Cyclic Quadrilaterals

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1 Lecture

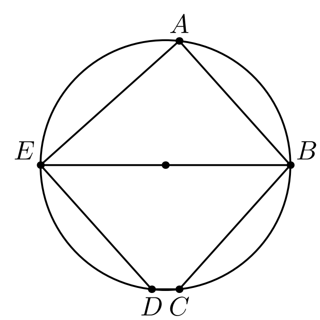
- A quadrilateral is said to be *cyclic* if it can be inscribed inside a circle.
- Let $ABCD$ be a cyclic quadrilateral. Then we have the following properties:
  - $\angle ABC + \angle ADC = \angle BCD + \angle BAD = 180°$
  - $\angle ABD = \angle ACD$, etc.
  - (Ptolemy) $AB \cdot CD + AD \cdot BC = AC \cdot BD$.
  - (Braghmaputa) Suppose a, b, c, and d are the side lengths of a cyclic quadrilateral $\mathcal{K}$, and set $s = \frac{a+b+c+d}{2}$. Then

$$[\mathcal{K}] = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

- The catch here is that these rules all go the other way around as well! So for example, if you can prove that $\angle ABD = \angle ACD$, then you know $ABCD$ is cyclic! This is very helpful, since it allows for transferring between different types of angle equalities.
- You can also use Power of a Point to determine whether four points lie on the same circle as well.

2 Problems

1. Suppose ABC is a right triangle with a right angle at B. Point D lies on side $\overline{AB}$, and E is the foot of the perpendicular from D to AC. If $\angle BAC = 17°$ and $\angle ABE = 23°$, compute $\angle DCB$.



2. [AMC 10B 2011] In the given circle, the diameter $\overline{EB}$ is parallel to $\overline{DC}$, and $\overline{AB}$ is parallel to $\overline{ED}$. The angles AEB and ABE are in the ratio $4:5$. What is the degree measure of angle BCD?
3. Suppose that P is a point on minor arc $\widehat{BC}$ of the circumcircle of equilateral triangle ABC. If $PB = 3$ and $PC = 7$, compute PA.
4. Let ABC be a right triangle with $\angle B = 90°$. Points D and E are placed such that $ACDE$ is a square. No part of the interior of the square lies inside $\triangle ABC$. Let O be the center of this square. Find $\angle OBC$.
5. Two related problems about angle bisectors.
   (a) [AIME 2016] In $\triangle ABC$ let I be the center of the inscribed circle, and let the bisector of $\angle ACB$ intersect AB at L. The line through C and L intersects the circumscribed circle of $\triangle ABC$ at the two points C and D. If $LI = 2$ and $LD = 3$, then $IC = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

(b) [Ray Li] In triangle ABC, $AB = 36$, $BC = 40$, $CA = 44$. The bisector of angle A meets BC at D and the circumcircle at E different from A. Calculate the value of DE^2.

6. [Bulgaria 1993] A parallelogram $ABCD$ with an acute angle BAD is given. The bisector of $\angle BAD$ intersects CD at point L, and the line BC at point K. Prove that the circumcenter of $\triangle LCK$ lies on the circumcircle of $\triangle BCD$.

7. [AMC 10B 2013] In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Distinct points D, E, and F lie on segments $\overline{BC}$, $\overline{CA}$, and $\overline{DE}$, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{DE} \perp \overline{AC}$, and $\overline{AF} \perp \overline{BF}$. The length of segment $\overline{DF}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

8. [USAMO 1992] Let $ABCD$ be a convex quadrilateral such that the diagonals AC and BD are perpendicular, and let P be their intersection. Prove that the reflections of P with respect to AB, BC, CD, DA lie on a circle.

9. Two more related problems.

(a) [AIME 1991] A hexagon is inscribed in a circle. Five of the sides have length 81 and the sixth, denoted by $\overline{AB}$, has length 31. Find the sum of the lengths of the three diagonals that can be drawn from A.

(b) [AMC 12B 2014] Let $ABCDE$ be a pentagon inscribed in a circle such that $AB = CD = 3$, $BC = DE = 10$, and $AE = 14$. The sum of the lengths of all diagonals of $ABCDE$ is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

10. [APMO 2007] Let ABC be an acute angled triangle with $\angle BAC = 60^\circ$ and $AB > AC$. Let I be the incenter and H the orthocenter of the triangle ABC. Prove that $2\angle AHI = 3\angle ABC$.

11. [David Altizio] Let $A_1A_2A_3A_4A_5A_6$ be a hexagon inscribed inside a circle of radius r. Furthermore, for each positive integer $1 \leq i \leq 6$ let M_i be the midpoint of the segment $\overline{A_iA_{i+1}}$, where $A_7 \equiv A_1$. Suppose that hexagon $M_1M_2M_3M_4M_5M_6$ can also be inscribed inside a circle. If $A_1A_2 = A_3A_4 = 5$ and $A_5A_6 = 23$, then r^2 can be written in the form $\frac{m}{n}$ where m and n are positive relatively prime integers. Find $m + n$.

12. [AIME 2016] Circles ω_1 and ω_2 intersect at points X and Y. Line ℓ is tangent to ω_1 and ω_2 at A and B, respectively, with line AB closer to point X than to Y. Circle ω passes through A and B intersecting ω_1 again at $D \neq A$ and intersecting ω_2 again at $C \neq B$. The three points C, Y, D are collinear, $XC = 67$, $XY = 47$, and $XD = 37$. Find AB^2.

13. [Balkan MO 1992] Let D, E, F be points on the sides BC, CA, AB respectively of a triangle ABC (distinct from the vertices). If the quadrilateral $AFDE$ is cyclic, prove that

$$\frac{4\mathcal{A}[DEF]}{\mathcal{A}[ABC]} \leq \left(\frac{EF}{AD}\right)^2.$$

14. [USAMO 2008] Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.