Similar Triangles

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1 Lecture

- We say (informally) that two triangles are *similar* to each other if one can be rotated, translated, or dilated to rest exactly on top of the other one. We say that $\triangle ABC \sim \triangle DEF$ to suggest that $\triangle ABC$ and $\triangle DEF$ are similar. Note that the order of points is important: in the statement above, we implicitly mean that under said transformations, A is taken to D, B is taken to E, and C is taken to F. (We usually say these pairs are *corresponding* parts of the triangle.)
- There are quite a few different ways to show that two triangles are similar.
  - **SSS Similarity:** If the three sides of one triangle are all in the same ratio as corresponding sides in another triangle, then the triangles are similar. (For example, if T_1 has side lengths of 3, 4, and 5, while T_2 has side lengths of 6, 8, and 10, then $T_1 \sim T_2$.
  - **SAS Similarity:** If two sides of a triangle are in the same ratio as two corresponding sides of another triangle, and furthermore the angle between these two sides is the same, then the triangles are similar. As an example of this, note that any two triangles with congruent legs must be similar to each other.
  - **AA Similarity:** If two angles of one triangle are congruent to two angles of another triangle, then the two triangles are similar to each other. (Note that we only need two conditions here as opposed to three; this is because if we know two pairs of angles are equal then the third pair must also be equal as well. As a result, this is probably the most common way you'll find yourself proving triangles similar to each other.)
- Why do we care about similarity? In short, similarity allows us to go back and forth between angle measures and length measures in a triangle. In other words, if we are able to prove that two triangles are similar using information about angles, then we automatically gain new information about some lengths in the diagram. This is especially useful in computational problems, where they might ask to compute the length of some segment when you're only given information about angles. The other direction is not as likely but also possible.
- Some common applications of similar triangles!
  - **Parallel lines:** In trapezoid $ABCD$ with $AB \parallel CD$, if P is the intersection of lines AD and BC, then $\triangle PAB \sim \triangle PDC$. This follows from using the fact that PC and PD are transversals with respect to the two parallel lines AB and CD. (Note: if instead we have $\triangle PAB \sim \triangle PCD$, we get a much different yet arguably much richer configuration. We'll be exploring this in a few weeks!)

- **Area ratios:** Suppose $\triangle T_1 \sim \triangle T_2$ with the ratio of similitude equal to K. Then

$$\frac{\text{Area}(T_1)}{\text{Area}(T_2)} = K^2.$$

This actually extends to general types of similar figures.

- **Spiral Similarity:** In general, if $\triangle PAB \sim \triangle PCD$, then $\triangle PAC \sim \triangle PBD$. This is of course assuming that none of these triangles are degenerate. Try proving this!

2 Problems

Note: Some of the problems toward the end of this set may creep into topics that we will be covering in future lectures.

1. [AHSME 1995] In $\triangle ABC$, $\angle C = 90°$, $AC = 6$ and $BC = 8$. Points D and E are on $\overline{AB}$ and $\overline{BC}$, respectively, and $\angle BED = 90°$. If $DE = 4$, then $BD =$

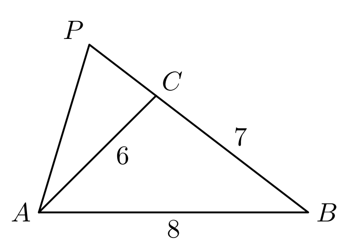
   (A) 5 **(B)** $\frac{16}{3}$ **(C)** $\frac{20}{3}$ **(D)** $\frac{15}{2}$ **(E)** 8

2. It is given that $\triangle ABC$ has $AB = 12$, $AC = 13$, and $BC = 15$. Points X and Y are placed on $\overline{AB}$ and $\overline{AC}$ respectively such that $\angle AXY = \angle ACB$. If $XY = 6$, what is $AX + AY$?

3. [Adapted from HMMT 2007] We are given four similar triangles whose areas are $1^2, 3^2, 5^2$, and 7^2. If the smallest triangle has a perimeter of 4, what is the sum of all the triangles' perimeters?

4. [HMMT Geometry 2002] Let $\triangle ABC$ be equilateral, and let D, E, and F be points on sides BC, CA, AB respectively, with $FA = 9$, $AE = EC = 6$, $CD = 4$. Determine the measure (in degrees) of $\angle DEF$.

5. [AHSME 1986] In $\triangle ABC$, $AB = 8$, $BC = 7$, $CA = 6$ and side BC is extended, as shown in the figure, to a point P so that $\triangle PAB$ is similar to $\triangle PCA$. What is the length of PC?



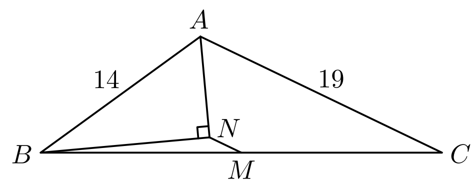
6. [Wikipedia, et. al.] A closed planar shape is said to be *equiable* if the numerical values of its perimeter and area are the same. For example, a square with side length 4 is equiable since its perimeter and area are both 16. Show that any closed shape in the plane can be stretched or shrunk to become equiable.

7. Let $\triangle ABC$ be a triangle with $AB = 13$, $BC = 14$, and $AC = 15$. Square $BCYX$ is erected outside $\triangle ABC$. Segment $\overline{AX}$ intersects $\overline{BC}$ at point P, while $\overline{AY}$ intersects it at point Q. Determine the length of $\overline{PQ}$.

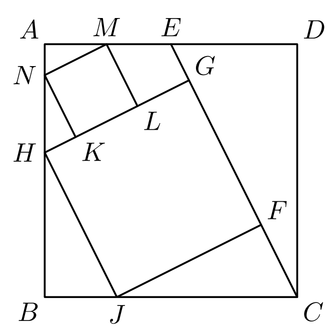
8. [Mandelbrot 2006-2007] Suppose that $ABCD$ is a trapezoid in which $\overline{AD} \parallel \overline{BC}$. Given $\overline{AC} \perp \overline{CD}$, $\overline{AC}$ bisects angle $\angle BAD$, and $area(ABCD) = 42$, then compute $area(ACD)$.

9. [OMO 2014] The points A, B, C, D, E lie on a line ℓ in this order. Suppose T is a point not on ℓ such that $\angle BTC = \angle DTE$, and $\overline{AT}$ is tangent to the circumcircle of triangle BTE. If $AB = 2$, $BC = 36$, and $CD = 15$, compute DE.

10. [AHSME 1981] In $\triangle ABC$, M is the midpoint of side BC, AN bisects $\angle BAC$, and $BN \perp AN$. If sides AB and AC have lengths 14 and 19, respectively, then find MN.



11. [AIME 2015] In the diagram below, $ABCD$ is a square. Point E is the midpoint of $\overline{AD}$. Points F and G lie on $\overline{CE}$, and H and J lie on $\overline{AB}$ and $\overline{BC}$, respectively, so that $FGHJ$ is a square. Points K and L lie on $\overline{GH}$, and M and N lie on $\overline{AD}$ and $\overline{AB}$, respectively, so that $KLMN$ is a square. The area of $KLMN$ is 99. Find the area of $FGHJ$.



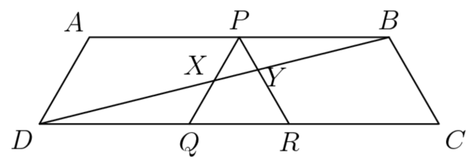
12. [AIME 1998] Let $ABCD$ be a parallelogram. Extend $\overline{DA}$ through A to a point P, and let $\overline{PC}$ meet $\overline{AB}$ at Q and $\overline{DB}$ at R. Given that $PQ = 735$ and $QR = 112$, find RC.

13. [Thomas Mildorf] ABC is an isosceles triangle with base $\overline{AB}$. D is a point on $\overline{AC}$ and E is the point on the extension of $\overline{BD}$ past D such that $\angle BAE$ is right. If $BD = 15, DE = 2$, and $BC = 16$, then CD can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Determine $m + n$.

14. [Math League HS 1977-1978] In $\triangle ABC$, $AC = 18$, and D is the point on $\overline{AC}$ for which $AD = 5$. Perpendiculars drawn from D to $\overline{AB}$ and $\overline{BC}$ have lengths 4 and 5 respectively. What is the area of $\triangle ABC$?

15. [Mandelbrot] Figure $ABCD$ below has sides $AB = 6, CD = 8, BC = DA = 2$, and $AB \parallel CD$. Segments are drawn from the midpoint P of AB to points Q and R on side

CD so that PQ and PR are parallel to AD and BC as shown. Diagonal DB intersects PQ at X and PR at Y. Evaluate PX/YR.



16. [AIME 1986] In $\triangle ABC$, $AB = 425$, $BC = 450$, and $AC = 510$. An interior point P is then drawn, and segments are drawn through P parallel to the sides of the triangle. If these three segments are of an equal length d, find d.

17. [AIME 2003] In $\triangle ABC$, $AB = 360$, $BC = 507$, and $CA = 780$. Let M be the midpoint of $\overline{CA}$, and let D be the point on $\overline{CA}$ such that $\overline{BD}$ bisects angle ABC. Let F be the point on $\overline{BC}$ such that $\overline{DF} \perp \overline{BD}$. Suppose that $\overline{DF}$ meets $\overline{BM}$ at E. The ratio $DE : EF$ can be written in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

18. [ISL 2005] Let $ABCD$ be a parallelogram. A variable line ℓ passing through the point A intersects the rays BC and DC at points X and Y, respectively. Let K and L be the centres of the excircles of triangles ABX and ADY, touching the sides BX and DY, respectively. Prove that the size of angle KCL does not depend on the choice of ℓ.

19. [All-Russian MO 2008] A circle ω with center O is tangent to the rays of an angle BAC at B and C. Point Q is taken inside the angle BAC. Assume that point P on the segment AQ is such that $AQ \perp OP$. The line OP intersects the circumcircles ω_1 and ω_2 of triangles BPQ and CPQ again at points M and N. Prove that $OM = ON$.