Short Math Contests	Misha Lavrov
Solutions to Problems from the 2012 St. Petersburg Math Olympiad	
Western PA ARML Practice	*October 23, 2016*

1. *Eeyore has 2012 sticks of length 1 cm, which he uses to make a rectangular grid in which each cell is a 1 cm by 1 cm square. If P is the perimeter of the grid and S its area, find $P + 4S$.*

   Solution 1: The grid is made up of S cells, each with 4 sticks along the sides, so $4S$ will count each stick as many times as it appears as the side of a cell. For sticks in the interior of the grid, this is twice (each stick is the boundary between two cells), but for sticks on the perimeter, this is only once. Therefore $P + 4S$ will count each stick twice; if there are 2012 sticks, then $P + 4S = 4024$.

   Solution 2: If the grid is $a \times b$, then it is made up of $a + 1$ horizontal lines of b sticks each, and $b + 1$ vertical lines of a sticks each, for a total of $(a + 1)b + (b + 1)a = 2ab + a + b$ sticks. So $2ab + a + b = 2012$. But $P + 4S = 2(a + b) + 4ab = 2(2ab + a + b) = 2 \cdot 2012 = 4024$.

2. *A sequence of k consecutive integers $a + 1, a + 2, \ldots, a + k$ is written on a chalkboard. If exactly 52% of the integers are even, find k.*

   The integers alternate between odd and even, so if k is even, 50% of the integers will be even; therefore k must be odd.

   The $1^{\text{st}}, 3^{\text{rd}}, 5^{\text{th}}, \ldots, k^{\text{th}}$ integers all have the same parity, and there are $\frac{k+1}{2}$ of them. The remaining integers all have the same parity, and there are $\frac{k-1}{2}$ of them. Therefore the fraction of odd integers is either $\frac{k+1}{2k} > \frac{1}{2}$ or $\frac{k-1}{2k} < \frac{1}{2}$.

   Since $52\% = \frac{52}{100} = \frac{13}{25} > \frac{1}{2}$, we must be in the first case, and can find k by solving $\frac{k+1}{2k} = \frac{13}{25}$, which yields $k = 25$.

3. *The roots of the quadratic equation $ax^2 + bx + c = 0$ are $\sin 42°$ and $\sin 48°$. Prove that $b^2 = a^2 + 2ac$.*

   Note that $\sin 48° = \cos 42°$, so if r and s are the two roots of the equation, then $r^2 + s^2 = \sin^2 42° + \cos^2 42° = 1$.

   If $ax^2+bx+c = 0$ has roots r and s, then it factors as $a(x-r)(x-s) = ax^2-a(r+s)x+ars = 0$, so $r + s = -\frac{b}{a}$ and $rs = \frac{c}{a}$. Therefore

   $$\frac{b^2}{a^2} = (r + s)^2 = r^2 + s^2 + 2rs = 1 + \frac{2c}{a},$$

   and multiplying by a^2 yields $b^2 = a^2 + 2ac$.

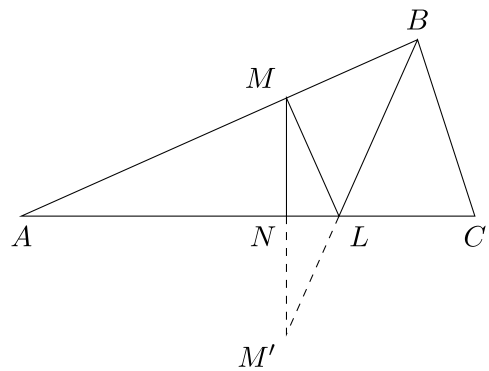
4. *Is it possible to arrange the integers $1, 2, \ldots, 100$ around a circle such that for every pair of adjacent integers x and y, at least one of the quantities $x - y, y - x, \frac{x}{y}, \frac{y}{x}$ is equal to 2?*

   This is impossible: under these rules, the integer 99 can only be adjacent to 97, but in any circular arrangement, it would have two neighbors.

5. *In $\triangle ABC$, BL is an angle bisector and $\angle C = 3\angle A$. Point M on AB and point N on AC are chosen so that $\angle AML = \angle ANM = 90°$. Prove that $BM + 2MN > BL + LM$.*

   We begin by observing some angle identities. Let α denote $\angle BAC$. Then $\angle ACB = 3\alpha$, so $\angle ABC = 180° - 4\alpha$. Since this angle is bisected by BL, $\angle ABL = 90° - 2\alpha$, and since $\angle BML = 90°$, we have $\angle BLM = 2\alpha$.

   Also, since $\angle AML = 90°$, $\angle ALM = 90° - \alpha$.



   Reflect $\triangle LMN$ over line AC, as shown above, to make $\triangle LM'N$. We notice that $\angle BLM' = \angle BLM + \angle MLN + \angle NLM' = 2\alpha + (90° - \alpha) + (90° - \alpha) = 180°$. Therefore L lies on the line BM'.

   By the triangle inequality, we have $BM + MM' > BM'$. But $MM' = MN + NM' = 2MN$, and $BM' = BL + LM' = BL + LM$, so we obtain the inequality we wanted.

6. *Prove that for distinct real numbers a, b, c, the system of equations*

$$\begin{cases} x^3 - ax^2 + b^3 = 0 \\ x^3 - bx^2 + c^3 = 0 \\ x^3 - cx^2 + a^3 = 0 \end{cases}$$

   has no real solutions.

   Suppose the contrary; that the three equations have a common real solution x. By taking $(b - c)$ times the first equation, $(c - a)$ times the second equation, and $(a - b)$ times the third equation, and adding these up, we get the equation

$$b^3(b - c) + c^3(c - a) + a^3(a - b) = 0,$$

   or $a^4 + b^4 + c^4 = a^3b + b^3c + c^3a$.

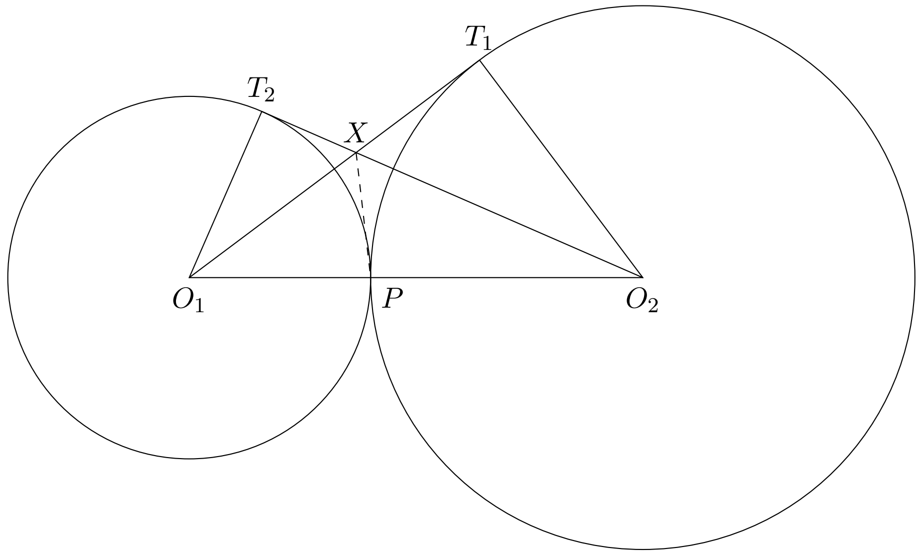
   By the AM-GM inequality, for $a \neq b$, $\frac{3a^4+b^4}{4} > \sqrt[4]{a^4 \cdot a^4 \cdot a^4 \cdot b^4} \geq a^3b$. Similarly, we get $\frac{3b^4+c^4}{4} > b^3c$, and $\frac{3c^4+a^4}{4} > c^3a$. Adding these inequalities, we get

$$a^4 + b^4 + c^4 > a^3b + b^3c + c^3a,$$

   which contradicts our earlier equation. (This inequality also follows from the rearrangement inequality).

   Since we arrived at a contradiction, the three equations cannot have a common solution x.

7. *Circles ω_1 and ω_2 are externally tangent at P. Line ℓ_1 passes through the center of ω_1 and is tangent to ω_2; similarly, line ℓ_2 passes through the center of ω_2 and is tangent to ω_1. If ℓ_1 and ℓ_2 intersect at X, prove that XP bisects one of the angles formed at X between ℓ_1 and ℓ_2.*



Let ω_i have center O_i, and T_i be the tangency point on ℓ_i, as shown in the diagram above. Also, let $O_1P = r_1$ and $O_2P = r_2$ be the lengths of the radii of ω_1 and ω_2.

We have $\angle O_1T_2X = \angle O_2T_1X = 90°$; also, $\angle O_1XT_2 = \angle O_2XT_1$. (In the diagram above, they are vertical angles; there is a second case, when they are the same angle.) So $\triangle O_1XT_2 \sim \triangle O_2XT_1$. Since $O_1T_2 = r_1$ and $O_2T_1 = r_2$, the ratio of similarity is $r_1 : r_2$, so $O_1X : O_2X = r_1 : r_2$.

But we also have $O_1P : O_2P = r_1 : r_2$ because those are actually the lengths of O_1P and O_2P. So $\frac{O_1P}{O_2P} = \frac{O_1X}{O2_X}$, and by the converse to the angle bisector theorem, XP bisects $\angle O_1XO_2$.

8. *Peter chose a natural number $n > 1$ and wrote the numbers*

$$1+n, 1+n^2, 1+n^3, \dots, 1+n^{15}$$

on a chalkboard. Then he erased some of the numbers so that among the remaining numbers, any two are relatively prime. At most how many numbers could Peter have left on the board?

Recall the factorization $x^k + 1 = (x+1)(x^{k-1} - x^{k-2} + x^{k-3} - \cdots + 1)$, valid when k is odd. Therefore:

- Any element of $\{n+1, n^3+1, n^5+1, \dots, n^{13}+1, n^{15}+1\}$ is divisible by $n+1$.
- Any element of $\{n^2+1, n^6+1, n^{10}+1, n^{14}+1\}$ is divisible by n^2+1.
- Any element of $\{n^4+1, n^{12}+1\}$ is divisible by n^4+1.
- There's only one element in $\{n^8+1\}$.

Accordingly, only one element from each group can be left on the board, so Peter must erase all but four integers.

9. *Let a and b be two distinct positive integers. The equations $y = \sin ax$ and $y = \sin bx$ are graphed in the same coordinate plane, and all of their intersection points are marked. Prove that there is a third positive integer c, distinct from a and b, such that the graph of $y = \sin cx$ passes through all the marked points.*

   Without loss of generality, $a > b$. In that case, choosing $c = 2(a+b)(a-b) + a$ will work, which we check below.

   Let (x, y) be a point such that $y = \sin ax = \sin bx$; then either $ax = bx + 2k\pi$, or else $ax = \pi - bx + 2k\pi$.

   In the first case, $x = \frac{2k\pi}{a-b}$. Then $cx = 2(a+b)(a-b)x + ax = ax + 4k(a+b)\pi$. In the second case, $x = \frac{(2k+1)\pi}{a+b}$. Then $cx = 2(a+b)(a-b)x + ax = ax + 2(2k+1)(a-b)\pi$.

   In either case, cx differs from ax by an integer multiple of 2π, so $\sin cx = \sin ax = y$, and the graph of $y = \sin cx$ passes through (x, y).