$  exists by [Pre88, 14.18] and  $\mathbf{K}$  has joint embedding and amalgamation because the class of flat modules is closed under direct sums and by Lemma 3.8.

The following example shows that Hypothesis 3.9 is not trivial, i.e., given  $T$  a first-order theory with an infinite model extending the theory of  $R$ -modules Hypothesis 3.9 does not necessarily hold.

**Example 3.11.** Let  $T = \text{Th}_{\mathbb{Z}} \cup \{\text{Inv}(-, x = x, 3x = 0) < 6\}$ .

Let  $A$  be an abelian group satisfying  $T$  and  $B$  the subgroup of  $A$  defined by  $3x = 0$ . Then  $|A/B| \in \{1, 2, 3, 4, 5\}$  and so  $A/B \cong A_0$ , where  $A_0$  is one of the finite groups  $\{0\}$ ,  $\mathbb{Z}/2$ ,  $\mathbb{Z}/2 \times \mathbb{Z}/2$ ,  $\mathbb{Z}/4$ ,  $\mathbb{Z}/5$ , or  $\mathbb{Z}/3$ .

In particular, if  $B = 0$ , observe that the first five  $A_0$ 's just listed are models of  $T$ . On the other hand, if  $B \neq 0$ , then  $B \cong (\mathbb{Z}/3)^{(\kappa)}$  for some finite or infinite cardinal  $\kappa$ , and since 3 is a prime, it has no non-trivial extensions by any of the groups  $A_0$ . There is one exceptional case, as  $\mathbb{Z}/9$  is an extension of  $\mathbb{Z}/3$  by itself.

Since the invariants multiply across direct sums (Fact 3.3), then all the models of  $T$  are  $\mathbb{Z}/9$  or of the form  $A_0$  or  $(\mathbb{Z}/3)^{(\kappa)} \oplus A_0$ , for some choice of  $A_0$  and  $\kappa$  a finite or infinite cardinal.

Therefore, there are many examples of failures of the joint embedding property: amongst them we have that  $\mathbb{Z}/2$  and  $\mathbb{Z}/5$  do not have a common extension to a model of  $T$ , and since the zero module is pure-injective, this is an example of the failure of amalgamation over pure-injectives. Since  $(\mathbb{Z}/3)^{(\aleph_0)}$  is pure-injective,  $(\mathbb{Z}/3)^{(\aleph_0)} \oplus \mathbb{Z}/2$  and  $(\mathbb{Z}/3)^{(\aleph_0)} \oplus \mathbb{Z}/5$  provide an infinite example.

It is worth pointing out that there is an easy first-order argument to find universal models if one assumes the hypothesis that  $\mathbf{K}^T$  is closed under direct sums.<sup>5</sup>

**Lemma 3.12.** If  $\mathbf{K}^T$  is closed under direct sums and  $\lambda^{|T|} = \lambda$ , then  $\mathbf{K}_{\lambda}^T$  has a universal model.

*Proof.* Observe that  $T$  has no more than  $2^{|T|}$  complete extensions. Each such extension is  $\lambda$ -stable, see Fact 2.15, and so has a saturated model of cardinality  $\lambda$ . Take the direct sum  $U$  of all of these; it has cardinality  $2^{|T|}\lambda = \lambda$ . We claim that  $U \in \mathbf{K}_{\lambda}^T$  and is universal in  $\mathbf{K}_{\lambda}^T$ . But  $\mathbf{K}^T$  is closed under direct sums, so  $U \in \mathbf{K}^T$ ; and we have already observed that  $\|U\| = \lambda$ .

If  $N \in \mathbf{K}_{\lambda}^T$ , then  $N$  is elementarily embedded in the  $\lambda$ -saturated model of  $\text{Th}(N)$  which is a summand of  $U$ , and hence  $N$  is purely embedded in  $U$ .  $\square$

**3.1. Galois-stability.** The following consequence of  $pp$ -quantifier elimination will be the key to the arguments in this subsection:

**Fact 3.13** ([Pre88, 2.17]). Let  $N \in \mathbf{K}^T$ ,  $A \subseteq N$  and  $\bar{b}_1, \bar{b}_2 \in N^{<\omega}$ . Then:

$$pp(\bar{b}_1/A, N) = pp(\bar{b}_2/A, N) \text{ if and only if } \text{tp}(\bar{b}_1/A, N) = \text{tp}(\bar{b}_2/A, N).$$

With this, we are able to show that  $pp$ -types and Galois-types are the same over models.

**Lemma 3.14.** Let  $M, N_1, N_2 \in \mathbf{K}^T$ ,  $M \leq_{pp} N_1, N_2$ ,  $\bar{b}_1 \in N_1^{<\omega}$  and  $\bar{b}_2 \in N_2^{<\omega}$ . Then:

$$\mathbf{gtp}(\bar{b}_1/M; N_1) = \mathbf{gtp}(\bar{b}_2/M; N_2) \text{ if and only if } pp(\bar{b}_1/M, N_1) = pp(\bar{b}_2/M, N_2).$$

<sup>5</sup>This was discovered after we had a proof using the theory of abstract elementary classes (see Lemma 3.17).

*Proof.  $\Rightarrow$ :* Suppose  $\mathbf{gtp}(\bar{b}_1/M; N_1) = \mathbf{gtp}(\bar{b}_2/M; N_2)$ . Since  $\mathbf{K}^T$  has amalgamation, there are  $N \in \mathbf{K}^T$  and  $f : N_1 \rightarrow N$  a  $\mathbf{K}^T$ -embedding such that  $f \upharpoonright_M = \text{id}_M$ ,  $f(\bar{b}_1) = \bar{b}_2$  and  $N_2 \leq_{pp} N$ . Then the result follows from the fact that  $\mathbf{K}^T$ -embeddings preserve and reflect  $pp$ -formulas by definition.

*$\Leftarrow$ :* Suppose  $pp(\bar{b}_1/M, N_1) = pp(\bar{b}_2/M, N_2)$ . Since  $M \in \mathbf{K}^T$  and  $\mathbf{K}^T$  has amalgamation, there is  $N \in \mathbf{K}^T$  and  $f : N_1 \rightarrow N$  a  $\mathbf{K}^T$ -embedding such that  $f \upharpoonright_M = \text{id}_M$  and  $N_2 \leq_{pp} N$ . Using that  $\mathbf{K}^T$ -embeddings preserve  $pp$ -formulas we have that  $pp(f(\bar{b}_1)/M, N) = pp(\bar{b}_2/M, N)$ .

Then by Fact 3.13 it follows that  $tp(f(\bar{b}_1)/M, N) = tp(\bar{b}_2/M, N)$ . Let  $N^*$  an elementary extension of  $N$  such that there is  $g \in \text{Aut}_M(N^*)$  with  $g(f(\bar{b}_1)) = \bar{b}_2$ . Observe that since  $\mathbf{K}^T$  is first-order axiomatizable  $N^* \in \mathbf{K}^T$ . Consider  $h := g \circ f : N_1 \rightarrow N^*$ .

It is clear that  $h(\bar{b}_1) = \bar{b}_2$ ,  $h \upharpoonright_M = \text{id}_M$  and since being an elementary substructure is stronger than being a pure substructure it follows that  $h : N_1 \rightarrow N^*$  is a  $\mathbf{K}^T$ -embedding and  $N_2 \leq_{pp} N^*$ . Therefore,  $\mathbf{gtp}(\bar{b}_1/M; N_1) = \mathbf{gtp}(\bar{b}_2/M; N_2)$ .  $\square$

The next corollary follows from the preceding lemma since we can witness that two Galois-types are different by a  $pp$ -formula.

**Corollary 3.15.**  $\mathbf{K}^T$  is  $(< \aleph_0)$ -tame.

The next theorem is the main result of this subsection.

**Theorem 3.16.** If  $\lambda^{|\mathbf{T}|} = \lambda$ , then  $\mathbf{K}^T$  is  $\lambda$ -Galois-stable.

*Proof.* Let  $M \in \mathbf{K}_\lambda^T$  and  $\{p_i : i < \alpha\}$  an enumeration without repetitions of  $\mathbf{gS}(M)$  where  $\alpha \leq 2^\lambda$ . Since  $\mathbf{K}^T$  has amalgamation, there is  $N \in \mathbf{K}^T$  and  $\{a_i : i < \alpha\} \subseteq N$  such that  $p_i = \mathbf{gtp}(a_i/M; N)$  for every  $i < \alpha$ .

Let  $\Phi : \mathbf{gS}(M) \rightarrow S_{pp}^{Th(N)}(M)$  be defined by  $p_i \mapsto pp(a_i/M, N)$ . By Lemma 3.14  $\Phi$  is a well-defined injective function. By Fact 3.13  $|S_{pp}^{Th(N)}(M)| = |S^{Th(N)}(M)|$ . Then it follows from Fact 2.15 that  $|S^{Th(N)}(M)| \leq \lambda$ , hence  $|\mathbf{gS}(M)| \leq \lambda$ .  $\square$

**3.2. Universal models.** It is straightforward to construct universal models in  $\mathbf{K}^T$  for  $\lambda$ 's satisfying that  $\lambda^{|\mathbf{T}|} = \lambda$ . This follows from Fact 2.9 and Remark 2.12.

**Lemma 3.17.** If  $\lambda^{|\mathbf{T}|} = \lambda$ , then  $\mathbf{K}_\lambda^T$  has a universal model.

The following lemma shows how to build universal models in cardinals where  $\mathbf{K}^T$  might not be  $\lambda$ -Galois-stable.

**Lemma 3.18.** If  $\forall \mu < \lambda(\mu^{|\mathbf{T}|} < \lambda)$ , then  $\mathbf{K}_\lambda^T$  has a universal model.

*Proof.* We may assume that  $\lambda$  is a limit cardinal, because if it is not the case then we have that  $\lambda^{|\mathbf{T}|} = \lambda$  and we can apply Lemma 3.17. Let  $\text{cf}(\lambda)\text{cf}(\lambda) = \kappa \leq \lambda$ . By using the hypothesis that  $\forall \mu < \lambda(\mu^{|\mathbf{T}|} < \lambda)$ , it is easy to build  $\{\lambda_i : i < \kappa\}$  an increasing continuous sequence of cardinals such that  $\forall i(\lambda_{i+1}^{|\mathbf{T}|} = \lambda_{i+1})$  and  $\sup_{i < \kappa} \lambda_i = \lambda$ .

We build  $\{M_i : i < \kappa\}$  an increasing continuous chain such that:

- (1)  $M_{i+1}$  is  $\|M_{i+1}\|$ -universal over  $M_i$ .
- (2)  $M_i \in \mathbf{K}_{\lambda_i}^T$ .

In the base step pick any  $M \in \mathbf{K}_{\lambda_0}^T$  and if  $i$  is limit, let  $M_i = \bigcup_{j < i} M_j$ .

If  $i = j + 1$ , by construction we are given  $M_j \in \mathbf{K}_{\lambda_j}^T$ . Using that  $\mathbf{K}^T$  has no maximal models, we find  $N \in \mathbf{K}_{\lambda_{j+1}}^T$  such that  $M_j \leq_{pp} N$ . Since  $\lambda_{j+1}^{|\mathbf{T}|} = \lambda_{j+1}$ , by Theorem 3.16  $\mathbf{K}^T$  is  $\lambda_{j+1}$ -Galois-stable. Then by Fact 2.9 applied to  $N$ , there is  $M_{j+1} \in \mathbf{K}_{\lambda_{j+1}}^T$  universal over  $N$ . Using that  $\mathbf{K}^T$  has amalgamation, it is straightforward to check that (1) holds.

This finishes the construction of the chain.

Let  $M = \bigcup_{i < \kappa} M_i$ . By (2)  $\|M\| = \lambda$ . We show that  $M$  is universal in  $\mathbf{K}_\lambda^T$ .

Let  $N \in \mathbf{K}_\lambda^T$  and  $\{N_i : i < \kappa\}$  an increasing continuous chain such that  $\forall i (N_i \in \mathbf{K}_{\lambda_i}^T)$  and  $\bigcup_{i < \kappa} N_i = N$ . We build  $\{f_i : i < \kappa\}$  such that:

- (1)  $f_i : N_i \rightarrow M_{i+1}$ .
- (2)  $\{f_i : i < \kappa\}$  is an increasing chain.

Observe that this is enough by taking  $f = \bigcup_{i < \kappa} f_i : N = \bigcup_{i < \kappa} N_i \rightarrow \bigcup_{i < \kappa} M_{i+1} = M$ .

Now, let us do the construction. In this case the base step is non-trivial. By joint embedding there is  $g : N_0 \rightarrow M^*$  with  $M_0 \leq_{pp} M^* \in \mathbf{K}_{\lambda_0}^T$ . Now, since  $M_1$  is  $\|M_1\|$ -universal over  $M_0$  there is  $h : M^* \xrightarrow{M_0} M_1$ . Let  $f_0 := h \circ g$  and observe that this satisfies the requirements.

We do the induction steps.

If  $i$  is limit, let  $f_i = \bigcup_{j < i} f_j : N_i = \bigcup_{j < i} N_j \rightarrow M_{i+1}$ .

If  $i = j + 1$ , by construction we have  $f_j : N_j \rightarrow M_{j+1}$  and  $N_j \leq_{pp} N_{j+1}$ . Since  $\mathbf{K}^T$  has amalgamation there is  $M' \in \mathbf{K}_{\lambda_{j+1}}^T$  and  $g : N_{j+1} \rightarrow M'$  such that  $M_{j+1} \leq_{pp} M'$  and  $f_j \upharpoonright_{N_j} = g \upharpoonright_{N_j}$ . Since  $M_{j+2}$  is  $\|M_{j+2}\|$ -universal over  $M_{j+1}$ , there is  $h : M' \xrightarrow{M_{j+1}} M_{j+2}$ . Let  $f_{j+1} := h \circ g$  and observe that this satisfies the requirements.  $\square$

Putting together Lemma 3.17 and Lemma 3.18 we get one of our main results.

**Theorem 3.19.** *If  $\lambda^{|T|} = \lambda$  or  $\forall \mu < \lambda (\mu^{|T|} < \lambda)$ , then  $\mathbf{K}_\lambda^T$  has a universal model.*

The proof of Lemma 3.17 and Lemma 3.18 can be extended in a straightforward way to the following general setting.

**Corollary 3.20.** *Let  $\mathbf{K}$  be an AEC with joint embedding, amalgamation and no maximal models. Assume there is  $\theta_0 \geq \text{LS}(\mathbf{K})$  and  $\kappa$  such that for all  $\theta \geq \theta_0$ , if  $\theta^\kappa = \theta$ , then  $\mathbf{K}$  is  $\theta$ -Galois-stable.*

*Suppose  $\lambda > \theta_0$ . If  $\lambda^\kappa = \lambda$  or  $\forall \mu < \lambda (\mu^\kappa < \lambda)$ , then  $\mathbf{K}_\lambda$  has a universal model.*<sup>6</sup>

**Remark 3.21.** *In [Vas16a, 4.13] it is shown that if  $\mathbf{K}$  is an AEC with joint embedding, amalgamation and no maximal models,  $\mathbf{K}$  is  $\text{LS}(\mathbf{K})$ -tame and  $\mathbf{K}$  is  $\lambda$ -Galois-stable for some  $\lambda \geq \text{LS}(\mathbf{K})$ , then there are  $\theta_0$  and  $\kappa$  satisfying the hypothesis of Corollary 3.20.*

**3.3. Reduced torsion-free abelian groups.** Recall that  $\mathbf{K}^{tf}$  has joint embedding and amalgamation, so it satisfies Hypothesis 3.9. Moreover,  $|T^{tf}| = \aleph_0$ , therefore the next assertion follows directly from Theorem 3.16 and Theorem 3.19.

**Corollary 3.22.**

- (1) *If  $\lambda^{\aleph_0} = \lambda$ , then  $\mathbf{K}^{tf}$  is  $\lambda$ -Galois-stable.*
- (2) *If  $\lambda^{\aleph_0} = \lambda$  or  $\forall \mu < \lambda (\mu^{\aleph_0} < \lambda)$ , then  $\mathbf{K}_\lambda^{tf}$  has a universal model.*

**Remark 3.23.** *In [BET07, 0.3] it is shown that:  $\mathbf{K}^{tf}$  is  $\lambda$ -Galois-stable if and only if  $\lambda^{\aleph_0} = \lambda$ . The argument given here differs substantially with that of [BET07, 0.3], their argument does not consider pp-formulas and instead exploits the property that  $\mathbf{K}^{tf}$  admits intersections.*

As mentioned in the introduction, Shelah's result [Sh17, 1.2] is concerned with reduced torsion-free groups instead of with torsion-free groups. The next two assertions show how we can recover his assertion from the above results. First let us introduce a new class of groups.

**Definition 3.24.** *Let  $\mathbf{K}^{rtf} = (K^{rtf}, \leq_{pp})$  where  $K^{rtf}$  is the class of reduced torsion-free abelian groups defined in the usual language  $L_{\mathbb{Z}}$  of  $\mathbb{Z}$ -modules, and  $\leq_{pp}$  is the pure subgroup relation. Recall that a group  $G$  is reduced if its only divisible subgroup is 0.*

<sup>6</sup>In Lemma 3.17 and Theorem 3.18  $\theta_0 = \text{LS}(\mathbf{K}^T) = |T|$  and  $\kappa = |T|$ .

**Fact 3.25.** *Let  $\lambda$  an infinite cardinal.  $\mathbf{K}_\lambda^{tf}$  has a universal model if and only if  $\mathbf{K}_\lambda^{rtf}$  has a universal model.*

*Proof.* The proof follows from the fact that divisible torsion-free abelian groups of cardinality  $\leq \lambda$  are purely embeddable into  $\mathbb{Q}^{(\lambda)}$  and that every group can be written as a direct sum of a unique divisible subgroup and a unique up to isomorphisms reduced subgroup (see [Fuc15, §4.2.4, §4.2.5]).  $\square$

The following is precisely [Sh17, 1.2].

**Corollary 3.26.**

- (1) *If  $\lambda^{\aleph_0} = \lambda$ , then  $\mathbf{K}_\lambda^{rtf}$  has a universal model.*
- (2) *If  $\lambda = \sum_{n < \omega} \lambda_n$  and  $\aleph_0 \leq \lambda_n = (\lambda_n)^{\aleph_0} < \lambda_{n+1}$ , then  $\mathbf{K}_\lambda^{rtf}$  has a universal model.*
- (3)  *$\mathbf{K}^{rtf}$  has amalgamation, joint embedding, is an AEC and is  $\lambda$ -Galois-stable if  $\lambda^{\aleph_0} = \lambda$ .*

*Proof.* For (1) and (2), realize that  $\lambda$  either satisfies the first or second hypothesis of Corollary 3.22.(2), hence  $\mathbf{K}_\lambda^{tf}$  has a universal model. Then by Fact 3.25 we conclude that  $\mathbf{K}_\lambda^{rtf}$  has a universal model in either case.

For (3), the first three assertions are easy to show. As for the last one, this follows from Corollary 3.22.(1) and the fact that if  $G, H \in \mathbf{K}^{rtf}$  and  $a, b \in H$  then:  $\mathbf{gtp}_{\mathbf{K}^{rtf}}(a/G; H) = \mathbf{gtp}_{\mathbf{K}^{rtf}}(b/G; H)$  if and only if  $\mathbf{gtp}_{\mathbf{K}^{tf}}(a/G; H) = \mathbf{gtp}_{\mathbf{K}^{tf}}(b/G; H)$ .  $\square$

**Remark 3.27.** *It is worth noticing that Corollary 3.22.(2) not only implies [Sh17, 1.2.1, 1.2.2] (Corollary 3.26.(1) and Corollary 3.26.(2)), but the two assertions are equivalent. The backward direction follows from the fact that if  $\lambda$  satisfies  $\text{cf}(\lambda) \geq \omega_1$  and  $\forall \mu < \lambda (\mu^{\aleph_0} < \lambda)$ , then  $\lambda^{\aleph_0} = \lambda$ .*

**Remark 3.28.** *It follows from Corollary 3.22.(2) that if  $2^{\aleph_0} < \aleph_\omega$ , then  $\mathbf{K}_{\aleph_\omega}^{tf}$  has a universal model. On the other hand, it follows from [KojSh95, 3.7] that if  $\aleph_\omega < 2^{\aleph_0}$ , then  $\mathbf{K}_{\aleph_\omega}^{tf}$  does not have a universal model. Hence the existence of a universal model in  $\mathbf{K}^{tf}$  of cardinality  $\aleph_\omega$  is independent of ZFC. Similarly one can show that the existence of a universal model in  $\mathbf{K}^{tf}$  of cardinality  $\aleph_n$  is independent of ZFC for every  $n \geq 1$ .*

#### 4. LIMIT MODELS IN CLASSES OF $R$ -MODULES

In this section we will begin the study of limit models in classes of  $R$ -modules under Hypothesis 3.9. The existence of limit models in  $\mathbf{K}^T$  for  $\lambda$ 's satisfying  $\lambda^{|T|} = \lambda$  follows directly from Theorem 3.16 and Fact 2.9.

**Corollary 4.1.** *If  $\lambda^{|T|} = \lambda$ , then there is a  $(\lambda, \alpha)$ -limit model in  $\mathbf{K}^T$  for every  $\alpha < \lambda^+$  limit ordinal.*

We first show that any two limit models are elementarily equivalent. In order to do that, we will use one more consequence of  $pp$ -quantifier elimination (Fact 2.14).

**Fact 4.2** ([Pre88, 2.18]). *Let  $M$  and  $N$   $R$ -modules.  $M$  is elementary equivalent to  $N$  if and only if  $\text{Inv}(M, \phi, \psi) = \text{Inv}(N, \phi, \psi)$  for every  $\phi, \psi$   $pp$ -formulas in one free variable such that  $\text{Th}_R \vdash \psi \rightarrow \phi$ .*

**Lemma 4.3.** *If  $M, N$  are limit models, then  $M$  and  $N$  are elementary equivalent.*

*Proof.* Assume  $M$  is a  $(\lambda, \alpha)$ -limit model for  $\alpha < \lambda^+$  and let  $\{M_i : i < \alpha\} \subseteq \mathbf{K}_\lambda^T$  be a witness for it. Similarly assume  $N$  is a  $(\mu, \beta)$ -limit model for  $\beta < \mu^+$  and let  $\{N_i : i < \beta\} \subseteq \mathbf{K}_\mu^T$  be a witness for it.

By Fact 4.2, it is enough to show that for every  $\phi, \psi$ ,  $pp$ -formulas in one free variable such that  $\text{Th}_R \vdash \psi \rightarrow \phi$ , and  $n \in \mathbb{N}$ :  $\text{Inv}(M, \phi, \psi) \geq n$  if and only if  $\text{Inv}(N, \phi, \psi) \geq n$ . By the symmetry

of this situation, we only need to prove one implication. So consider such  $pp$ -formulas  $\phi, \psi$  and  $n \in \mathbb{N}$  such that  $\text{Inv}(M, \phi, \psi) \geq n$ . We show that  $\text{Inv}(N, \phi, \psi) \geq n$ .

If  $n = 0$ , the result is clear. So assume that  $n \geq 1$ . Then since  $\text{Inv}(M, \phi, \psi) \geq n$ , there are  $m_0, \dots, m_{n-1} \in M$  such that:

$$M \models \bigwedge_i \phi(m_i) \wedge \bigwedge_{i \neq j} \neg \psi(m_i - m_j).$$

Applying the downward Löwenheim-Skolem-Tarski axiom inside  $M$  to  $\{m_i : i < n\}$ , we get  $M^* \leq_{pp} M$  such that  $M^* \in \mathbf{K}_{\text{LS}(\mathbf{K})}^T$  and  $\{m_i : i < n\} \subseteq M^*$ . Then it is still the case that

$$M^* \models \bigwedge_i \phi(m_i) \wedge \bigwedge_{i \neq j} \neg \psi(m_i - m_j).$$

By joint embedding there is  $g$  and  $M^{**} \in \mathbf{K}_\mu^T$  such that  $g : M^* \rightarrow M^{**}$  and  $N_0 \leq_{pp} M^{**}$ . Then since  $N_1$  is universal over  $N_0$ , there is  $h : M^{**} \xrightarrow[N_0]{} N_1$ . Finally, observe that:

$$N \models \bigwedge_i \phi(h \circ g(m_i)) \wedge \bigwedge_{i \neq j} \neg \psi(h \circ g(m_i) - h \circ g(m_j)).$$

Hence  $\text{Inv}(N, \phi, \psi) \geq n$ . □

**Remark 4.4.** *Observe that in the proof of the above lemma we only used that  $\mathbf{K}^T$  is an AEC of modules with the joint embedding property.*

As in [Maz20, §4], limit models with chains of big cofinality are easier to understand than those of small cofinalities. Due to this we begin by studying the former.

**Theorem 4.5.** *Assume  $\lambda \geq |T|^+ = \text{LS}(\mathbf{K}^T)^+$ . If  $M$  is a  $(\lambda, \alpha)$ -limit model and  $\text{cf}(\alpha) \geq |T|^+$ , then  $M$  is pure-injective.*

*Proof.* Fix  $\{M_i : i < \alpha\}$  a witness to the fact that  $M$  is a  $(\lambda, \alpha)$ -limit model. We show that  $M$  is pure-injective using the equivalence of Fact 2.17.

Let  $p(x)$  be an  $M$ -consistent  $pp$ -type over  $A \subseteq M$  and  $|A| \leq |R| + \aleph_0 = |T|$ . Then there is a module  $N$  and  $b \in N$  with  $M \preceq N \in \mathbf{K}_{\|M\|}^T$  and  $b$  realizing  $p$ . Since  $|A| \leq |T|$  and  $\text{cf}(\alpha) \geq |T|^+$ , there is  $i < \alpha$  such that  $A \subseteq M_i$ .

Note that  $M_i \leq_{pp} N$ . Then there is  $f : N \xrightarrow[M_i]{} M_{i+1}$ , because  $M_{i+1}$  is universal over  $M_i$ . Since  $A$  is fixed by the choice of  $M_i$ , it is easy to see that  $f(b) \in M_{i+1} \leq_{pp} M$  realizes  $p(x)$ . Therefore,  $M$  is pure-injective. □

The following fact about pure-injective modules is a generalization of Bumby's result [Bum65]. A proof of it (and a discussion of the general setting) appears in [GKS18, 3.2]. We will use it to show uniqueness of limit models of big cofinalities.

**Fact 4.6.** *Let  $M, N$  be pure-injective modules. If there is  $f : M \rightarrow N$  a  $\mathbf{K}^{\text{Th}_R}$ -embedding and  $g : N \rightarrow M$  a  $\mathbf{K}^{\text{Th}_R}$ -embedding, then  $M \cong N$ .*

**Corollary 4.7.** *Assume  $\lambda \geq |T|^+ = \text{LS}(\mathbf{K}^T)^+$ . If  $M$  is a  $(\lambda, \alpha)$ -limit model and  $N$  is a  $(\lambda, \beta)$ -limit model such that  $\text{cf}(\alpha), \text{cf}(\beta) \geq |T|^+$ , then  $M$  is isomorphic to  $N$ .*

*Proof.* It is straightforward to check that  $M$  and  $N$  are universal models in  $\mathbf{K}_\lambda^T$  (see Remark 2.12). Since  $M$  and  $N$  are pure-injective by Theorem 4.5, then the result follows from Fact 4.6 because  $\mathbf{K}^T$ -embeddings and  $\mathbf{K}^{\text{Th}_R}$ -embeddings are the same. □

Dealing with limit models of small cofinality is complicated. We will only be able to describe limit models of countable cofinality under the additional assumption that  $\mathbf{K}^T$  is closed under direct sums. All the examples of Example 3.10, except Example 3.10.(2), satisfy this additional hypothesis.

**Lemma 4.8.** *Assume  $\mathbf{K}^T$  is closed under direct sums. If  $M \in \mathbf{K}_\lambda^T$  is pure-injective and  $U \in \mathbf{K}_\lambda^T$  is a universal model in  $\mathbf{K}_\lambda^T$ , then  $M \oplus U$  is universal over  $M$ .*

*Proof.* It is clear that  $M \leq_{pp} M \oplus U$  and that both modules have the same cardinality, so take  $N \in \mathbf{K}_\lambda^T$  such that  $M \leq_{pp} N$ . Since  $M$  is pure-injective we have that  $N = M \oplus M'$  for some  $M' \in \mathbf{K}_{\leq \lambda}^T$ . Using that  $\mathbf{K}^T$  has no maximal models and that  $U$  is universal in  $\mathbf{K}_\lambda^T$ , there is  $f' : M' \rightarrow U$  a  $\mathbf{K}^T$ -embedding. Let  $f : M \oplus M' \rightarrow M \oplus U$  be given by  $f(a + b) = a + f'(b)$ . It is easy to check that  $f$  is a  $\mathbf{K}^T$ -embedding that fixes  $M$ .  $\square$

**Theorem 4.9.** *Assume  $\lambda \geq |T|^+ = \text{LS}(\mathbf{K}^T)^+$  and  $\mathbf{K}^T$  is closed under direct sums. If  $M$  is a  $(\lambda, \omega)$ -limit model and  $N$  is a  $(\lambda, |T|^+)$ -limit model, then  $M \cong N^{(\aleph_0)}$ .*

*Proof.* For every  $i < \omega$ , let  $N_i$  be given by  $i$ -many direct copies of  $N$ . Consider the increasing chain  $\{N_i : i < \omega\} \subseteq \mathbf{K}_\lambda^T$ .

By Theorem 4.5  $N \in \mathbf{K}^T$  is pure-injective. Since pure-injective modules are closed under finite direct sums,  $N_i$  is pure-injective for every  $i < \omega$ . Moreover, for each  $i < \omega$ ,  $N_{i+1} = N_i \oplus N$  is universal over  $N_i$  because  $N$  is universal in  $\mathbf{K}_\lambda^T$ ,  $N_i$  is pure-injective and by Lemma 4.8. Therefore,  $N_\omega := \bigcup_{i < \omega} N_i$  is a  $(\lambda, \omega)$ -limit model.

Since  $N_\omega$  and  $M$  are limit models with chains of the same cofinality, a back-and-forth argument shows that  $N_\omega \cong M$ . Hence  $M \cong N^{(\aleph_0)}$ .  $\square$

Lemma 4.8 can also be used to characterize Galois-stability in classes closed under direct sums.

**Corollary 4.10.** *Assume  $\mathbf{K}^T$  is closed under direct sums and  $\lambda \geq |T|^+$  is an infinite cardinal.  $\mathbf{K}^T$  is  $\lambda$ -Galois-stable if and only if  $\mathbf{K}_\lambda^T$  has a pure-injective universal model.*

*Proof.* The forward direction follows from the fact that  $(\lambda, |T|^+)$ -limit models are pure-injective by Theorem 4.5. So we sketch the backward direction. Let  $M \in \mathbf{K}_\lambda^T$  and  $U \in \mathbf{K}_\lambda^T$  a pure-injective universal model. By universality of  $U$  we may assume that  $M \leq_{pp} U$ . Then by minimality of the pure hull we have that  $\overline{M} \leq_{pp} U$ , thus  $\overline{M} \in \mathbf{K}_\lambda^T$ . So by Lemma 4.8  $\overline{M} \oplus U$  is universal over  $\overline{M}$ . Therefore, every type over  $M$  is realized in  $\overline{M} \oplus U$ . Hence  $|\mathbf{gS}(M)| \leq \|\overline{M} \oplus U\| = \lambda$ .  $\square$

**Remark 4.11.** *Observe that by Corollary 4.7 we know that for every cardinal  $\lambda$  the number of non-isomorphic limit models is bounded by  $|\{\alpha : \alpha \leq |T|, \alpha \text{ is limit and } \text{cf}(\alpha) = \alpha\}| + 1$ . So for example, when  $R$  is countable, we know that there are at most two non-isomorphic limit models.*

We believe the following question is very interesting (see also Conjecture 2 of [BoVan]):

**Question 4.12.** *Let  $\mathbf{K}^T$  as in Hypothesis 3.9. How does the spectrum of limit models look like? More precisely, given  $\lambda$ , how many non-isomorphic limit models are there of cardinality  $\lambda$  for a given  $\mathbf{K}^T$ ? Is it always possible to find  $T$  such that  $\mathbf{K}^T$  has the maximum number of non-isomorphic limit models?*

We will be able to answer Question 4.12 when the ring is countable.

**Theorem 4.13.** *Let  $R$  be a countable ring. Assume  $\mathbf{K}^T$  satisfies Hypothesis 3.9.*

- (1) If  $\mathbf{K}^T$  is Galois-superstable<sup>7</sup>, then there is  $\mu < \beth_{(2^{\aleph_0})^+}$  such for every  $\lambda \geq \mu$  there is a unique limit model of cardinality  $\lambda$ .
- (2) If  $\mathbf{K}^T$  is not Galois-superstable, then  $\mathbf{K}^T$  does not have uniqueness of limit models in any infinite cardinal  $\lambda \geq \text{LS}(\mathbf{K}^T)^+ = \aleph_1$ . More precisely, if  $\mathbf{K}^T$  is  $\lambda$ -Galois-stable there are exactly two non-isomorphic limit models of cardinality  $\lambda$ .

*Proof sketch.*  $\mathbf{K}^T$  has joint embedding, amalgamation and no maximal models and by Corollary 3.15  $\mathbf{K}^T$  is  $(< \aleph_0)$ -tame. Due to this we can use the results of [GrVas17] and [Vas18].

- (1) This follows on general grounds from [Vas18, 4.24] and [GrVas17, 5.5].
- (2) Let  $\lambda \geq \aleph_1$  such that  $\mathbf{K}^T$  is  $\lambda$ -Galois-stable. As in [Maz20, 4.19, 4.20, 4.21, 4.23] one can show that the limit models of countable cofinality are not pure-injective. Since we know that limit models of uncountable cofinality are pure-injective by Theorem 4.5, we can conclude that the  $(\lambda, \omega)$ -limit model and the  $(\lambda, \omega_1)$ -limit model are not isomorphic. Moreover, given  $N$  a  $(\lambda, \alpha)$ -limit model,  $N$  is isomorphic to the  $(\lambda, \omega)$ -limit model if  $cf(\alpha) = \omega$  (by a back-and-forth argument) or  $N$  is isomorphic to the  $(\lambda, \omega_1)$ -limit model if  $cf(\alpha) > \omega$  (by Corollary 4.7).

□

**4.1. Torsion-free abelian groups.** In this section we will show how to apply the results we just obtained to answer Question 4.25 of [Maz20].

Recall that a group  $G$  is *algebraically compact* if given  $\mathbb{E} = \{f_i(x_{i_0}, \dots, x_{i_{n_i}}) = a_i : i < \omega\}$  a set of linear equations over  $G$ ,  $\mathbb{E}$  is finitely solvable in  $G$  if and only if  $\mathbb{E}$  is solvable in  $G$ . It is well-known that an abelian group  $G$  is algebraically compact if and only if  $G$  is pure-injective (see e.g. [Fuc15, 1.2, 1.3]). The following theorem answers Question 4.25 of [Maz20].

**Theorem 4.14.** *If  $G \in \mathbf{K}^{tf}$  is a  $(\lambda, \omega)$ -limit model, then  $G \cong \mathbb{Q}^{(\lambda)} \oplus \prod_p \overline{\mathbb{Z}}_{(p)}^{(\lambda)}^{(\aleph_0)}$ .*

*Proof.* The amalgamation property together with the existence of a limit model imply that  $\mathbf{K}^{tf}$  is  $\lambda$ -Galois-stable. Then by Remark 3.23  $\lambda^{\aleph_0} = \lambda$ , so by Corollary 4.1 there is  $H$  a  $(\lambda, \omega_1)$ -limit model. Since  $\mathbf{K}^{tf}$  is closed under direct sums, we have that  $G \cong H^{(\aleph_0)}$  by Theorem 4.9.

In view of the fact that  $H$  is a  $(\lambda, \omega_1)$ -limit model, by Fact 2.20  $H \cong \mathbb{Q}^{(\lambda)} \oplus \prod_p \overline{\mathbb{Z}}_{(p)}^{(\lambda)}$ . Therefore we have:

$$G \cong \left( \mathbb{Q}^{(\lambda)} \oplus \prod_p \overline{\mathbb{Z}}_{(p)}^{(\lambda)} \right)^{(\aleph_0)} \cong \mathbb{Q}^{(\lambda)} \oplus \prod_p \overline{\mathbb{Z}}_{(p)}^{(\lambda)}^{(\aleph_0)}.$$

□

In [Maz20, 4.22] it was shown that limit models of countable cofinality are not pure-injective. The argument given there uses some deep facts about the theory of AECs. Here we give a new argument that relies on some well-known properties of abelian groups.

**Corollary 4.15.** *If  $G \in \mathbf{K}^{tf}$  is a  $(\lambda, \omega)$ -limit model, then  $G$  is not pure-injective.*

*Proof.* By Theorem 4.14 we have that  $G \cong \mathbb{Q}^{(\lambda)} \oplus \prod_p \overline{\mathbb{Z}}_{(p)}^{(\lambda)}^{(\aleph_0)}$ . For every  $p$ , one can show that  $\overline{\mathbb{Z}}_{(p)}^{(\lambda)}^{(\aleph_0)}$  is not pure-injective by a similar argument to the proof that  $\overline{\mathbb{Z}}_{(p)}^{(\aleph_0)}$  is not pure-injective

<sup>7</sup>We say that  $\mathbf{K}$  is *Galois-superstable* if there is  $\mu < \beth_{(2^{\text{LS}(\mathbf{K})})^+}$  such that  $\mathbf{K}$  is  $\lambda$ -Galois-stable for every  $\lambda \geq \mu$ . Under the assumption of joint embedding, amalgamation, no maximal models and  $\text{LS}(\mathbf{K})$ -tameness by [GrVas17] and [Vas18] the definition of the previous line is equivalent to any other definition of Galois-superstability given in the context of AECs.

(an argument for this is given in [Pre88, §2]). Then using that a direct product is pure-injective if every component is pure-injective (see [Fuc15, §6.1.9]), it follows that  $G$  is not pure-injective.  $\square$

Combining the results of this section with the ones of the previous section we obtain:

**Corollary 4.16.** *If  $\forall \mu < \lambda (\mu^{\aleph_0} < \lambda)$ , then for any  $G \in \mathbf{K}_\lambda^{tf}$  pure-injective there is a universal model over it.*

*Proof.* Let  $G \in \mathbf{K}_\lambda^{tf}$  be pure-injective. Since  $\lambda$  satisfies the hypothesis of Corollary 3.22, there is  $U \in \mathbf{K}_\lambda^{tf}$  universal model in  $\mathbf{K}_\lambda^{tf}$ . Then by Lemma 4.8  $G \oplus U$  is a universal model over  $G$ .  $\square$

By the above corollary, given  $G \in \mathbf{K}_{\beth_\omega}^{tf}$  pure-injective, for example  $G = \mathbb{Q}^{(\beth_\omega)}$ , there is  $H \in \mathbf{K}_{\beth_\omega}^{tf}$  such that  $H$  is universal over  $G$ . Since  $\beth_\omega^{\aleph_0} > \beth_\omega$ , by Remark 3.23 we have that  $\mathbf{K}^{tf}$  is not  $\beth_\omega$ -Galois-stable. This is the first example of an AEC with joint embedding, amalgamation and no maximal models in which one can construct universal extensions of cardinality  $\lambda$  without the hypothesis of  $\lambda$ -Galois-stability.

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