Combinatorics — Misha Lavrov

Markov Chains

Western PA ARML Practice — *March 12, 2017*

1 Warm-up

Two cows stand on opposite faces of a cube and take turns moving to a random adjacent face. A cow moving into the same face as another cow knocks the other cow over. What is the probability that the first cow to move eventually knocks over the second cow?

2 Problems

2.1 Hitting probabilities

1. Annie, C.J., and Emily take turns (in that order) flipping a fair coin. The first to flip heads wins; if none of them flip heads, the game restarts from Annie.

   What is the probability that Emily wins?

2. Suppose that the two cows from the warm-up problem decide to make their game more exciting and play it on the vertices of the cube, starting on opposite vertices. What is the probability that the first cow to move eventually knocks over the second cow?

3. In a gambling game, you start with \$1, and every time you play, you are equally likely to win or lose \$1.

   (a) What is the probability that you reach \$10 before you go broke?

   (b) What are your expected winnings in this game, assuming you decide to play until you reach \$10 or you go broke?

4. (AIME 1995) If you flip a fair coin repeatedly, what is the probability you see a run of 5 consecutive heads before seeing a run of 2 consecutive tails?

5. (AIME 2009) Dave rolls a fair six-sided die until a six appears for the first time. Independently, Linda also rolls a fair six-sided die until a six appears for the first time. What is the probability that the number of times Dave rolls his die is equal to or within one of the number of times Linda rolls her die?

2.2 Limiting distributions

1. Every day that Victor's computer is working, it breaks down with a 10% probability. Every day that Victor's computer is broken, he fixes it with a 50% probability. What fraction of the time does Victor's computer spend broken?

2. C.J. is very bad at the game "Snakes and Ladders". His version of the game has a snake with 10 spaces; his playing piece starts on space #1. Each turn, C.J. rolls a die; on a 1, 2, 3, or 4, he moves his piece back 1 space, and on a 5 or 6, he moves his piece forward one space.

   (If C.J.'s playing piece is on space #1 and he rolls 1, 2, 3, or 4, or if the playing piece is on space #10 and he rolls 5 or 6, the piece doesn't move.)

   C.J. doesn't know how to win the game, so he just keeps playing. After many turns, what is the probability that C.J. is on space $\#k$, in terms of k?

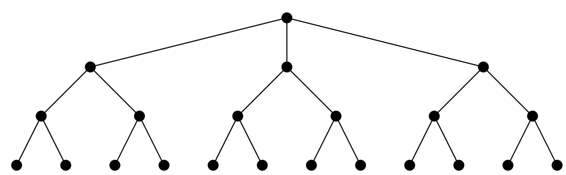
3. Misha is very good at the game "Win, Lose, or Banana", and wins $\frac{2}{3}$ of the time. Misha records the length of the winning streak: how many times he has just won in a row.

   (a) What fraction of the time is the length of Misha's winning streak exactly k?

   (b) What is the average length of Misha's winning streak?

4. A flea is placed on one vertex of the graph pictured below. Every second, the flea jumps to one of the adjacent vertices (ones which are connected to its current location by an edge). If there are multiple adjacent vertices, the flea chooses one uniformly at random.

   What fraction of the time does the flea spend on one of the 12 bottom vertices?



2.3 Hitting times

1. A monkey sitting at a typewriter types a single letter every second, which is chosen randomly from the set $\{A, B, C\}$. (All three of these letters are equally likely.)

   (a) What is the average time before the monkey types "C"?

   (b) What is the average time before the monkey types "AAA"?

   (c) What is the average time before the monkey types "ABACAB"?

2. Patrick is standing on a corner square of a human-sized 8×8 chessboard. He is very good at jumping; every second, he jumps to a randomly chosen square that's either in the same row or in the same column as his current position.

   On average, how long will it take until Patrick lands in the corner square opposite from where he started?

3. The US government releases 100 collectible trading cards with the 100 US Senators. You can buy a random card for a dollar, or you can buy the entire collection for \$400. If you want to collect all 100 cards, and duplicates are worthless to you, which is the better deal?

4. If you roll a fair six-sided die and keep a tally of how many times each value comes up, what is the expected number of rolls until each value has occurred an *odd* number of times?