Team Round

Problems from BMT 2016.

1. Define a_n such that $a_1 = \sqrt{3}$ and for all integers i, $a_{i+1} = a_i^2 - 2$. What is a_{2016}?

2. Jennifer wants to do origami, and she has a square of side length 1. However, she would prefer to use a regular octagon for her origami, so she decides to cut the four cornerse of the square to get a regular octagon. Once she does so, what will be the side length of the octagon that she obtains?

3. Let ABC be a right triangle with $AB = BC = 2$. Let ACD be a right triangle with $\angle DAC = 30°$ and $\angle DCA = 60°$. Given that ABC and ACD do not overlap, what is the area of triangle BCD?

4. How many integers less than 400 have exactly 3 factors that are perfect squares?

5. Suppose that f is a function that takes in two integers and outputs a real number, and suppose further that it satisfies

$$f(x, y) = \frac{f(x, y+1) + f(x, y-1)}{2}$$
$$f(x, y) = \frac{f(x+1, y) + f(x-1, y)}{2}.$$

   What is the minimum number of pairs (x, y) we need to evaluate to be able to uniquely determine f?

6. How many ways are there to divide 10 candies between 3 Berkeley students and 4 Stanford students, if each Berkeley student must get at least one candy? All students are distinguishable and all candies are indistinguishable.

7. How many subsets of $\{1, 2, 3, 4, 5, 6\}$ do not contain three consecutive integers?

8. What is the smallest possible perimeter of a triangle with integer coordinate vertices, area $1/2$, and no side parallel to an axis?

9. Circles C_1 and C_2 intersect at points X and Y. Point A is a point on C_1 such that the tangent line with respect to C_1 passing through A intersects C_2 at B and C, with A closer to B than C, such that $2016 \cdot AB = BC$. Line XY intersects line AC at D. If circles C_1 and C_2 have radii of 20 and 16, respectively, find the ratio of $\sqrt{1 + BC/BD}$.

10. Consider an urn containing 51 white and 50 black balls. Every turn, we randomly pick a ball, record the color of the ball, and then put the ball back into the urn. We stop picking when we have recorded n black balls, where n is an integer randomly chosen from $\{1, 2, \ldots, 100\}$. What is the expected number of turns in this game?

11. What are the last two digits of $9^{8^{\cdots^{2}}}$?

12. Consider the set of axis-aligned boxes in $\mathbb{R}^d, B(a,b) = \{x \in \mathbb{R}^d \mid \forall\, i, a_i \le x_i \le b_i\}$ for some $a, b \in \mathbb{R}^d$. In terms of d, what is the maximum number n such that there exists a set of n points $S = \{x_1, \ldots, x_n\}$ such that no matter how one partitions S into (possibly empty) sets P, Q, there exists a box B such that all the points in P are contained in B and all the points in Q are outside B?

13. Let s_1, s_2, s_3 be the three roots of $x^3 + x^2 + \frac{9}{2}x + 9$. Then
$$\prod_{i=1}^{3}(4s_i^4 + 81)$$
can be written as $2^a 3^b 5^c$. Find $a + b + c$.

14. Triangle ABC has side lengths $AB = 5, BC = 9$, and $AC = 6$. Define the incircle of ABC to be C_1, then define C_i for $i > 1$ to be externally tangent to C_{i-1} and tangent to AB and BC, Compute the sum of the areas of all circles C_n.

15. When expressed in decimal form, $(\sqrt{6} + \sqrt{7})^{1000}$ has a tens digit of a and a ones digit of b. Determine $10a + b$.