Miscellaneous Functions

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Logarithms

For any $b > 0$, the function $\log_b$ is defined by the property that $\log_b(b^x) = b^{log_b(x)} = x$. In other words, $\log_b(x)$ is the inverse function of b^x. The basic properties of log are:

- $\log_b(x^y) = y \log_b(x)$.
- $\log_b(xy) = \log_b(x) + \log_b(y)$ and $\log_b(x/y) = \log_b(x) - \log_b(y)$.
- $\log_b(c) = \frac{\log_a(c)}{\log_a(b)}$ for any $a > 0$.

The following useful properties can be derived from the above:

- $\log_a b = \frac{1}{\log_b a}$.
- $\log_{a^n} b = \frac{1}{n} \log_a b$.

Floor/Greatest Integer Function

The floor function (also called the greatest integer function) is defined by $\lfloor x \rfloor$ equals the greatest integer y such that $y \le x$. For example $\lfloor 1 \rfloor = 1$, $\lfloor 2.3 \rfloor = 2$, $\lfloor -10.2 \rfloor = -11$. An analogous but less common function is the ceiling function, defined by $\lceil x \rceil$ equals the least integer y such that $x \le y$. For example, $\lceil \pi \rceil = 4$, $\lceil -3.333 \rceil = -3$.

The most common application of the function $\lfloor \cdot \rfloor$ is the following:

- If n is any nonnegative integer and p is any prime, then the greatest k such that $p^k \mid n!$ is given by $k = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor$. Note that for $j > \log_k n$, the term $\left\lfloor \frac{n}{p^j} \right\rfloor = 0$, so this is always a finite sum.

1 Problems

Logarithms

1. Prove the last two log identities.
2. (AIME 00) Compute $\frac{2}{\log_4 2000^6} + \frac{3}{\log_5 2000^6}$.
3. (ARML 90) Compute the $k > 2$ such that $\log_{10}(k-2)! + \log_{10}(k-1)! + 2 = 2\log_{10} k!$.
4. (ARML 80) Compute $(\log_2 25)(\log_5 27)(\log_3 16)$.
5. (ARML 77) Find all real x such that $x^{2\log_2 x} = 8$.
6. (HMMT 02) Given that a, b, c are positive real numbers such that $\log_a b + \log_b c + \log_c a = 0$, compute $(\log_a b)^3 + (\log_b c)^3 + (\log_c a)^3$.

Greatest integer

1. How many 0s are at the end of the decimal expansion of 100!? What about the base 12 expansion?
2. (HMMT 10) Compute the sum of the positive solutions to $2x^2 - x\lfloor x \rfloor = 5$.
3. (HMMT 02) Determine all L for which $\sum_{n=1}^{L} \left\lfloor \frac{n^3}{9} \right\rfloor$ is a perfect square. Hint: use the formula $\sum_{n=1}^{k} n^3 = \frac{k^2(k+1)^2}{4}$.
4. (IMO 68) Find a closed form for $\sum_{k=0}^{\infty} \left\lfloor \frac{n+2^k}{2^{k+1}} \right\rfloor$ in terms of n.

Application of Greatest Integer (ARML 83)

The goal is to use the greatest integer function to show that for all positive integers a, b, $\frac{(2a)!(2b+1)!}{a!b!(a+b+1)!}$ and $\frac{(2a)!(2b)!}{2 \cdot a!b!(a+b)!}$ are integers. For a challenge, don't read any further and prove this directly. For a guided solution, prove the following.

1. For any $x \in \mathbb{R}$, $a \in \mathbb{Z}$, $\lfloor a + x \rfloor = a + \lfloor x \rfloor$.
2. If $x < y$ then $\lfloor x \rfloor \leq \lfloor y \rfloor$.
3. If $r, s \in [0, 1)$ then $\lfloor 2r \rfloor + \lfloor 2s \rfloor \geq \lfloor r \rfloor + \lfloor s \rfloor + \lfloor r + s \rfloor$. The same holds for all $r, s \in \mathbb{R}$.
4. If $a, b, c \in \mathbb{Z}$ and $c > 0$, then $\left\lfloor \frac{2a}{c} \right\rfloor + \left\lfloor \frac{2b+1}{c} \right\rfloor \geq \left\lfloor \frac{a}{c} \right\rfloor + \left\lfloor \frac{b}{c} \right\rfloor + \left\lfloor \frac{a+b+1}{c} \right\rfloor$.
5. If a, b are positive integers, then $\frac{(2a)!(2b+1)!}{a!b!(a+b+1)!}$ and $\frac{(2a)!(2b)!}{2 \cdot a!b!(a+b)!}$ are integers.

Logarithms and Greatest Integer

1. (AIME 1994) Find the positive integer n for which $\sum_{k=1}^{n} \lfloor \log_2 k \rfloor = 1994$.
2. (AIME 04) Let S be the set of ordered pairs (x, y) such that $x, y \in [0, 1]$, and $\lfloor \log_2 \frac{1}{x} \rfloor$ and $\lfloor \log_5 \frac{1}{y} \rfloor$ are both even. Find the area of S.
3. (IMO 76) Let $u_0 = 2$, $u_1 = \frac{5}{2}$, $u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1$ for all $n \geq 1$. Prove that $3 \log_2 \lfloor u_n \rfloor = 2^n - (-1)^n$.