Writing Proofs	Misha Lavrov

Analyzing Games

Western PA ARML Practice *March 13, 2016*

Warm-up

Two players play a game with a pile of 13 beans. They take turns picking up some number of beans; on his or her turn, a player can pick up exactly 1, exactly 4, or exactly 6 beans. The player who takes the last bean wins.

Which player (the first or the second to move) wins this game, and how?

Can you solve this problem for 333 beans instead?

Problems

1. A box contains 300 matches. Two players take turns taking some matches from the box; each player must take at least one match, but no more than half the matches. When only one match is left, the player whose turn it is (with no legal move to make) loses. Who has the winning strategy?

2. Two players play a game with a stack of 1000 pennies. They take turns taking some pennies from the top of the stack; on his or her turn, a player can either take 1 penny, or half the pennies (if the number of pennies in the stack is odd, the player takes half the pennies, rounded up).

   If the player taking the last penny wins, which player has the winning strategy?

3. (Martin Gardner) A game is played with a length of string exactly 10π inches long, tied so that it makes a loop.

   Two players take turns cutting a length of exactly 1 inch from somewhere in the string, and picking it up. Eventually, the string may end up in several pieces. In that case, the players may either pick up a piece exactly 1 inch long, or cut out a 1-inch length from the middle or end of a piece longer than 1 inch.

   The player who cannot take a turn, because all remaining pieces of string are shorter than 1 inch, loses. Which player has the winning strategy?

4. (USAMO 2004) Alice and Bob play a game on a 6×6 grid. On his or her turn, a player chooses a rational number not yet appearing in the grid and writes it in an empty square of the grid. Alice goes first and then the players alternate.

   When all squares have numbers written in them, in each row, the square with the greatest number in that row is colored black. Alice wins if she can then draw a path from the top of the grid to the bottom of the grid that stays in black squares, and Bob wins if she can't. (If two squares share a vertex, Alice can draw a path from one to the other that stays in those two squares.)

Find, with proof, a winning strategy for one of the players.

5. (BMO 2003) Alice and Barbara play a game with a pack of $2n$ cards, on each of which is written a positive integer. (The integers can be arbitrary.) The pack is shuffled and the cards laid out in a row, with the numbers facing upwards. Alice starts, and the girls take turns to remove one card from either end of the row, until Barbara picks up the final card. Each girl's score is the sum of the numbers on her chosen cards at the end of the game.

   Prove that Alice can always obtain a score at least as great as Barbara's.

6. (Putnam 1993) Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$. The last person to discard wins the game. Assuming optimal strategy by both A and B, what is the probability that A wins?

7. (Germany 1984) Two players take turns writing an intger between 1 and 6 on the board. When $2n$ numbers have been written, the game ends; the second player wins if the sum of the numbers is divisible by 9.

   For which values of n does the second player have a winning strategy?

8. (USAMO 1999) The Y2K game is played on a 1×2000 grid as follows. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.